A brief Introduction to Dispersion Relations and Analyticity[1]

Roman Zwicky,

Higgs Centre for Theoretical Physics, School of Physics and Astronomy,
University of Edinburgh, Edinburgh EH9 3JZ, Scotland

E-Mail: roman.zwicky@ed.ac.uk.

In these lectures we provide a basic introduction into the topic of dispersion relation and analyticity. The properties of 2-point functions are discussed in some detail from the viewpoint of the Källén-Lehmann and general dispersion relations. The Weinberg sum rules figure as an application. The analytic structure of higher point functions in perturbation theory are analysed through the Landau equation and the Cutkosky rules.

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Contents

1. Prologue

Dispersion relations are a powerful non-perturbative tool which have originated in classical electrodynamics in the theory of Kramers-Kronig dispersion relations. Analytic properties follow from causality and the use of Cauchy's theorem allows to obtain the physical amplitude from the knowledge of the singularities which are often physical and better accessible. This is the idea of the S-matrix program from the fifties and sixties. Dispersion relations are sparsely discussed in modern textbooks as the focus is on other aspects of Quantum Field Theory (QFT). There are some excellent older textbooks on analyticity e.g. [1,2], some modern textbooks devote some chapters to the topic e.g. [3, 4] and parts of these lectures with more emphasis on application can be found in [5]. I would hope that a student who has followed an introductory course on QFT or has read some chapters of a QFT textbook would be able to largely follow the presentation below.

2. Introduction

In the fifties and sixties QFT has found a big success in describing quantum electrodynamics (QED) thanks to the successful renormalisation program carried out by Dyson, Feynman, Schwinger, Tomonaga and others [6]. The description of the strong force with QFT proved to be difficult and there was some prejudice that a solution outside field theory had to be found. Two such approaches are dispersion theory using analytic properties [1] (Heisenberg, Chew, ..) and Wilson's operator product expansion [7]. As Weinberg remarks in his book [4] both of these approaches later became a part of QFT! By analytic properties we mean analyticity

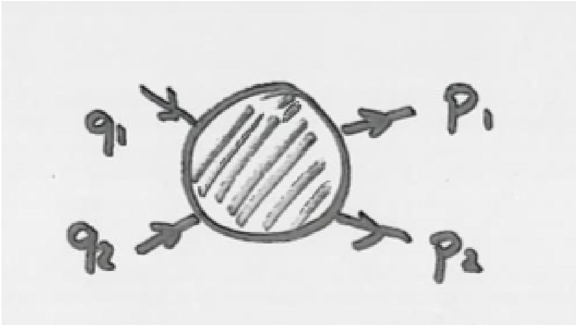

Figure 1: Schematic diagram for $2 \to 2$ scattering.

in the external momenta which in QFT have their roots in describing particles through fields (second quantisation).

A primary goal of particle physics is to describe scattering of n-particles via the so-called S-matrix. For the scattering of $2 \to 2$ particles this reads (cf. Fig.1)

$$ {}_{\rm out}\langle p_1, p_2 | q_1 q_2 \rangle_{\rm in} = {}_{\rm out}\langle p_1, p_2 | S | q_1 q_2 \rangle_{\rm out} \ , \tag{1} $$

where we have assumed the particles to be of spin 0. In the case where they are all of equal mass this implies the following on-shell conditions: $p_1^2 = p_2^2 = q_1^2 = q_2^2 = m^2$. Hence one might wonder how analytic properties come into play. The answer is through the celebrated *Lehman-Symmanzik-Zimmermann (LSZ) formula* whose derivation can be found in most textbooks e.g. [3]. For our case it reads

$$ \begin{aligned} {}_{\rm out}\langle p_1, p_2 | q_1 q_2 \rangle_{\rm in} &= -(iZ^{-1/2})^4 \int_{x_1,x_2,y_1,y_2} e^{-i(q_1\cdot x_1 + q_2 \cdot x_2 - p_1 \cdot y_1 - p_2 \cdot y_2)} K_{x_1} K_{x_2} K_{y_1} K_{y_2} \times \\ & \langle T \phi(x_1) \phi(x_2) \phi^\dagger(y_1) \phi^\dagger(y_1) \rangle + \text{disconnected terms} \ , \end{aligned} \tag{2} $$

where $\int_x = \int d^4x$, T is the time ordering, $\langle .. \rangle$ is the vacuum expectation value (VEV), the quanta are assumed to carry a charge (complex conjugation for outgoing particle), K is the Klein-Gordon operator $K_{x_1} + m^2 \to -(q_1^2 - m^2)$ and the Z factor results from the *asymptotic condition*,

$$ \langle 0 | \phi(x) | q_1 \rangle \overset{x_0 \to \mp\infty}{\to} Z^{1/2} \langle 0 | \phi_{\rm in(out)}(x) | q_1 \rangle \ , \tag{3} $$

The asymptotic condition is the key idea of the LSZ-approach. Namely that when the particle are well separated from each other all that remains is the self-interaction which is parameterised by the renormalisation factor Z. The field ϕ is what is known as an interacting field whereas $\phi_{\rm in(out)}$ are free fields in which case the right-hand side of the equation above equals $\sqrt{Z/(2\pi)^3} e^{-iq_1 \cdot x}$.[23] The disconnected part correspond, for example, to the case where particle $q_1 \to p_1$ and $q_2 \to p_2$ without any interaction which is of no interest to us.

From (2) we conclude that

[2] The LSZ formalism, in its elegancy and efficiency, also allows for the description of composite particles. For example for a pion of $SU(2)$-isospin quantum number a may be described by $\phi \to \varphi^a = \phi \bar{q} T^a \gamma_5 q$ in the sense that $\langle 0 | \varphi^a | \pi^b \rangle = g_\pi \delta^{ab}$. In such a case φ^a is referred to as an interpolating field.

[3] It is crucial that this condition is only imposed on the matrix element (weak topology) as otherwise one runs into Haag's theorem.

a) The scattering of n-particles ($n = n_{\text{in}} + n_{\text{out}}$) is described by n-point functions (or n-point correlators). The study of the latter is therefore of primary importance.

b) The n-point correlators are functions of the external momenta e.g. $p_{1,2}^2, q_{1,2}^2, p_1 \cdot p_2, ...$ First and foremost they are defined for real values or more precisely for real values with a small imaginary part e.g. $p_1^2 = \text{Re}[p_1^2] + i0$.[4] From there they can analytically continued into the complex plane. Hence it is the second quantisation, describing particles with fields, that allows to go off-shell.[5]

The course consists of three parts. Analytic properties of 2-point functions (section 3), which comes with definite answer in terms of the non-perturbative Källén-Lehmann spectral representation. Applications of 2-point function in section 4. Last a short discussion of the analytic properties of higher point function in perturbation theory (PT) e.g. Landau equations and Cutkosky rules in section 5.

3. 2-point function

3.1. Dispersion relation from 1st-principles : Källén-Lehmann representation

Let us define the Fourier transform of the 2-point correlator as follows

$$\Gamma(p^2) = i \int_x e^{ip\cdot x} \langle T\phi(x)\phi^\dagger(0)\rangle \ . \tag{4}$$

What determines the analytic structure of $\Gamma(p^2)$? By analytic structure we mean the singularities e.g. poles, branch points and the associated branch cuts. The Källén-Lehmann representation [8,9] gives a very definite answer to this question. The presentation is straightforward and can be found in most textbooks e.g. [4].

The 2-point function in the free and interacting can be written as

$$\Gamma(p^2) = \begin{cases} \frac{1}{m^2-p^2-i\epsilon} = -\Delta_F(p^2,m^2) & \text{free} \\ \frac{Z(\lambda)}{m^2-p^2-i\epsilon} + f(\lambda,p^2) & \text{interacting} \end{cases} \ . \tag{5}$$

The function $Z(\lambda)$ and $f(\lambda, p^2)$, where λ is the coupling constant e.g. $\mathcal{L}_{\text{int}} = \lambda\phi^3 + \text{h.c.}$, obey

$$Z(\lambda) \overset{\lambda\to 0}{\to} 1 \ , \quad f(\lambda,p^2) \overset{\lambda\to 0}{\to} 0 \ , \tag{6}$$

in order to obey the smooth free field theory limit. In what follows it is our goal to determine the properties of $f(\lambda, p^2)$ more precisely. At the end of this section we are going to make remarks about the possible ranges of the $Z(\lambda)$-function. The first lesson to be learnt from the free field theory case is that it is the mass (i.e. the spectrum) which determines the analytic properties cf. Fig. 2(left). As we shall see this generalises to the interacting case.

[4]When doing perturbation the reality of the momenta is implicitly used when doing the shift of momenta when completing the squares for example.

[5]In it's most standard formulation string theory is first quantised and does not allow this analytic continuation. String field theory does exist but is less developed than first quantised string theory for technical reasons.

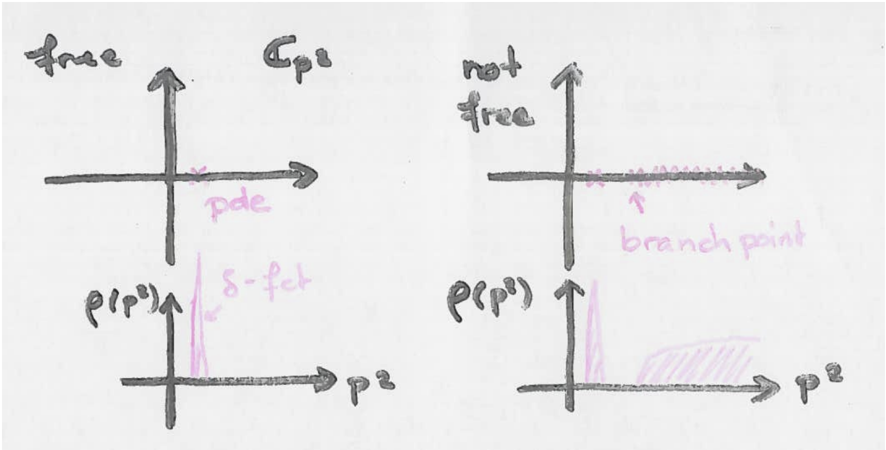

Figure 2: (left) analytic structure for free field theory propagator with spectral function underneath (right) idem for an interacting theory with a stable 1-particle state and a multiparticle-threshold

For technical reason it is advantageous to first study the positive frequency distribution

$$\langle \phi(x)\phi^\dagger(0)\rangle = \begin{cases} \Delta_+(x^2,m^2) = \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot x}\delta^+(p^2-m^2) & \text{free} \\ (*) & \text{interacting} \end{cases} \tag{7}$$

where $\delta^+(p^2-m^2) \equiv \delta(p^2-m^2)\Theta(p_0)$ assures that energies are positive and that the momenta are on the mass-shell. It is the quantity $(*)$ that we intend to study. First we use the formal decomposition of the identity into a complete set of states $\mathbb{1} = \sum_n |n\rangle\langle n|$ which follow from unitarity. Inserting this relation and using translation invariance one gets

$$(*) = \sum_n e^{-ip_n\cdot x} |\underbrace{\langle 0|\phi(0)|n(p_n)\rangle}_{\equiv f_n}|^2 \; . \tag{8}$$

Further using $1 = \frac{1}{(2\pi)^4}\int_p e^{-ip\cdot x}\int_x e^{ip\cdot x}$ and interchanging the $\sum_n$ and the $\int_x$[6] leads to

$$(*) = \int d^4p e^{-ip\cdot x} \underbrace{\equiv \sum_n \delta(p-p_n)|f_n|^2}_{(2\pi)^{-3}\rho(p^2)\Theta(p_0)} \; , \tag{9}$$

where $\rho(p^2)$ is known as the *spectral function*, $(2\pi)^{-3}$ a convenient normalisation factor and $\Theta(p_0)$ assures positive energies which come from the positive energy condition on the external momentum. Upon using $\int_p F(p) = \int_p \int ds\delta(s-p^2)F(s)$ and exchanging the ds and d^4p integration one finally gets

$$(*) = \int_0^\infty ds\rho(s)\Delta_+(x^2,s) \; , \tag{10}$$

a spectral representation.

[6]We will come back to these interchanges which are ill-defined when there are UV-divergences.

From (5) and (7) it seems plausible that this spectral representation generalises to the time ordered 2-point function as follows

$$\Gamma(p^2) = \int_0^\infty ds\rho(s)(-\Delta_F(s,p^2)) = \int_0^\infty ds \frac{\rho(s)}{s-p^2-i0} \; . \tag{11}$$

Eq. (11) is most referred to as the *Källén-Lehmann (spectral) representation.*

At this stage we can make many relevant comments.

1. The analytic properties of $\Gamma(p^2)$ are in one-to-one correspondence with the spectrum of the theory which is the answer to the question what determines the analytic properties of the 2-point function. Hence for the 2-point function there are no other singularities on the first sheet (known as the physical sheet)[7] other than on the positive real axis determined by the spectrum. The analytic structure is depicted in Fig. 2(right).

2. The spectral function $\rho(s) \geq 0$ is positive definite as a direct consequence of unitarity. [As a homework question you could try to show that for a non-unitary theory with negative normed states (i.e. $\langle gh|gh\rangle = -1$ where "gh" stands for ghost) $\rho(s)$ is looses positive definiteness.]

3. Often the spectral function is decomposed into a pole part[8]

$$\rho(s) = Z\delta(s-m^2) + \Theta(s-s_0)\sigma(s) \tag{12}$$

and continuum part $\sigma(s)$. The latter is the concrete realisation of the function $f(\lambda, s)$ in (5). In many applications f_0, the residue of the lowest state,

$$\Gamma(p^2) = \frac{|f_0|^2}{m^2-p^2-i0} + \int_{s_0}^\infty \frac{\sigma(s)}{m^2-p^2-i0} \tag{13}$$

is the non-perturbative quantity that is aimed to be extracted. The left-hand side is computed and the σ-part is then either estimated or suppressed by applying an operation to the equation. This technique is the basis of QCD sum rules [10] and lattice QCD. In the former case the σ-part is suppressed by a Borel-transformation and in lattice QCD the latter decay exponentially since euclidian correlation function are used.

4. The Källén-Lehmann representation straightforwardly applies to the case of a non-diagonal correlation function e.g. $\langle \phi_A(x)\phi_B^\dagger(0)\rangle$ but clearly positive definiteness is, in general, lost since $|f_n|^2 \to f_n^A (f_n^B)^*$.

5. As promised we return to the issue of interchanging various sums and integrals. This is of no consequence as long as there are no UV-divergences. As is well-known most field theories show UV-divergences so care has to be taken. UV-divergences demand regularisations and a prescription to renormalise the ambiguities which arise from removing the

[7] More precisely the 2-point function is at first defined for real $p^2 + i0$ with $p^2 > 0$. Analytic continuation proceeds from the upper half-plane to the left and passes below zero for negative p^2 into the lower half-plane.

[8] When the particle becomes unstable and acquires a width then the pole wanders on the second sheet since the principle that there are no singularities on the first sheet holds up e.g. [1]. This would have been an interesting additional topic which we can unfortunately not cover in these short lectures.

infinities. There two ways to write it. First, assuming a logarithmic divergence, we may amend (11) as

$$\Gamma(p^2) = \int_0^{\Lambda_{\rm UV}} ds \frac{\rho(s)}{s - p^2 - i0} + A \; , \tag{14}$$

where the so-called subtraction constant is adjusted to cancel the logarithmic divergence coming form the integral: $A = A_0 \ln(\Lambda_{\rm UV}^2/\mu_0^2) + A_1$ with μ_0 being some arbitrary reference scale. The constant A_1 has either to be taken from experiment in the case where $\Gamma(p^2)$ is physical (which implies scheme-independence) or is dependent on the scheme. The dependence in the latter case has to disappear when physical information is extracted from $\Gamma(p^2)$. A more elegant way, in our opinion, is to handle the problem with a once subtracted dispersion

$$\Gamma(p^2) = \Gamma(p_0^2) + (p^2 - p_0^2) \int_0^{\infty} ds \frac{\rho(s)}{(s - p^2 - i0)(s - p_0^2)} \; . \tag{15}$$

It is observed that the integral is now convergent due to the extra $1/(s - p_0^2)$ factor. The same remarks apply to $\Gamma(p_0^2)$ as for the previously discussed A_1.

6. Imposing the canonical commutation relation $[\partial_t \phi^\dagger(x), \phi(0)]_{x_0=0} = -i\delta(\vec{x})$ leads to the sum rule

$$\int_0^{\infty} ds \rho(s) = 1 \; , \tag{16}$$

from where one deduces that:

- $Z = 1$ for a free theory
- $0 \leq Z \leq 1$ for an interacting theory
- $Z = 0$ if ϕ is a confined field

The last case does not follow directly from (16) but is an important result due to Weinberg. An example is given by the quark propagator for which we do not expect a residue since it is a confined (coloured) particle. The fact that $Z_{\rm quark} \neq 0$ in each order in PT is a sign that the latter is not suited to describe the phenomenon of confinement.

7. By using causality, i.e. $\langle [\phi(x), \phi^\dagger(0)] \rangle = 0$ for $x^2 < 0$ spacelike, it follows that $\bar{\rho}(s) = \rho(s)$ where $\bar{\rho}(s)$ is the antiparticle spectral function associated with $\Delta_-(x^2, m^2) = \langle \phi^\dagger(x)\phi(0) \rangle$. This is a special case of the CPT theorem. Related to this matter it was Gell-Mann, Goldberger and Thirring [11] in 1954 who derived analyticity properties from causality, for $\gamma + N \to \gamma + N$, justifying dispersion relations from a non-perturbative viewpoint.

3.2. Dispersion relation and Cauchy theorem

It is our goal to characterise the spectral function $\rho(s)$ in other ways than through the spectrum. In preparation to the general case we are going to recite the optical theorem in for the S-matrix. The S-matrix (1) is conveniently parameterised as

$$S = \mathbb{1} + iT \; , \tag{17}$$

where T is the non trivial part of the scattering operator. From the unitarity of the S-matrix it follows that

$$\mathbb{1} = SS^\dagger = \mathbb{1} + \underbrace{i(T - T^\dagger)}_{-2{\rm Im}[T]} + |T|^2 \; , \tag{18}$$

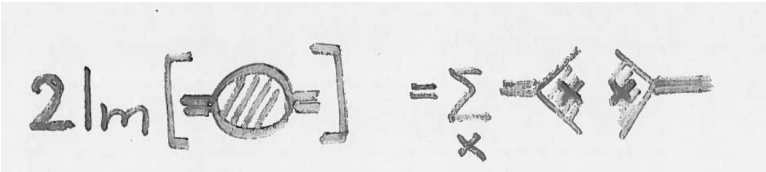

Figure 3: Standard sketch of optical theorem. The right-hand side is the sum over all intermediate states. It is a particular case of the cutting rules to be discussed in section 5.3.

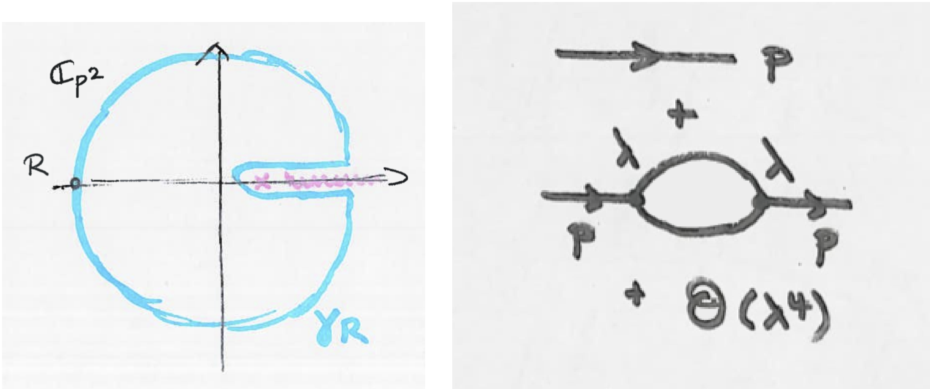

Figure 4: (left) Integration contour for 2-point function dispersion representation. (right) 2-point function in ϕ^3 theory in perturbation theory.

if and only if

$$2\text{Im}[T] = |T|^2 = T^\dagger \sum_n |n\rangle\langle n|Tx^2 \ . \tag{19}$$

Eq. (19) is the celebrated *the optical theorem* depicted in Fig. 3.

The right-hand side of the equation above is reminiscent of the spectral function in the case where T is associated with ϕ which in principle could be the case. Hence the expectation that $\rho(s)$ is related to an imaginary part is not unexpected form the viewpoint of the optical theorem. Below we are going to see that this is the case on very general grounds.

To do so we first take a little detour to discuss integral representations of arbitrary analytic functions by the use of Cauchy's theorem. Let $f(p^2)$ be analytic functions then by Cauchy's theorem the following integral representation holds

$$f(p^2) = \frac{1}{2\pi i} \int_\gamma \frac{ds\, f(s)}{s - p^2} \ , \tag{20}$$

provided that i) p^2 is inside the contour of γ , ii) the contour of γ does not cross any singularities.

Applying this techniques to 2-point function in QFT one makes use of the knowledge of the analytic structure and chooses a contour γ_R as in Fig. 4. Taking $R \to \infty$ and assuming that the arc at infinity does not contribute, which may or may not result in polynomial subtraction terms which we parameterise with $P(p^2)$, the integral can then be written as

$$\begin{aligned}\Gamma(p^2) &= \frac{1}{2\pi i}\int_{\gamma_R} \frac{ds\,\Gamma(s)}{s-p^2} \overset{R\to\infty}{\to} \frac{1}{2\pi i}\int_{s_1}^{\infty} \frac{ds\,(\Gamma(s+i\epsilon)-\Gamma(s-i\epsilon))}{s-p^2} + P(p^2) \\ &= \frac{1}{2\pi i}\int_{s_1}^{\infty} \frac{ds\,(\mathrm{disc}[\Gamma(s)]}{s-p^2-i\epsilon} + P(p^2) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{ds\,(\mathrm{Im}[\Gamma(s)]}{s-p^2-i\epsilon} + P(p^2)\ , \end{aligned} \tag{21}$$

where the second line is the definition of what is called the discontinuity along the branch cut. The last equality follows from $2i\mathrm{Im}[\Gamma(s)] = \mathrm{disc}[\Gamma(s)]$. This formula is known to practitioners of PT but can be justified on general grounds by the Schwartz reflection principle (cf. appendix A). In summary we then have that the spectral function is related to the the imaginary part and the discontinuity by

$$\rho(s) = \frac{1}{\pi}\mathrm{Im}[\Gamma(s)] = \frac{1}{2\pi i}\mathrm{disc}[\Gamma(s)]\ . \tag{22}$$

This equation follows from equating (11) and (21) and the knowledge that the subtraction constant are the same in both cases since they originate from UV divergences. Hence eliminating the contributions from the arc may result in UV-divergences and subtraction constants.

3.3. Dispersion relation in perturbation theory

This sections aims to illustrate (22) from the viewpoint of PT. In order to do PT one needs to specify a theory for which we may think of $\mathcal{L}_{\mathrm{int}} = \lambda\phi^3 + \mathrm{h.c.}$. The pole contribution is then just the propagator and the first non-trivial interaction is generated by the diagram in Fig. 4(right) and when computed leads to a term

$$\Gamma(p^2) = \frac{Z(\lambda)}{m^2-p^2-i\epsilon} - \lambda^2|A|\ln\left(1-\frac{p^2}{4m^2-i0}\right) + ... \tag{23}$$

and the corresponding imaginary part divided by π must be the spectral function

$$\rho(p^2) \overset{(22)}{=} \frac{1}{\pi}\mathrm{Im}[\Gamma(p^2)] \overset{(23)}{=} Z(\lambda)\delta(p^2-m^2) + \lambda^2|A|\Theta(p^2-4m^2) + ... \tag{24}$$

The propagator term is a pole singularity with a delta function in the spectral function and the logarithm corresponds to a branch cut singularity resulting in a Θ-function part. By the spectral representation (33) this branch cut must correspond to some physical intermediate state. This state is a 2-particle state starting at the minimum centre of mass energy $4m^2$ ranging all the way up to infinity. The precise value depends on the corresponding momentum configuration. For example let the two particle momenta be parameterised by

$$p_{1,2} = (\sqrt{m^2+x^2}, 0, 0, \pm x)\ , \quad x > 0\ , \quad p_{1,2}^2 = m^2\ , \quad p^2 = (p_1+p_2)^2 = 4m^2 + 4x^2 \tag{25}$$

and therefore $4x^2 = p^2 - 4m^2$ which can be satisfied for any (arbitrarily large) $p^2 \geq 4m^2$.

4. Application of 2-point functions

There are numerous applications of 2-point functions and dispersion relations. For example deep-inelastic scattering, QCD sum rules which we have alluded to in and below (13),

$e^+e^- \to$ hadrons and inclusive $b \to X_{u,c}\ell\nu$ decays with the additional assumption of analytic continuation to Minkowski-space.[9] We choose to present to Weinberg sum rules (WSR).

4.1. Weinberg sum rules

The Weinberg sum rules are an ingenious construction involving a few ideas. They were proposed in 1967 by Weinberg [12] in the pre-QCD era but we are going to present them from the viewpoint of QCD e.g. [5, 13]. One considers the correlation function of left and right-handed current with two massless quark flavours

$$i \int d^4x e^{iq\cdot x} \langle T J^{a,L}_\mu(x) J^{b,L}_\nu(x)\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{a,b}_{\rm LR}(q^2) \, , \tag{26}$$

where

$$J^{a,(L,R)}_\mu = \bar{q} T^a \gamma_\mu q_{L,R} \, , \tag{27}$$

with $q_{L,R} = P_{L,R}q$, $P_{L,R} = 1/2(1 \mp \gamma_5)$, T^a being an $SU(2)$-generator (Pauli-matrix). The Lorentz-decomposition in (26) is valid in the limit of massless quarks. According to the previous sections the function $\Pi^{a,b}_{\rm LR}(-Q^2)$, with $-q^2 = Q^2 > 0$ satisfies a dispersion relation of the form

$$\Pi^{a,b}_{\rm LR}(-Q^2) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{ds\, {\rm Im}[\Pi^{a,b}_{\rm LR}(s)]}{s+Q^2} + A \tag{28}$$

where A is a subtraction constant due to the potential logarithmic divergence which may arise since $\Pi^{a,b}_{\rm LR}$ is of mass dimension zero.

The peculiarity of the WSR relies on the absence of lower order corrections to the OPE. This can be seen in an elegant manner using group theory, that is the say representation theory of $SU(2)$. We denote by $1, F$ and A the trivial, fundamental and adjoint representation of $SU(2)$ which are of dimension $1, 2$ and 3. The correlation function is in the (A,A)-representation of the $(SU(2)_L, SU(2)_R)$ global flavour symmetry. Unless the contribution match this global flavour symmetry they their respective contribution has to vanish.

One considers Wilson OPE in momentum space, valid for $Q^2 = -q^2 \gg \Lambda^2_{\rm QCD}$

$$\Pi^{a,b}_{\rm LR}(-Q^2) = C_{\mathbb{1}}(Q^2)\langle \mathbb{1}\rangle + C_{\bar{q}q}(Q^2)\frac{(\langle \bar{q}^\alpha_L q^\beta_R\rangle + {\rm h.c.})}{Q^3} + C_{JJ}(Q^2)\frac{\langle J^{L,a}_\mu J^{R,b}_\mu\rangle}{Q^6} + \mathcal{O}\left(\frac{\Lambda^8_{\rm QCD}}{Q^8}\right) \, . \tag{29}$$

The functions $C(Q^2)$ are known as Wilson coefficients and carry logarithmic correction in QCD. As can be seen from the formula above the condensate terms of dimension d are suppressed by $1/Q^d$ relative to the identity term. The condensates, i.e. VEVs of operators, are non-perturbative objects and $\langle \mathbb{1}\rangle$ corresponds to PT. The latter is in $(1, 1)$-representation and therefore absent.[10] A quark bilinear term is in the (F, F)-representation and absent for the same reason. The dimension six operator, is somewhat trivially, in the (A, A)-representation and therefore the leading term at $1/Q^6$. Therefore the $1/Q^2$- and $1/Q^4$-terms have to vanish

[9] Without going into any details let us mention that it is in particular inclusive decay rate and amplitudes of exclusive decays which are amenable to a dispersive treatment. It is the amplitude and not the rate which has the simple analytic properties. The inclusive case is special in that the rate can be written as an amplitude!

[10] The practitioner will notice the absence of PT from $P_L P_R = 0$ which necessarily arises in the a perturbative computation in the massless quark limit.

and this leads to constraints. The latter can be obtained by expanding the denominator in inverse powers of Q^2,

$$\frac{1}{s+Q^2} = \frac{1}{Q^2}\frac{1}{1+s/Q^2} = \frac{1}{Q^2} - \frac{s}{Q^4} + \frac{s^2}{Q^6} + \ldots \, . \tag{30}$$

The exact sum rules on the spectral function

$$\int_{s_1}^{\infty} ds\, \Pi^{a,b}_{\rm LR}(s) = 0 \, , \qquad \int_{s_1}^{\infty} ds\, \Pi^{a,b}_{\rm LR}(s) s = 0 \tag{31}$$

known as the *first and second Weinberg sum rule* follow.

Note the absence of the perturbative term means in particular that there is no UV-divergence and hence $A = 0$. Since the convergence is even a power higher (2nd WSR) one often speaks of a superconvergent dispersion relation in this context.

The WSR (31) are a powerful non-perturbative constraint. We present the original application pursued by Weinberg. First we notice that the left-right correlator can be written as a difference of the vector and axial correlator

$$\Pi^{a,b}_{\rm LR}(s) = \frac{1}{4}\left(\Pi^{a,b}_{\rm VV}(s) - \Pi^{a,b}_{\rm AA}(s)\right) \, , \tag{32}$$

where $J^{V(A),a}_\mu \equiv \bar{q} T^a \gamma_\mu (\gamma_5) q$. Taking into account the lowest lying particles π, ρ and a_1 in the narrow width approximation and assuming isospin symmetry (i.e. global $SU(2)_V$ -flavour symmetry) one arrives at[11]

$$\begin{aligned} \rho^{a,b}_V(s) &= \frac{1}{\pi}{\rm Im}[\Pi^{a,b}_{\rm VV}](s) = \delta^{ab}(f_\rho^2 \delta(s-m_\rho^2) + \Theta(s-s_1)\sigma_V) \, , \\ \rho^{a,b}_A(s) &= \frac{1}{\pi}{\rm Im}[\Pi^{a,b}_{\rm AA}](s) = \delta^{ab}(f_\pi^2 \delta(s) + f_{a_1}^2 \delta(s-m_{a_1}^2) + \Theta(s-s_1)\sigma_A) \, , \end{aligned} \tag{33}$$

π, ρ and a_1 . The functions $\sigma_{V,A}$ contain any higher states and multiparticle states. If one assumes that around s_1 perturbation theory is valid then $\rho_{\rm LR}(s) = 0$ for $s > s_1$ which in turn implies $\sigma_V = \sigma_A$

Hence using (33) the two WSR (31) read

$$f_\rho^2 = f_\pi^2 + f_{a_1}^2 \, , \qquad m_\rho^2 f_\rho^2 = m_{a_1}^2 f_{a_1}^2 \, , \tag{34}$$

where the decay constant are defined as

$$\langle \rho[a_1]^b(p) | J^{V[A],a}_\mu | 0 \rangle = \delta^{ab} \eta_\mu(p) m_{\rho[a_1]} f_{\rho[a_1]} \, , \quad \langle \pi^b(p) | J^{A,a}_\mu | 0 \rangle = \delta^{ab} p_\mu f_\pi \, . \tag{35}$$

with η being the polarisation vector.

In his original paper Weinberg used the KSFR relation $f_\rho^2 = 2 f_\pi^2$ which then leads to $m_{a_1} = \sqrt{2} m_\rho$. This relation is reasonably satisfied experimentally: $m_{a_1}/m_\rho \simeq 1.63 \simeq 1.15\sqrt{2}$.

Let us end this section with mentioning two further application of this reasoning.

[11]Note since we work in the massless limit the pion is massless as it is the goldstone boson of broken chiral symmetry $SU(2)_L \otimes SU(2)_R \to SU(2)_V$. The spin parity quantum numbers J^P of the particles are as follows $0^-, 1^+, 1^-$ for the π, ρ and a_1 respectively.

- Being related to chirality the WSR, or the Π_{LR} function, is a measure of contributions to electroweak precision measurement in the case of physics beyond the standard model coupling to new fermions. The WSR serve to estimate the contribution of strongly coupled extension of the standard model such as technicolor and the composite Higgs model.

- The inverse moments of the spectral function, with pion pole subtracted, is related to the low energy constant L_{10} of chiral perturbation theory. Note, chiral perturbation theory is an expansion in Q^2, and not $1/Q^2$ as the OPE, and thus leads to inverse moments rather than moments themselves. It is not the WSR per se which is important in this respect but the onset of the duality threshold of PT-QCD which allows to estimate L_{10} in terms of f_{π,ρ,a_1}. The estimate of L_{10} obtained is in reasonable agreement with experiment.

5. Analytic properties of higher point functions

There are many motivations to study higher point functions and their analytic structure amongst which we quote the following

a) As seen in the introduction they describe the scattering of n-particles.

b) From the discussion in section 3.2 it is clear that to write down dispersion relations one needs to know first and foremost the analytic structure of the amplitude in question.

b) 3-point functions are relevant for the study of form factors. Consider for example the $B \to \pi$ form factor, relevant for the determination of the CKM-element $|V_{\rm ub}|$ defined by

$$\langle \pi(p)|V_\mu|B(p_B)\rangle = (p_B)_\mu f_+^{B\to\pi}(q^2) + \dots \ , \tag{36}$$

where the dots stand for the other Lorentz structure and $V_\mu = \bar{b}\gamma_\mu u$ is the weak current (the axial part does not contribute in QCD by parity conservation). Then the form factor can be extracted from the following 3-point function, by using a double dispersion relation (dispersion relation in the p_B^2 and p^2-variable)

$$\begin{aligned} \Gamma(p^2,p_B^2,q^2) &= i^2 \int_{x,y} e^{-i(p_B\cdot x - p\cdot y)} \langle T J_B(x) J_\pi(y) V_\mu(0)\rangle \ , \\ &= (p_B)_\mu \left(\frac{f_\pi\, f_B\, f_+^{B\to\pi}(q^2)}{(p_B^2 - m_B^2)(p^2-m_\pi^2)} + \text{higher} \right) + \dots \ , \end{aligned} \tag{37}$$

where "higher" stands for higher contributions in the spectrum (the analogue of $\sigma(s)$ in (12)) and

$$\begin{aligned} J_B &= \bar{q} i\gamma_5 b \ , \quad \langle B|J_B|0\rangle = f_B \ , \\ J_\pi &= \bar{q} i\gamma_5 q \ , \quad \langle 0|J_\pi|\pi\rangle = f_\pi \ , \end{aligned} \tag{38}$$

play the role of the interpolating operators of the LSZ-formalism cf. footnote 2. As previously mentioned the key is then to compute $\Gamma(p^2,p_B^2,q^2)$ in some formalism and to find ways to either estimate or suppress the higher states in order to extract the form factor where f_π and f_B are assumed to be known quantities.

In fact if we were able to compute $\Gamma(p^2, p_B^2, q^2)$ with arbitrary precision then the function would assume the form in (37) and we could simply extract the form factor from the limiting expression

$$f_+^{B\to\pi}(q^2) = \frac{1}{f_\pi f_B} \lim_{p_B^2 \to m_B^2, p^2 \to m_\pi^2} (p_B^2 - m_B^2)(p^2 - m_\pi^2)\Gamma(p^2, p_B^2, q^2) \;, \tag{39}$$

which makes the connection to the LSZ-formalism (2) apparent. Unfortunately at present we cannot hope to do so and therefore we have to resort to the approximate techniques alluded to above.

We have seen that for the 2-point function the analytic structure of the first sheet (physical sheet) is fully understood through the Källén-Lehmann representation. Moreover the singularities on the physical sheet are in one-to-one correspondence with the physical spectrum. For higher point function less is known in all generality. We refer the reader to the works of Källén-Wightman [14] and Källén [15] for some general studies of 3- and 4-point functions using first principles and the summary by Martin for a comparatively recent survey of rigorous results [16].[12]

Hence one has to become immediately more modest! We are going to restrain ourselves to analyse the singularities in PT for physical (real) momenta. This is done in two major steps:

i) Landau equations: where are the singularities (and on which sheet) cf. section 5.2

ii) Cutkosky rules: how to compute the discontinuity of an amplitude cf. section 5.3

Before analysing these matters in more details let us first consider the so-called normal-thresholds for higher point functions.

5.1. Normal thresholds: cutting diagrams into two pieces

The so-called normal thresholds are those associated with unitarity in the sense of cutting a diagram into two pieces. They are a slight generalisation of the case of the optical theorem in the sense that the diagram can be cut into two unequal pieces. We are going to look at 2,3,4-point functions depicted in Fig. 5. Cutting a diagrams into two pieces is equivalent to the combinatorial problem of grouping the external momenta into two sets. Tab. 1 provides the overview of the number of cuts versus number of independent kinematic variables. For the two lowest functions there are no constraints whereas for all higher point functions there are constraints due to momentum conservation. For the 4-point functions this constraint is known as the famous Mandelstam constraint

$$\sum_{i=1}^{4} p_i^2 = s + t + u \;, \quad s = (p_1 + p_2)^2 \;, t = (p_1 + p_3)^2 \;, u = (p_1 + p_4)^2 \;. \tag{40}$$

The corresponding constraints for 5- and higher point functions are known as the Steinmann relations. This is already an indication that higher point functions are much more complex

[12] Am important topic was the conjecture by Mandelstam of a double dispersion relation for $2 \to 2$ scattering (i.e. 4-point function) which was consistent with known results but never proven in all generality not even in perturbation theory. From this the so-called Froissart bound was derived which states that the scattering of two particles cannot grow faster than $\ln^2 s$ where s is the centre of mass energy.

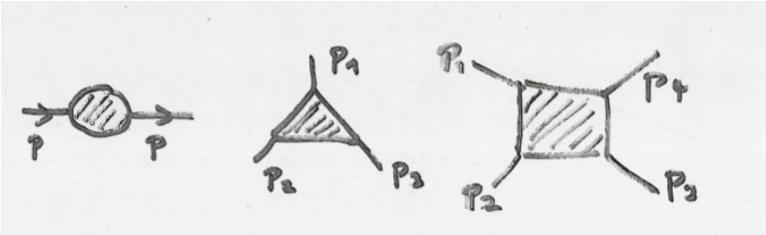

Figure 5: Sketch of generic $2, 3, 4$-point function.

than the previously studied 2-point function. In addition there are cuts which are not directly associated with unitarity, so-called anomalous thresholds, associated with cutting the diagram into more than 2 pieces. We will return to the latter briefly when discussing the Landau equations and Cutkosky rules.

n-point function	#cuts	#variables	#constrains
2	1	1	0
3	3	3	0
4	7	6	1

Table 1: The # (=number) of cuts equals the number of independent variables plus constraints. For the 2- and 3-point functions there are no constraints whereas for the 4-point function there is the famous Mandelstam constraint (40).

5.2. Landau equations

Before stating the Landau equations it is useful to look at singularities of a one-variable integral representation where the integrand has pole singularities as a function of external parameters. The Landau equations originate form analysing this problem for the integrals of several variables appearing in PT.

5.2.1. Singularities of one-variable integrals representations

Consider the following integral representation of a analytic function $f(z)$

$$f(z) = \int_{\gamma_{ab}} g(z, w) dw \; , \tag{41}$$

where the integrand $g(z, w)$ contains pole singularities $w_i(z)$ which depend on z. The path γ_{ab} ranges from a point a to b and does not cross any singularities for some $z = z_0$ as shown in Fig. 6(left). The analytic properties of $f(z)$ depend on whether or not the path γ_{ab} can be smoothly deformed not to cross any of those pole singularities.

If z is to be deformed smoothly from z_0 to z_1 and z_1 crosses the path γ_{ab} as in Fig. 6(middle) then the path γ_{ab} can be smoothly deformed as in Fig. 6(right) and this constitutes an analytic continuation of the function $f(z)$. There are though instances when this is not possible:

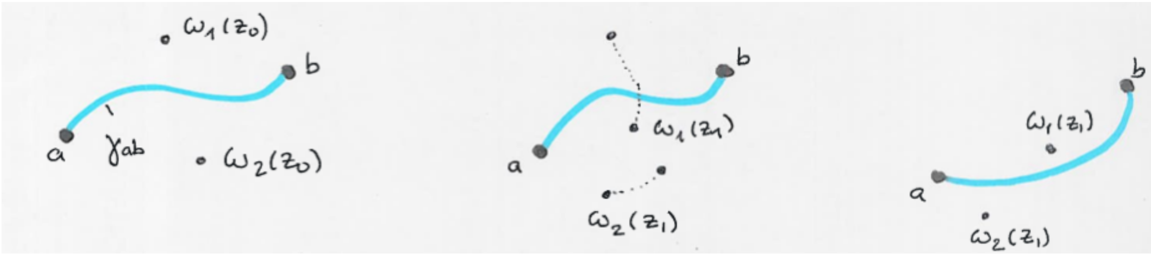

Figure 6: (left) Poles and path at $z = z_0$ (middle) path of poles while deformed by parameter z_0 to z_1 (right) deformation of path γ_{ab} serves as a legitimate analytic continuation of the function f(z) in (41).

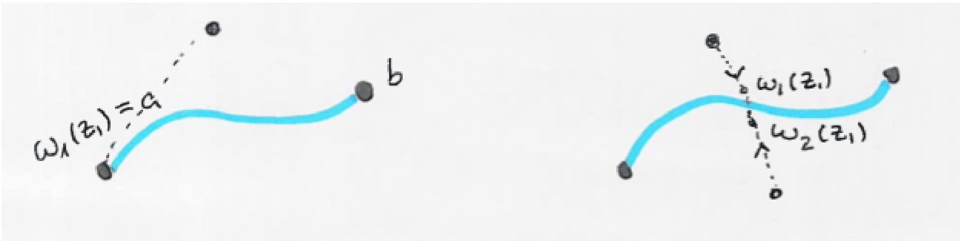

Figure 7: (left) endpoint singularity (right) pinch singularity

a) When a singularity $w_i(z)$ approaches one of the endpoints a or b; e.g. $w_i(z_s) = a$ Fig. 7(left). This case is known as an *endpoint singularity*.

b) When two singularities approach each other, $w_1(z_s) = w_2(z_s)$ from different direction of the integration path as depicted in Fig. 7(right). This case is known as a *pinch singularity*.

c) When the path needs to be deformed to infinity (can be reduced to case b).

In PT it is the pinch singularity type that gives rise to the singularity.

5.2.2. Landau equations = several variable case

Landau [17] and others (cf. for further references [1]) has analysed the problem of singularities, discussed for a single integral, for the case of several variables. A generic Feynman graph of L-loop of momenta k_i $(i = 1..L)$, N-propagators, external momenta p_i (for which we present one representative in Fig. 8) can be written as follows

$$I = \int Dk \frac{1}{\prod_{i=1}^{N}(q_i^2 - m_i^2 + i\epsilon)} \,, \qquad Dk = \prod_{i=1}^{l} d^4 k_i \,, \tag{42}$$

where $q_i = q_i(p_j, k_l)$ are the momenta of the propagators. By the technique of Feynman parameters (generalisation of $(AB)^{-1} = \int_0^1 d\alpha(\alpha A + (1-\alpha)B)^{-2}$) one may rewrite I as follows

$$I = \int Dk \int_0^1 D\alpha \frac{1}{(F + i0)^N} \,, \qquad D\alpha = \prod_{i=1}^{N} d\alpha_i \delta(1 - \sum_{i=1}^{N} \alpha_i) \,, \tag{43}$$

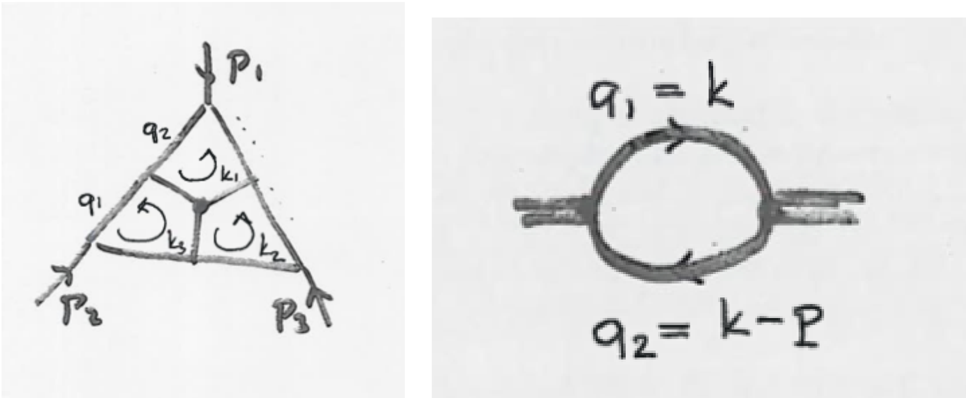

Figure 8: (left) Generic Feynman diagram aimed to clarify notation used in text (right) bubble graph discussed in the text

where most crucially the denominator F reads

$$F = \sum_{i=1}^{N} \alpha_i (q_i^2 - m_i^2) \; . \tag{44}$$

Is seems worthwhile to emphasise that even though these formulae look rather involved they are completely straightforward.

We are not going to show the proof but state the result and argue for its plausibility. The key idea is that there are different type of singularities depending on how many of the N propagators serving in addition as a classification of the singularities.

Landau equations/conditions There are singularities if and only if

$$\text{[i)] either } q_i^2 = m_i^2 \text{ or } \alpha_i = 0 \; , \tag{45}$$

$$\text{[ii)]} \sum_{i \in \text{loop}(l)} \alpha_i (q_i)^\mu = 0 \text{ for } l = 1..L. \tag{46}$$

Let us emphasise that the Landau equation neither tell us on which sheet the singularities are (cf. section 5.2.3 for the refinement in this direction) nor how to compute the discontinuity relevant for the dispersion relations (cf. Cutkosky rules cf. section 5.3). The first condition assures that $F = 0$ by demanding that each summand is zero in (44). The interpretation of $q_i^2 = m_i^2$ is of course that the corresponding propagator is on-shell and contributes to the singularity. Correspondingly $\alpha_i = 0$ means that the corresponding line does not enter the singularity. In Fig. 9 we give an example of a 3-point function cut. The second condition (46) has a geometric interpretation. It means that the corresponding singularity surfaces are parallel to each other and that the hypercontour can therefore not be deformed away from the singularity surfaces. This is the analogy of the pinch singularity discussed in section 5.2.1. Eq. (46) can be cast into a more convenient form by contracting the equation by a vector $(q_j)_\mu$ which leads to

$$\text{[ii')]} \quad Q\vec{\alpha} = 0 \; , \qquad (Q)_{ij} = q_i \cdot q_j \; , \quad (\vec{\alpha})_i = \alpha_i \tag{47}$$

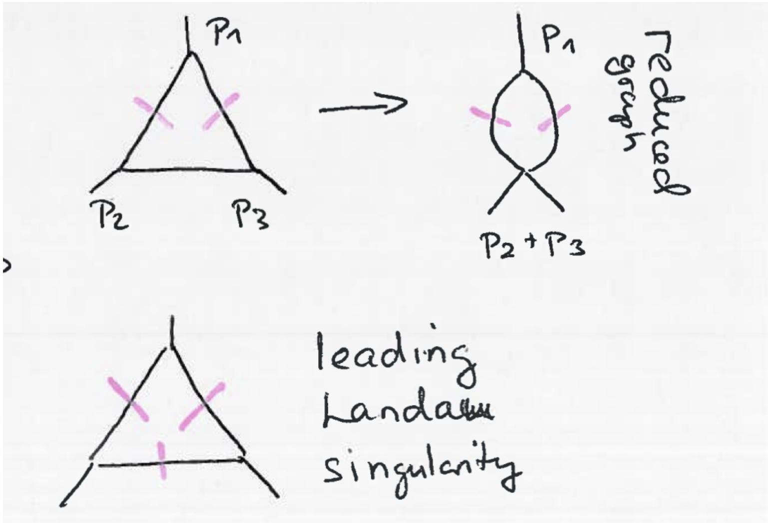

Figure 9: (top) normal threshold cut with reduced graph to the right (bottom) leading Landau singularity

Terminology of singularity As discussed in section 5.1 the singularities which arise from putting on-shell (cutting) propagators such that the diagram is cut into two pieces are known as the *normal thresholds*. For the 1-loop triangle graph this corresponds to setting two propagators on shell and a for which a reduced graph is associated cf. Fig. 9(top). Clearly one can also put all three propagators on-shell cf. Fig. 9(bottom). This contribution is known as an *anomalous threshold* whose physical interpretation is discussed below.[13] They can appear on the first sheet depending on the momentum configuration and have to be taken into account in dispersion relations. The singularity with the maximal number of on-shell propagators is usually referred to as the *leading Landau singularity.* In the case of the triangle diagram the anomalous threshold is the leading Landau singularity. It would be very interesting to pursue the significance of anomalous thresholds further but time does unfortunately not permit.

Physical interpretation of second Landau equations (46,47) In addition the second Landau equation has a direct physical interpretation due to Norton and Coleman [18] which states that (46,47) assures that the corresponding diagram can occur as a real process where $\alpha_i \sim \tau_i/m_i$ with τ_i is the proper time. This is a very nice and reassuring result in view of the interpretation of the imaginary part as the discontinuity in connection with the optical theorem (19). This means when $\alpha_i = 0$ that the corresponding particles does not propagate at all and gives the reduced graph a more profound meaning at the same time.

5.2.3. An example of Landau equation: 1-loop 2-point function (bubble graph)

Consider the bubble graph depicted in Fig. 8(right) with external momenta p, loop momenta k and momenta $q_1 = k$ and $q^2 = k - p$. The first Landau equation (45) tells us that [As a homework you could ask yourself why the case $\alpha_1 = 0$ and $q_2^2 = m_2^2$ is not an option for a

[13]Their existence can be deduced from hermitian analyticity [1] which in our case corresponds to the property that the imaginary part is proportional to the discontinuity.

singularity] $q_1^2 = m_1^2$ and $q_2^2 = m_2^2$. The second Landau equation (47) can be cast into the form $\det Q = 0$

$$\det Q = \det \begin{pmatrix} m_1^2 & q_1 \cdot q_2 \\ q_1 \cdot q_2 & m_2^2 \end{pmatrix} = 0 \quad \Leftrightarrow q_1 \cdot q_2 = \pm m_1 m_2 \tag{48}$$

which we may reinsert back into

$$p^2 = (q_1 - q_2)^2 = q_1^2 - 2q_1 \cdot q_2 + q_2^2 = (m_1 \mp m_2)^2 \ , \tag{49}$$

and Hence there are singularities starting at $p_+^2 = (m_1 + m_2)^2$ *and* $p_-^2 = (m_1 - m_2)^2$. This might surprise us at first since from unitarity we expect there to be a branch point at p_+^2 but the point $p_-^2 < p_+^2$ has no place in this picture. The resolution comes upon recalling that the Landau equations inform us about the singularities but do not tell us on which sheet they are! In order to learn more we may solve $Q\vec{\alpha} = 0$ with $\vec{\alpha} = (\alpha, (1-\alpha))^T$ which gives

$$\alpha_\pm = \frac{m_2}{m_2 \pm m_1} \Rightarrow \quad 0 < \alpha_+ < 1 \ , \quad \alpha_- > 1 \text{ or } \alpha_- < 0 \ , \tag{50}$$

for $m_{1,2} > 0$. From this we learn that α_+ is within the integration region (recall $\int_0^1 \alpha$) and p_+^2 is therefore on the first sheet, wheres α_- is outside the integration region necessitating the deformation of the α-contour. This indicates that α_- may not lay on the first sheet in the case where the contour crosses singularities in the course of deformation.

Refinement of Landau equations For physical configuration, by which we mean the real external momenta, the Landau singularities are

- on the first (physical) sheet when $\alpha_i \in [0, 1]$
- may or may not lay the first sheet when $\alpha_i \notin [0, 1]$

For non physical configuration, complex momenta, the situation is far from straightforward to say the least. The method of choice is often deformation to a case of a physical configuration and then deform back checking whether or not singularities are crossed in that process. In the latter case this signals that the singularity is not on the first sheet. Alternatively one can deform the masses to complex values keeping the $\alpha_i \in [0, 1]$ and then deform back.

5.3. Cutkosky rules

The question on how to compute the actual singularities for physical configurations is answered by the cutting rules stated by Cutkosky [19] shortly after the Landau equation were formulated. This is by no means accidental as they are closely related. The Landau equation tell us that there is a singularity if either $q_i^2 = m_i^2$ or $\alpha_i = 0$ (45) and the Cutkosky rules state that the corresponding singularity can be computed by replacing each on-shell (or cut propagator)

$$\frac{1}{q_i^2 - m_i^2 - i0} \to -2\pi i \delta^+(q_i^2 - m_i^2) \ , \tag{51}$$

with the $\delta(p^2 - m^2)^+ \equiv \delta(p^2 - m^2)\Theta(p_0)$-distribution. Before we motivate this rather elegant and surprisingly simple prescription let us state the result more explicitly.

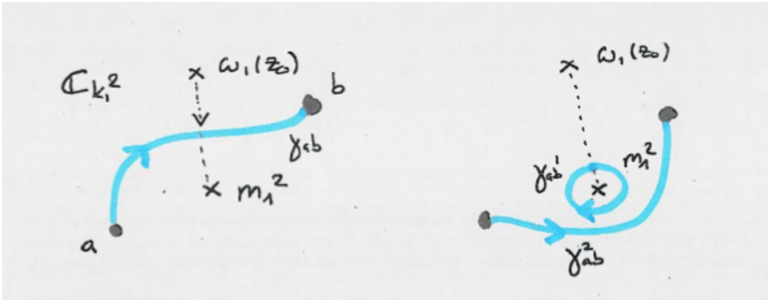

Figure 10: (left) pinch singularity (right) equivalent path $\gamma_{ab} \sim \gamma^1_{ab} + \gamma^2_{ab}$ with isolated pinch singularity

The discontinuity of I (42) with propagators $i = 1..r \leq N$ cut is given by

$$\mathrm{disc}[I] = (-2\pi i)^r \int Dk \frac{\prod_{i=1}^{r} \delta^+(q_i^2 - m_i^2)}{\prod_{i=1}^{N-r} (q_{r+i}^2 - m_{r+i}^2)} \,, \tag{52}$$

known as the Cutkosky or cutting rule!

Before trying to make plausible the formula (52) let us state the obvious. The rule (51) certainly gives the discontinuity of the propagator. The somewhat surprising fact is that this seems to be the recipe in any diagram. In the book of Peskin and Schröder [20] one can find the bubble graph evaluated in this way.

In order to motivate the Cutkosky rules we are going to sketch an argument given in the original paper [19] which is also reproduced in [1]. One considers an integral representation of the form

$$I(z) = \int dk_1^2 \frac{F(k_1^2, z)}{k_1^2 - m_1^2 - i\epsilon} \,, \tag{53}$$

where the variable z is a function of the other momenta external and internal. Let the integrand F contain a pole $w_1(z)$ which approaches m_1^2 for some z such that there is going to be a pinch singularity as shown in Fig.10(left). One then switches to the equivalent configuration where the contour is deformed below the mass m_1^2 at the cost of encircling the singularity m_1^2. In the next step the latter integral is performed using Cauchy's theorem which is equivalent to replace the denominator by $\delta(k_1^2 - m_1^2)$. This argument falls short in justifying the additional $\Theta[(k_1)_0]$-rule yet that's what we expect from the optical theorem. Repeated use of the argument above, for each propagator gives the celebrated Cutkosky rules.

6. Outlook

Even though dispersion relation are an old subject and a pure dispersive approach to particle physics proved rather complicated, it is and will remain a powerful tool in QFT as it follows from first principles and is intrinsically non-perturbative. The latter makes it particularly useful for hadronic physics but dispersion relation have also seen a major revival in evaluating perturbative diagrams in the last few years. Furthermore dispersion relation can serve to proof positivity. For example when a physical quantity can be expressed as an unsubtracted dispersion integral with positive integrand (discontinuity). Examples of which are the so-called c- and a-theorem which characterise the irreversibility of the renormalisation group flows in 2D and 4D. The dispersive proofs are given in [21,22] in two and four dimension by looking at two an four-point functions respectively.

Last but not least I would like to thank the organisers of the “Strong Fields and Heavy Quarks” as well as the participants for a stimulating atmosphere. I did really enjoy my trip to and around Dubna! I'd be grateful in case you note types for writing to me.

A. Schwartz reflection principle

Consider an analytic function $f(z)$ with $f(z) \in \mathbb{R}$ for $z \in I_R$ where I_R is an interval on the real line. Then the following relation holds

$$f(z) = f(z^*)^* \;, \tag{54}$$

which can be analytically continued to the entire plane. Note that analytic continuation is unique from any set with accumulation point for which an interval is a special case. Hence Eq. (54) implies that

$$\mathrm{Re}[f(z)] = \mathrm{Re}[f(z^*)] \;, \quad \mathrm{Im}[f(z)] = -\mathrm{Im}[f(z^*)] \;. \tag{55}$$

Choosing $z = s + i0$ with $s \in \mathbb{R}$ it then follows that

$$\mathrm{disc}[\Gamma(s)] = 2i\mathrm{Im}[\Gamma(s)] \;, \tag{56}$$

which is a result known from experience with 2-point functions and intuitively in accordance with the optical theorem.

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