Optical signatures of Quantum Vacuum Nonlinearity in strong electromagnetic fields

($\overset{\text{here}}{\dashrightarrow}$ Heisenberg Euler eff. action
& photon prop. effects $\rightsquigarrow$ vacuum birefringence)

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Felix Karbstein

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felix.karbstein@uni-jena.de

Introduction

Classical understanding : vacuum is empty

$\equiv$ absence of anything

(true / pure vacuum : no fields, boundary conditions, temperature

$\rightsquigarrow \vec{E}, \vec{B}, \vec{g}, T \to 0 \quad , V \to \infty \qquad \Leftrightarrow$ Lab)

Vacuum + electromagnetic fields

$$\mathcal{L}_{MW} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_j = -j_\mu A^\mu \qquad\qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathcal{L}_{ED} = \mathcal{L}_{MW} + \mathcal{L}_j \qquad\qquad S_{ED} = \int d^4x \, \mathcal{L}_{ED}$$

$\mathcal{L}_{ED} = \mathcal{L}(A_\nu, \partial_\mu A_\nu)$ or equivalently $\mathcal{L}(A_\nu, F_{\mu\nu})$

$\downarrow$ EOM

$$\frac{\delta S_{ED}}{\delta A_\nu} = 0 \rightsquigarrow \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\Leftrightarrow \quad \frac{\partial \mathcal{L}}{\partial A_\nu} - 2\partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = 0 \qquad \leftarrow \text{EOM}$$

dh. für $\mathcal{L}_{MW} \rightsquigarrow \partial_\mu F^{\mu\nu} = 0$ Maxwell Eqs.

$\rightsquigarrow$ superposition principle holds ;

if A_i^ν with associated $F_i^{\mu\nu}$ are solutions

then clearly also $\sum_i A_i^\nu$.

Different in <u>quantum vacuum</u> ← vacuum of QFT
↓
<u>here</u> QED

The QVac is not empty but rather permeated by fluctuations / "virtual processes" of the fields of the considered theory. In QED : electrons positrons & photons

These virtual processes can be excited / influenced by external influences, e.g. <u>strong (classical) electromagnetic fields</u> (Heisenberg & Euler '36) or boundary conditions (→ Casimir type experiments) (Casimir '48)

⤳ response encodes the <u>signal</u> / imprint of the fluctuating fields.

Quantum electrodynamics: $(\hbar = c = 1)$

$$\mathcal{L}_{QED} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

with $\not{D} = \gamma_\mu D^\mu$ & $D^\mu = \partial^\mu - ie A^\mu$

Ψ: 4-component complex Dirac spinor (anti-commuting Grassmann-valued field)

γ matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = -2 g_{\mu\nu}$

here $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$

basic principle : "all processes that can be drawn can happen (if they are not forbidden)"

generically more suppressed with increasing loop order

In a next step we focus on QED in an external / classical field $A^\mu \to A^\mu_{cl}$ and (for the moment) completely ignore dynamical / quantized photons.

→ We are only interested in the dominant / leading order effect & it can be shown that virtual dynamical photons only contribute at higher loop order.

Hence, from now on we focus on

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where now $D^\mu = \partial^\mu - ieA^\mu_{cl.}$

$$F^{\mu\nu} = \partial^\mu A^\nu_{cl} - \partial^\nu A^\mu_{cl.}$$

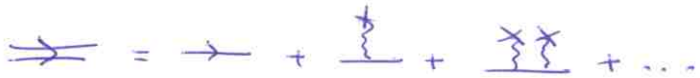
$\longrightarrow$ 

no propagating photons

Here, we will only be interested in effective interactions among electromagnetic fields (→ lasers $^{(\text{as probe})}_{\gamma}$, "optical signatures") and do not consider situations / signatures with <u>real</u> e^+/e^- (in the initial & final states) → "~~all~~-optical".

Note that it is immediately clear that all possible Feynman diagrams (connected), which can be drawn for this theory are without external e^+/e^- lines

with "dressed propagator"



In order to evaluate ◎ explicitly we turn to the partition function Z and (in a first step) integrate out the Dirac field providing us with an <u>effective action</u> without any expl. reference to the Dirac field.

To this end, recall that $\int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi \, e^{\bar{\Psi} M \Psi} = \det M$.

$$Z[A^\mu_{cl}] \sim \int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi \, e^{iS[A^\mu_{cl}]}$$

$$= \int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi \, e^{i\int d^4x \, \bar{\Psi}(i\not{D}-m)\Psi + iS_{MW}}$$

$$S_{MW} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\right)$$

$$= e^{iS_{MW}} \det(-\not{D} - im) = e^{i[S_{MW} - i\ln\det(-\not{D}-im)]}$$

$$\sim e^{iS_{eff}[A^\mu_{cl}]} \leftarrow \text{defined up to a constant}$$

It is convenient to demand that $Z[A^\mu_{cl}=0] = 1 \Leftrightarrow S_{eff}[A^\mu_{cl}=0] = 0$

$\rightarrow$ In turn $Z[A^\mu_{cl}] = \dfrac{e^{iS_{MW}} \det(-\not{D}-im)}{\det(-\not{\partial}-im)}$

$$= e^{iS_{MW}} \frac{\det(-i\not{D}+m)}{\det(-i\not{\partial}+m)}$$

and $S_{eff}[A^\mu_{cl}] = S_{MW} - i\ln\det(-i\not{D}+m)$

$$+ i\ln\det(-i\not{\partial}+m)$$

$$\equiv S_{MW}[A^\mu_{cl}] + S^1[A^\mu_{cl}]$$

with $S^1[A^\mu_{cl}] = -i\ln\det(-i\not{D}+m) + i\ln\det(-i\not{\partial}+m)$

"  "

In even spacetime dimensions we can define matrix γ_5 which fulfills $\{\gamma^\mu, \gamma_5\} = 0$; $\gamma_5^2 = \mathbb{1}$

$(\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3)$

Therewith, $\det(-i\not{D}+m) = \det[(-i\not{D}+m)\gamma_5^2]$

$$= \det[\gamma_5(+i\not{D}+m)\gamma_5] = \det(i\not{D}+m)$$

$$\Leftrightarrow \det{}^2(-i\not{D}+m) = \det(-i\not{D}+m)\det(i\not{D}+m)$$
$$= \det(\not{D}^2+m^2)$$

$$\rightarrow \ln\det(-i\not{D}+m) = \frac{1}{2}\ln\det(\not{D}^2+m^2)$$

such that $S^1 = -\frac{i}{2}\ln\det(\not{D}^2+m^2) + \frac{i}{2}\ln\det(\not{\partial}^2+m^2)$

$$\not{D}^2 = D_\mu D_\nu \gamma^\mu\gamma^\nu = D_\mu D_\nu \frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right)$$

$$\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu}\mathbb{1}$$
$$[\gamma^\mu,\gamma^\nu] \equiv -2i\sigma^{\mu\nu} \quad \leftarrow \text{antisym.}$$

$$= -D^2 - i\sigma^{\mu\nu}D_\mu D_\nu$$
$$= -D^2 - \frac{i}{2}\sigma^{\mu\nu}\underbrace{[D_\mu,D_\nu]}_{=\frac{e}{i}F_{\mu\nu}}$$

$$= -D^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$$

and hence $S^1 = -\frac{i}{2}\ln\det(-D^2+m^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu})$

$$+\frac{i}{2}\ln\det(-\partial^2+m^2)$$

$\ln\det\ldots = \mathrm{Tr}\ln\ldots \quad (\ln\prod_i \lambda_i = \sum_i \ln\lambda_i)$

$$\rightarrow S^1 = -\frac{i}{2}\mathrm{Tr}\ln(-D^2+m^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}) + \frac{i}{2}\mathrm{Tr}\ln(-\partial^2+m^2)$$

In the next step we employ the "propertime representation" of the log ; Schwinger '51

$$\ln M - \ln M_0 = \lim_{\Lambda\to\infty}\left\{-\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\left(e^{-MT}-e^{-M_0T}\right)\right\}$$

for $\mathrm{Re}\{M, M_0\} > 0$; sufficient here, cf. below.

Here, $\Lambda :=$ <u>regulator</u> for UV-divergences $\Leftrightarrow$ large energies

This follows straightforwardly from

$$\lim_{\Lambda\to\infty} \int_{1/\Lambda^2}^{\infty} dT\, e^{-\tilde{M}T} = \frac{1}{\tilde{M}} \quad \text{by integration } \int_{M_0}^{M} d\tilde{M}$$

and exchange of int.

$$\rightsquigarrow S^1 = \frac{i}{2}\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\left\{ \mathrm{Tr}\, e^{-(-D^2+m^2-\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu})T} - \mathrm{Tr}\, e^{-(-\partial^2+m^2)T}\right\}$$

where $\lim\limits_{\Lambda\to\infty}$ is implicitly understood (to be taken at the very end of the calculation).

Ende 1. Vorl.

This expression is still valid for arbitrary A^μ_{cl} fields (At least $\mathrm{Tr}\{\ldots\}$) can be evaluated explicitly for several special cases:

- $\rightsquigarrow$ constant electromagnetic fields, arbitrary orientation
- $\rightsquigarrow$ plane wave "null" fields $A^\mu_{cl} \sim \cos(k^\mu x_\mu)$ with $k^2=0$ and $\vec{E} \perp \vec{B}$, $|\vec{E}| = |\vec{B}|$. $\rightsquigarrow S^1 = 0$
- $\rightsquigarrow$ 1d magnetic field inhomogeneity $\sim \mathrm{sech}^2(x) = \frac{1}{\mathrm{ch}^2(x)}$
- $\rightsquigarrow$ starting point of worldline formalism to QED $\rightsquigarrow$ numerics

Subsequently we stick to a <u>constant magnetic field</u>

$\rightsquigarrow$ explicit evaluation of $S^1[B]$

choose $A^\mu_{cl} = (0,0,Bx,0) \rightsquigarrow \vec{B} = B\vec{e}_z$

$\rightsquigarrow F_{12} = -F_{21} = B$; all other components vanish

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & & \vec{E}^T & \\ \hline & 0 & B_3 & -B_2 \\ -\vec{E} & -B_3 & 0 & B_1 \\ & B_2 & -B_1 & 0 \end{array}\right)$$

To this end we use the following conventions for the γ-Matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

with Pauli matrices σ^i; $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\frac{1}{2}[\sigma^i, \sigma^j] = i\epsilon_{ijk}\sigma^k$$

$$\rightsquigarrow \sigma^{12} = \frac{i}{2}\begin{pmatrix} [\sigma^2,\sigma^1] & 0 \\ 0 & [\sigma^2,\sigma^1] \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\rightsquigarrow \frac{e}{2} F^{\mu\nu}\sigma_{\mu\nu} = e F^{12}\sigma_{12} = eB \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$D^2 = (\partial^\mu - ieA^\mu_{cl})(\partial_\mu - ieA_{cl,\mu})$$

$$= -\partial_t^2 + \partial_z^2 + D_\perp^2 \qquad \text{with } D_\perp^2 = \sum_{j=1}^{2}(\partial^j - ieA^j_{cl})(\partial_j - ieA_{cl,j})$$

$$\rightsquigarrow D_\perp^2 = \partial^j\partial_j - ie(\partial_j A^j_{cl}) - 2ie A^j_{cl}\partial_j - e^2 A^j_{cl} A_{cl,j}$$

$$= \partial_x^2 + \partial_y^2 - 2ieBx\,\partial_y - (eBx)^2$$

$$= \partial_x^2 + (\partial_y - ieBx)^2$$

In order to perform the trace, we need the eigenvalues of D^2.

$\rightsquigarrow$ eigenvalues of $-(-\partial_t^2 + \partial_z^2)$ are plane wave eigenvalues $-p_t^2 + p_z^2$ & after rotation to Euclidean $p_t \rightarrow ip_\tau$

$+p_\tau^2 + p_z^2$

$\rightsquigarrow$ eigenvalues for $-D_\perp^2$?

$$-D_\perp^2 f(x,y) = \lambda f(x,y)$$

$$-[\partial_x^2 + (\partial_y - ieBx)^2] f(x,y) = \lambda f(x,y)$$

ansatz $f(x,y) = e^{ip_y y} f(x)$

$$-\left[\partial_x^2 - (p_y - eBx)^2\right] f(x) = \lambda f(x)$$

$$-\left[\partial_x^2 - \left(eB\left(x - \frac{p_y}{eB}\right)\right)^2\right] f(x) = \lambda f(x)$$

defining $\varrho = \sqrt{eB}\left(x - \frac{p_y}{eB}\right)$, $\frac{\partial}{\partial x} = \frac{\partial \varrho}{\partial x}\frac{\partial}{\partial \varrho} = \sqrt{eB}\,\frac{\partial}{\partial \varrho}$

$$-eB\left[\partial_\varrho^2 - \varrho^2\right]\tilde{f}(\varrho) = \lambda \tilde{f}(\varrho)$$

$$\left[(-i\partial_\varrho)^2 + \varrho^2\right]\tilde{f}(\varrho) = \frac{\lambda}{eB}\tilde{f}(\varrho)$$

cf. harmonic oscillator in quantum mechanics (1d)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \quad , \quad \hat{p} = -i\hbar\partial_x$$

$\to$ spectrum $\hat{H}|\psi\rangle = E|\psi\rangle$

(pos. space $\hat{H}\psi(x) = E\psi(x)$)

$\to$ eigenvalues $E_n = \hbar\omega(n + 1/2)$, $n \in \mathbb{N}_0^+$

$\to$ $2\hat{H}\psi(x) = \left[\hat{p}^2 + \hat{x}^2\right]\psi(x) = 2E\psi(x)$

for $\hbar = m = \omega = 1$

$\leftrightarrow$ $\frac{\lambda_n}{eB} = 2E_n = 2\left(n + \frac{1}{2}\right)$

$\to$ $\lambda_n = 2eB\left(n + \frac{1}{2}\right)$ Landau levels

Hence, $\mathrm{Tr} \to i\int \frac{dp_\tau}{\frac{2\pi}{L}} \int \frac{dp_z}{\frac{2\pi}{L}} \sum_{n=0}^{\infty} g(n)\, \mathrm{tr}_\gamma$

$\uparrow$ density of states for Landau levels

$$-D^2 + m^2 \to p_\tau^2 + p_z^2 + 2eB\left(n + \frac{1}{2}\right) + m^2$$

1st term of S^1.

$$\left.\begin{array}{l} \text{while } \mathrm{Tr} \to 4i \displaystyle\int \frac{dp_\tau}{\frac{2\pi}{L}} \int \frac{dp_x}{\frac{2\pi}{L}} \int \frac{dp_y}{\frac{2\pi}{L}} \int \frac{dp_z}{\frac{2\pi}{L}} \\ \\ \qquad -\partial^2 + m^2 \to p_\tau^2 + p_x^2 + p_y^2 + p_z^2 + m^2 \end{array}\right\} \begin{array}{l}\text{2nd} \\ \text{term of } S^1\end{array}$$

$g(n)$ can be worked out from limit $eB \to 0$

(and accounting for $S_{eff} \simeq$ extensive quantity $\sim V$)

$$\lim_{eB\to 0} \sum_{n=0}^{\infty} g(n) \overset{!}{=} \int \frac{dp_x}{\frac{2\pi}{L}} \int \frac{dp_y}{\frac{2\pi}{L}} = \frac{L^2}{(2\pi)^2} \int_0^\infty dp_\perp p_\perp \int_0^{2\pi} d\varphi$$

$$= \frac{L^2}{4\pi} \int_0^\infty dp_\perp^2 = \frac{L^2}{4\pi} \lim_{eB\to 0} \sum_{n=0}^{\infty} \Delta p_\perp^2$$

as $p_\perp^2 = p_x^2 + p_y^2 \Leftrightarrow \Delta p_\perp^2 = 2eB$

$$= \frac{L^2}{2\pi} \lim_{eB\to 0} \sum_{n=0}^{\infty} eB$$

we obtain $g(n) = \frac{L^2}{2\pi} eB$.

And with
- $\displaystyle\int \frac{dp}{\frac{2\pi}{L}} e^{-p^2 T} = L \frac{1}{\sqrt{4\pi T}}$

- $$\sum_{n=0}^{\infty} e^{-eBT(2n+1)} = \frac{e^{-eBT}}{1 - e^{-2eBT}} = \frac{1}{e^{eBT} - e^{-eBT}}$$
$$= \frac{1}{2\sinh(eBT)}$$

- $$\mathrm{tr}_\gamma\, e^{\frac{e}{2} G^{\mu\nu} F_{\mu\nu} T} \underset{\substack{\uparrow \\ \text{B field}}}{=} \mathrm{tr}_\gamma\, e^{eBT \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}}$$
$$= 2\left(e^{eBT} + e^{-eBT}\right) = 4\cosh(eBT)$$

we finally obtain

$$S^1 = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T^2} e^{-m^2 T} \frac{L^2}{4\pi} \left\{ \frac{L^2}{2\pi} eB\, 2\coth(eBT) - \frac{L^2}{\pi} \frac{1}{T} \right\}$$

$$S^1 = -\frac{L^4}{8\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^3}\,e^{-m^2T}\left\{(eBT)\coth(eBT)-1\right\}$$

$\cong$ Heisenberg-Euler action (unren.) for purely B-field

Note that $\chi\coth\chi - 1 = \underline{\frac{1}{3}\chi^2} - \frac{1}{45}\chi^4 + \mathcal{O}(\chi^6)$

$\rightsquigarrow$ log-type divergence for $\Lambda\to\infty$

We write (subtracting & adding)

$$S^1 = -\frac{L^4}{8\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^3}\,e^{-m^2T}\left\{eBT\coth(eBT)-\frac{1}{3}(eBT)^2-1\right\}$$

$$-\frac{L^4}{24\pi^2}(eB)^2\underbrace{\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\,e^{-m^2T}}_{\text{subst. } m^2T=\tilde T}\rightsquigarrow\int_{(\frac{m}{\Lambda})^2}^{\infty}\frac{d\tilde T}{\tilde T}\,e^{-m^2\tilde T}$$

$$\overset{\substack{v\ll 1\\ \text{fixed}}}{=}\int_v^{\infty}\frac{d\tilde T}{\tilde T}\,e^{-m^2\tilde T}+\int_{(\frac{m}{\Lambda})^2}^{v}\frac{d\tilde T}{\tilde T}\left(1-\tilde T+\mathcal{O}(\tilde T^2)\right)$$

$$=\ln\frac{\Lambda^2}{m^2}+\text{const.}+\mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)$$

$\rightsquigarrow$ the divergent contribution is $\sim B^2$

cf. $S_{MW} = -L^4\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -L^4\frac{1}{2}(B^2-E^2) \underset{E=0}{=} -L^4\frac{1}{2}B^2 \quad \sim B^2$

$\rightsquigarrow S_{eff} = S_{MW} + S^1 \quad ; \quad \mathcal{L}_{eff} = \frac{S_{eff}}{L^4}$

$$\mathcal{L}_{eff}(B) = -\frac{1}{2}B^2\left\{1+\frac{e^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{\mu^2}+\text{const.}\right]\right\}$$

$$-\frac{e^2}{24\pi^2}B^2\ln\frac{\mu^2}{m^2}$$

$$-\frac{1}{8\pi^2}\int_0^{\infty}\frac{dT}{T^3}\,e^{-m^2T}\left\{eBT\coth(eBT)-\frac{1}{3}(eBT)^2-1\right\}$$

Recall that so far we have worked with unrenormalized fields and not bothered about renormalization.

→ Renormalization: $\xrightarrow{!}$ the term $\sim B^2$ should match the physical (measurable) Maxwell term

To this end we introduce wave function renormalization

$$Z^{-1} \equiv 1 + \frac{e^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{\mu^2} + \text{const.}\right]$$

and define $B_R^2 = Z^{-1}B^2$. (Note that $A^\mu_{cl} \sim B \leftrightarrow A^\mu_{cl,R} = Z^{-1/2}A^\mu_{cl}$.)
With this rescaling, the vertex/interaction in the Lagrangian of the microscopic theory $e A^\mu_{cl}\bar\Psi\gamma_\mu\Psi \to e Z^{1/2} A^\mu_{cl,R}\bar\Psi\gamma_\mu\Psi$.
Demanding it to be given by $e_R A^\mu_{cl,R}\bar\Psi\gamma_\mu\Psi$ we infer $e_R^2 = Z e^2$.
In turn, we have

$$B_R = Z^{-1/2}B = B_R(\mu)$$
$$e_R = Z^{1/2}e = e_R(\mu)$$

but $e_R B_R = eB$, μ-indep.

Correspondingly, the renormalized H.E. Lagrangian reads

$$\mathcal{L}^{ren}_{eff}(B) = -\frac{1}{2}B_R^2 - \frac{e_R^2}{24\pi^2}B_R^2\ln\frac{\mu^2}{m^2}$$
$$-\frac{1}{8\pi^2}\int_0^\infty\frac{dT}{T^3}e^{-m^2T}\left\{e_RB_RT\coth(e_RB_RT) - \frac{1}{3}(e_RB_RT)^2 - 1\right\}.$$

Conventionally the "on-shell" renormalization condition $\mu\equiv m$ is adopted

$$\frac{e_R^2(\mu=m)}{4\pi} = \alpha_R(\mu=m) \simeq \frac{1}{137}$$

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$\overset{\mu=m}{\rightsquigarrow}$ $\mathcal{L}^{ren}_{eff}(B) = -\frac{1}{2}B_R^2 - \frac{1}{8\pi^2}\int_0^\infty\frac{dT}{T^3}e^{-m^2T}\left\{e_RB_RT\coth(e_RB_RT) - \frac{1}{3}(e_RB_RT)^2 - 1\right\}$

It can be shown that for arbitrary/generic constant electromagnetic fields $\mathcal{L}_{eff} \equiv \mathcal{L}_{eff}(\mathcal{F},\mathcal{G}^2)$ with $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $\mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde F^{\mu\nu}$

with $\tilde F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

$$\left(\quad \mathcal{F} = \frac{1}{2}(\vec B^2 - \vec E^2)\ ,\quad \mathcal{G} = -\vec E\cdot\vec B\quad\right)$$

In order to write it compactly we moreover define

$$a = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2} - \mathcal{F}\right)^{1/2},\quad b = \left(\sqrt{\mathcal{F}^2+\mathcal{G}^2} + \mathcal{F}\right)^{1/2}.$$

The result is

$$\mathcal{L}^{\text{ren}}_{\text{eff}}(\mathcal{F},\mathcal{G}^2) = -\mathcal{F} - \frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2 T}\Big\{(eT)^2 ab \coth(Teb)\cot(eTa) + \frac{1}{3}(eT)^2(a^2-b^2) - 1\Big\}$$

→ contains all orders in $\mathcal{F}, \mathcal{G}^2$

→ is even in e ↔ charge conjugation invariance of QED   Furry '37

To lowest order – counting $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\mathcal{G}) = \mathcal{O}(F^2)$ – we obtain

$$\mathcal{L}_{\text{eff}} = -\mathcal{F} + \underbrace{\frac{8}{45}\frac{\alpha^2}{m^4}}_{\equiv c_1}\mathcal{F}^2 + \underbrace{\frac{14}{45}\frac{\alpha^2}{m^4}}_{\equiv c_2}\mathcal{G}^2 + \mathcal{O}\Big(\big(\tfrac{eF}{m^2}\big)^6\Big).$$

Application(s)?

Let us now study light propagation

The results derived for constant electromagnetic fields can be adopted for "weakly varying" fields also.

(Atleast for weak fields) The QED scale is m ↔ length scale $\lambda_c = \frac{1}{m} = 3{,}8\cdot 10^{-13}\,m$. Consider derivative expansion around const.-field result → derivatives are rendered dimensionless by λ_c ~ $(\lambda_c \partial_x)^n \sim \left(\frac{\omega}{m}\right)^n$ where ω is the typical frequency scale of variation of the considered field configuration. For $\left(\frac{\omega}{m}\right) \ll 1$ these can be neglected and we can employ the substitution $F^{\mu\nu} \to F^{\mu\nu}(x)$ in constant-field-result for $\mathcal{L}_{\text{eff}}$.

Let us stick to this assumption and consider the equations of motion (cf. begin of lecture)

Here $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(F^{\mu\nu}) = \mathcal{L}_{\text{eff}}(\mathcal{F},\mathcal{G}^2)$; $\mathcal{L}_{\text{eff}} = -\mathcal{F} + \mathcal{L}^1$

$$\rightarrow \partial_\mu \frac{\partial \mathcal{L}_{\text{eff}}}{\partial F_{\mu\nu}} = 0 \quad \longrightarrow \quad \partial_\mu\left(F^{\mu\nu} - \frac{\partial \mathcal{L}^1}{\partial \mathcal{F}}F^{\mu\nu} - \frac{\partial \mathcal{L}^1}{\partial \mathcal{G}}\tilde{F}^{\mu\nu}\right) = 0$$

$$\frac{\partial \mathcal{L}^1}{\partial \mathcal{F}} = 2c_1\mathcal{F}\Big(1+\mathcal{O}\big(\big(\tfrac{eF}{m^2}\big)^2\big)\Big), \quad \frac{\partial \mathcal{L}^1}{\partial \mathcal{G}} = 2c_2\mathcal{G}\Big(1+\mathcal{O}\big(\big(\tfrac{eF}{m^2}\big)^2\big)\Big)$$

$$\rightsquigarrow \partial_\mu \left\{ (1-2c_1 \mathcal{F}) F^{\mu\nu} - 2c_2 \mathcal{G} \tilde{F}^{\mu\nu} \right\} = \mathcal{O}(F^5)$$

In next step we decompose $F^{\mu\nu}(x) \to \underset{\substack{\uparrow \\ \text{const.}}}{F^{\mu\nu}} + \underset{\substack{\uparrow \\ \text{probe}}}{f^{\mu\nu}(x)}$

and linearize in $f \ll F$.

Note that $\mathcal{F} \to \mathcal{F} + \frac{1}{2} F_{\mu\nu} f^{\mu\nu} + \mathcal{O}(f^2)$

$$\mathcal{G} \to \mathcal{G} + \frac{1}{2} \tilde{F}_{\mu\nu} f^{\mu\nu} + \mathcal{O}(f^2)$$

Generic $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ fulfills Bianchi Id.

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \quad \leftarrow \text{checked by insertion}$$

$$\Leftrightarrow \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\overset{\text{Bianchi}}{\rightsquigarrow} \partial_\mu \left\{ (1-2c_1 \mathcal{F}) F^{\mu\nu} \right\} - 2c_2 (\partial_\mu \mathcal{G}) \tilde{F}^{\mu\nu} = \mathcal{O}(F^5)$$

$$\overset{\text{lin.}}{\rightsquigarrow} \quad (1-2c_1 \mathcal{F})\, \partial_\mu f^{\mu\nu} - 2c_2 \underbrace{(\partial_\mu \mathcal{G})}_{\to 0} \tilde{f}^{\mu\nu}$$

$$- 2c_1 \partial_\mu \left(\tfrac{1}{2} F_{\alpha\beta} f^{\alpha\beta} \right) F^{\mu\nu} - 2c_2 \partial_\mu \left(\tfrac{1}{2} \tilde{F}_{\alpha\beta} f^{\alpha\beta} \right) \tilde{F}^{\mu\nu} = 0$$

$$\Leftrightarrow \boxed{\begin{array}{c} (1-2c_1 \mathcal{F})\, \partial_\mu f^{\mu\nu} - c_1 F_{\alpha\beta} F^{\mu\nu} \partial_\mu f^{\alpha\beta} - c_2 \tilde{F}_{\alpha\beta} \tilde{F}^{\mu\nu} \partial_\mu f^{\alpha\beta} = 0 \\ \text{lin. equations of motion} \quad (\text{pos. space}; \; f \equiv f(x)) \end{array}}$$

In order to solve them we first go to momentum space

$$a^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} a^\mu(k)$$

$$\rightsquigarrow f^{\mu\nu}(x) = i \int \frac{d^4k}{(2\pi)^4} e^{ikx} \left[k^\mu a^\nu(k) - k^\nu a^\mu(k) \right]$$

and work in Lorenz gauge $\partial_\mu a^\mu(x) = 0 \quad \Leftrightarrow \quad k_\mu a^\mu(k) = 0$

$\cancel{\rightsquigarrow k_\mu f^{\mu\nu}(x) = i}$

$$\rightsquigarrow \partial_\mu f^{\mu\nu}(x) = - \int \frac{d^4k}{(2\pi)^4} e^{ikx} \left[k^2 a^\nu(k) \right]$$

$$(1-2c_1\mathcal{F})k^2 a^\nu(k) - c_1 F_{\alpha\beta} F^{\mu\nu} k_\mu \left[k^\alpha a^\beta(k) - k^\beta a^\alpha(k)\right]$$
$$- c_2 \tilde{F}_{\alpha\beta} \tilde{F}^{\mu\nu} k_\mu \left[k^\alpha a^\beta(k) - k^\beta a^\alpha(k)\right] = 0$$

$F_{\alpha\beta} = -F_{\beta\alpha}$

$$\rightarrow \boxed{\begin{aligned}(1-2c_1\mathcal{F})k^2 a^\nu(k) &- 2c_1 F_{\alpha\beta} F^{\mu\nu} k_\mu k^\alpha a^\beta(k) \\ &- 2c_2 \tilde{F}_{\alpha\beta} \tilde{F}^{\mu\nu} k_\mu k^\alpha a^\beta(k) = 0\end{aligned}} \quad (*)$$

$\simeq$ Eq. of Mot. of type $M^\nu{}_\mu a^\mu = 0$

$\rightarrow$ nontrivial solutions ?

Strategy: use ansatz $a_1{}^\mu \sim F^{\mu\nu} k_\nu \equiv (Fk)^\mu = (\vec{k}.\vec{E}, \vec{k}\times\vec{B} + \omega\vec{E})$
$a_2{}^\mu \sim \tilde{F}^{\mu\nu} k_\nu \equiv (\tilde{F}k)^\mu = (\vec{k}.\vec{B}, -\vec{k}\times\vec{E} + \omega\vec{B})$

and use contraction ids.

$$F^{\mu\alpha} F^\nu{}_\alpha - \tilde{F}^{\mu\alpha} \tilde{F}^\nu{}_\alpha = 2\mathcal{F} g^{\mu\nu}$$
$$F^{\mu\alpha} \tilde{F}^\nu{}_\alpha = \tilde{F}^{\mu\alpha} F^\nu{}_\alpha = \mathcal{G} g^{\mu\nu}$$

obviously $k_\mu a_i{}^\mu = 0$ due to symmetry

$$a_1{}^\mu a_{2\mu} = F^{\mu\nu} k_\nu \tilde{F}_\mu{}^\alpha k_\alpha = \mathcal{G} g^{\nu\alpha} k_\nu k_\alpha = \mathcal{G} k^2$$

From $(*)$ we infer $k^2 = 0 + \mathcal{O}\left(\left(\frac{eF}{m^2}\right)^2\right)\alpha$

With Maxwell's eqs. in Vac, the electric/magnetic fields associated with $a_i{}^\mu$ read

$$\left.\begin{aligned}\vec{e}_i &= -\vec{\nabla} a_i^0 - \partial_t \vec{a}_i \\ \vec{b}_i &= \vec{\nabla}\times\vec{a}_i\end{aligned}\right\} \text{ in position space}$$

$$\rightarrow \left.\begin{aligned}\vec{e}_i(k) &\sim -\vec{k}\, a_i^0(k) + \omega\vec{a}_i(k) \\ \vec{b}_i(k) &\sim \vec{k}\times\vec{a}_i(k)\end{aligned}\right\} \text{ in mom. space} \quad k^0 \equiv \omega$$

$\vec{e}_1(k) \sim -\vec{k}(\vec{k}.\vec{E}) + \omega(\vec{k}\times\vec{B}) + \omega^2\vec{E}$
$\vec{b}_1(k) \sim \omega\vec{k}\times\vec{E} - \vec{k}^2\vec{B} + \vec{k}(\vec{k}.\vec{B})$

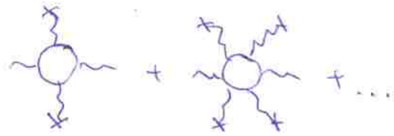
$\vec{e}_2(k) \sim -\vec{k}(\vec{k}.\vec{B}) - \omega(\vec{k}\times\vec{E}) + \omega^2\vec{B}$
$\vec{b}_2(k) \sim \omega\vec{k}\times\vec{B} + \vec{k}^2\vec{E} - \vec{k}(\vec{k}.\vec{E})$

$$(*):\quad \boxed{\begin{aligned}(1-2c_1\mathcal{F})\,k^2 a^\nu(k) &+ 2c_1 (Fk)^\nu k^\alpha F_{\alpha\beta}\, a^\beta(k)\\ &+ 2c_2 (\tilde{F}k)^\nu k^\alpha \tilde{F}_{\alpha\beta}\, a^\beta(k) = 0\end{aligned}} \qquad (\square)$$

$\underline{a_1:}$ $(1-2c_1\mathcal{F})\,k^2 (Fk)^\nu - 2c_1 (Fk)^\nu (Fk)^2 + 2c_2 (\tilde{F}k)^\nu k^\alpha k^\rho (-1)\,\mathcal{G}\, g_{\alpha\rho} = 0$

$$\Longleftrightarrow \left[(1-2c_1\mathcal{F})\,k^2 - 2c_1 (Fk)^2\right](Fk)^\nu - 2c_2\, \mathcal{G}\, k^2 (\tilde{F}k)^\nu = 0$$

solved with ansatz $k^2 = 0 + \underset{\substack{\uparrow \\ = \alpha\, \mathcal{O}\left(\left(\frac{eF}{m^2}\right)^2\right)}}{\delta_1}$



$\rightsquigarrow$ can be systematically (recursively) extended to higher orders

$$\rightsquigarrow \Big[\underbrace{(1-2c_1\mathcal{F})\,\delta_1}_{\alpha\mathcal{O}\left(\left(\frac{eF}{m^2}\right)^4\right)} - 2c_1 (Fk)^2\Big](Fk)^\nu - \alpha\, \mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)(\tilde{F}k)^\nu = 0$$

$$\rightsquigarrow \delta_1 = 2c_1 (Fk)^2 + \mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)\alpha$$

$\underline{a_2:}$ $(1-2c_1\mathcal{F})\,k^2 (\tilde{F}k)^\nu - 2c_1 (Fk)^\nu k^\alpha k^\rho\, \mathcal{G}\, g_{\alpha\rho} - 2c_2 (\tilde{F}k)^\nu (\tilde{F}k)^2 = 0$

$$\left|\;\begin{aligned}(\tilde{F}k)^2 &= \tilde{F}^{\mu\nu} k_\nu \tilde{F}_\mu{}^\alpha k_\alpha = k_\nu k_\alpha \left(F^{\mu\nu} F_\mu{}^\alpha - 2 g^{\nu\alpha}\mathcal{F}\right)\\ &= (Fk)^2 - 2\mathcal{F} k^2\end{aligned}\right.$$

$$\Longleftrightarrow \left[(1-2c_1\mathcal{F} + 2c_2\mathcal{F})\,k^2 - 2c_2 (Fk)^2\right](\tilde{F}k)^\nu - 2c_1\, \mathcal{G}\, k^2 (Fk)^\nu = 0$$

$$\rightsquigarrow \delta_2 = 2c_2 (Fk)^2 + \mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)\alpha$$

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the dispersion relations for modes $a_i{}^\mu(k)$ are given by

$$\boxed{k^2 = 2c_i (Fk)^2 + \alpha\, \mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)}$$

$$(Fk)^2 = \omega^2 \vec{E}^2 + \vec{k}^2 \vec{B}^2 - (\vec{k}\cdot\vec{E})^2 - (\vec{k}\cdot\vec{B})^2 + 2(\vec{k}\times\vec{B})\cdot\vec{E}\,\omega$$

$$\rightsquigarrow \text{obviously } (Fk)^2 = (Fk)^2\big|_{k^2=0} + F^2\, \mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^2\right)$$

$$(Fk)^2\big|_{k^2=0} = (\vec{k}\times\vec{E})^2 + (\vec{k}\times\vec{B})^2 - 2\,\vec{k}\cdot(\vec{E}\times\vec{B})\,|\vec{k}|$$

$$= \vec{k}^2\left[E^2\sin^2\measuredangle(\vec{k},\vec{E}) + B^2\sin^2\measuredangle(\vec{k},\vec{B}) - 2EB\,\hat{\vec{k}}\cdot\vec{s}\right]$$

with $\hat{\vec{k}} = \frac{\vec{k}}{|\vec{k}|}$, $E = |\vec{E}|$, $B = |\vec{B}|$, $\vec{s} \equiv \frac{\vec{E}\times\vec{B}}{|\vec{E}||\vec{B}|}$ unit Poynting vector

$\rightsquigarrow$ dispersion relations

$$\omega^2(|\vec{k}|) = \vec{k}^2\left\{1 - 2c_i\left[E^2\sin^2\measuredangle(\vec{k},\vec{E}) + B^2\sin^2\measuredangle(\vec{k},\vec{B}) - 2EB\,\hat{\vec{k}}\cdot\vec{s}\right]\right\} + \alpha\,\mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)$$

$$\rightsquigarrow\ \omega = |\vec{k}|\left\{1 - c_i\left[\ \text{"}\ \right]\right\} + \alpha\,\mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)$$

(to this order)

$v_{gr_i} = \frac{d\omega}{d|\vec{k}|} = v_{ph_i} = \frac{\omega}{|\vec{k}|} \equiv v_i$ $\rightsquigarrow$ index of refraction $n_i = \frac{1}{v_i}$

$$\rightsquigarrow \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 1 - \frac{\alpha}{\pi}\frac{1}{90}\begin{Bmatrix} 7 \\ 4 \end{Bmatrix}\left[\left(\frac{eE}{m^2}\right)^2\sin^2\measuredangle(\vec{k},\vec{E}) + \left(\frac{eB}{m^2}\right)^2\sin^2\measuredangle(\vec{B},\vec{k}) - 2\frac{eE}{m^2}\frac{eB}{m^2}\,\hat{\vec{k}}\cdot\vec{s}\right] + \alpha\,\mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)$$

magnetic field

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = 1 - \frac{\alpha}{\pi}\frac{1}{90}\begin{Bmatrix} 7 \\ 4 \end{Bmatrix}\left(\frac{eB}{m^2}\right)^2\sin^2\measuredangle(\vec{B},\vec{k}) + \alpha\,\mathcal{O}\left(\left(\tfrac{eF}{m^2}\right)^4\right)$$

- $\vec{e}_1\big|_{E=0} \sim \omega(\vec{k}\times\vec{B})$ $\perp$ plane spanned by $\vec{k},\vec{B}$
- $\vec{e}_2\big|_{E=0} \sim -\vec{k}(\vec{k}\cdot\vec{B}) + \omega^2\vec{B}$ $\in$ plane spanned by $\vec{k},\vec{B}$

"1" $\rightsquigarrow$ $\perp$ mode , "2" $\rightsquigarrow$ $\parallel$ mode

magnetized quantum vacuum is birefringent [Toll '52]

$\rightsquigarrow$ Signal: ellipticity specified by angle $\Delta\Phi := 2\pi\frac{L}{\lambda}\Delta n$; $\Delta n = n_1 - n_2$ (L: optical path, λ: probe wavelength)