J.G. Körner

Institute of Physics
PRISMA Cluster of Excellence,
University of Mainz

Quantum Field Theory at the Limits:
from Strong Fields to Heavy Quarks
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The many facets of the decay

$$\boxed{H \to \tau^+\tau^-\tau^+\tau^-}$$

Higgs properties

- $m_H = 125$ **GeV**
- **The Higgs is a very narrow particle (Γ_H(theoretical)$= 4.07$ MeV)**

$$\Gamma_H/m_H = 3.3 \cdot 10^{-5}$$

Compare to

$$\begin{aligned} \Gamma_W/m_W &= 2.59 \cdot 10^{-2} \\ \Gamma_Z/m_Z &= 2.74 \cdot 10^{-2} \\ \Gamma_t/m_t &\approx 8 \cdot 10^{-3} \end{aligned}$$

- **Production rate**

Present: LHC at 14 GeV produces $\approx 3 \cdot 10^6$/**year**
Future: HL-LHC at 14 GeV produces $\approx 30 \cdot 10^6$/**year**

- **Branching ratios** ($2m_Z > m_H$, $2m_W > m_H$)

$$\begin{aligned} BR(H \to Z^* Z^*) &= 2.64 \cdot 10^{-2} \\ BR(H \to W^{*+} W^{*-}) &= 2.15 \cdot 10^{-1} \\ BR(H \to \tau^+\tau^-\tau^+\tau^-) &= 3.02 \cdot 10^{-5} \end{aligned}$$

Sample of other branching ratios

Compare to other branchng ratios that were recently calculated (König, Neubert, Santorelli, Colangelo et al.)

$$\begin{array}{lcll} H & \to & \rho^0 + \gamma & 1.68 \cdot 10^{-5} \\ H & \to & \omega + \gamma & 1.48 \cdot 10^{-6} \\ H & \to & \phi + \gamma & 2.31 \cdot 10^{-6} \\ H & \to & J/\psi + \mu^+\mu^- & 9.1 \cdot 10^{-8} \\ H & \to & J/\psi + \gamma & 2.95 \cdot 10^{-6} \\ H & \to & \Upsilon + \gamma & 4.61 \cdot 10^{-9} \end{array}$$

This is not the whole story. Relevant for the detection of a decay mode is the product

Branching Ratio $\times$ Detection Efficiency

τ–modes can only be detected through their hadronic decays, e.g. $\tau^- \to (\pi^-, \rho^-, a_1^-) + \nu_\tau$ which make up 45.6 % of the τ^- decay rate. Dominant modes are leptonic modes $\tau^- \to \ell^- \bar{\nu}_\ell \nu_\tau$.

Lepton mass effects and identical particle effects

- **Lepton mass effects**
  - **Scale of lepton mass effects is set by off-shellness of the Z-boson**

    $$4m_\ell^2 \leq q^2 \leq (m_H - m_Z)^2$$

    and not by the Higgs mass m_H.
  - **τ mass effects are not negligible. ($m_{e,\mu} = 0$ is a good approximation).**
  - **Numerical example:**

    $$\Gamma(H \to Z + Z^*(\to \tau\tau))/\Gamma(H \to Z + Z^*(\to \mu\mu)) = 0.96 \quad (-4.0\,\%)$$
  - **Angular decay distribution of leptons changes. Can mimic the contributions of new effective operators. Later.**
  - **Test of lepton universality**
- **Identical particle effects**
  - **In the decay $Z \to (\tau^+\tau^+)(\tau^-\tau^-)$ the two tauons in the two pairs $(\tau^+\tau^+)$ and $(\tau^-\tau^-)$ are undistinguishable.**
  - **Statistical factor of $1/4$**
  - **Quantum interference effects, i.e. more Feynman diagrams**

Two Feynman diagrams

There are two Feynman diagrams that contribute to $H \to \tau^+\tau^-\tau^+\tau^-$

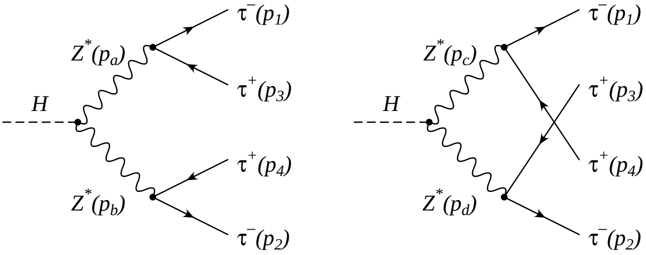


Figure: Feynman diagrams (A) and (B) contributing to $H \to \tau^+\tau^-\tau^+\tau^-$.

They contribute to the rate as follows

$$|M_A + M_B|^2 = |M_A|^2 + 2Re(M_A M_B^*) + |M_B|^2$$

The diagonal contributions

- **Diagrams describing the contributions from $|M_A|^2$ and $|M_B|^2$ are topologically equivalent, i.e. $|M_A|^2 = |M_B|^2$**
- **Diagonal terms contribute as (add statistical factor of $1/2!\,2!$)**

$$\frac{1}{4}\,(|M_A|^2 + |M_B|^2) = \frac{1}{2}\,|M_A|^2$$

- **If interference contribution $2Re(M_A M_B^*)$ (and lepton mass effects) are neglected one finds**

$$\Gamma(H \to \tau^+\tau^-\tau^+\tau^-) \quad = \quad \frac{1}{2} \quad \Gamma(H \to \tau^+\tau^-\mu^+\mu^-)$$

Sign of interference contribution

The interference contribution is given by the absorptive part of a one-loop contribution compared to the two-loop contributions of the diagonal graphs. Take a minus sign into account.

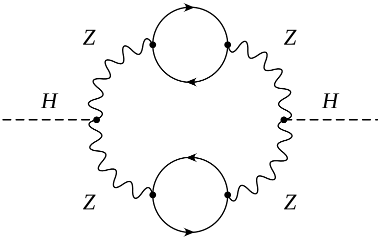


Figure: Squared Feynman diagrams $\sim |M_A|^2$ and $\sim Re(M_A M_B^*)$ contributing to $H \to \tau^+\tau^-\tau^+\tau^-$.

Including the dynamics the interference contribution adds constructively.

A calculational detail

NB: Calculating the diagonal diagram $Re(M_A M_B^*)$ involves the trace

$$
\begin{aligned}
|M_A|^2 &\propto tr\Big\{(\not{p}_1 - m_\tau)\gamma^\mu(g_V + g_A\gamma_5)(\not{p}_3 - m_\tau)\gamma^\nu(g_V + g_A\gamma_5)\Big\} \\
&\qquad \otimes \; tr\Big\{(\not{p}_2 - m_\tau)\gamma_\mu(g_V + g_A\gamma_5)(\not{p}_4 - m_\tau)\gamma_\nu(g_V + g_A\gamma_5)\Big\} \\
&= L^{\mu\nu}(p)L_{\mu\nu}(q)
\end{aligned}
$$

NB: Calculating the nondiagonal diagram $Re(M_A M_B^*)$ involves the trace

$$
\begin{aligned}
Re(M_A M_B^*) &\propto tr\Big\{(\not{p}_1 - m_\tau)\gamma^\mu(g_V + g_A\gamma_5)(\not{p}_3 - m_\tau)\gamma^\nu(g_V + g_A\gamma_5) \\
&\qquad (\not{p}_2 - m_\tau)\gamma_\mu(g_V + g_A\gamma_5)(\not{p}_4 - m_\tau)\gamma_\nu(g_V + g_A\gamma_5)\Big\}
\end{aligned}
$$

Even though the rate is only contributed to by γ_5– and γ–even terms the trace calculation cannot be done by hand.

Width dependence of interference contribution

- **As the width of the Z becomes smaller and smaller the momentum mismatch of the leptons in the interference contribution will become bigger and bigger. One expects**

$$\lim_{\Gamma_Z \to 0} \quad \frac{\Gamma_{\rm interference}}{\Gamma_{\rm diagonal}} \quad \Longrightarrow \quad 0$$

- **Question: Does the relative suppression go like (Γ_Z/m_Z) or like $(\Gamma_Z/m_Z)^2$?**

Some numerical results

Numerically one has (m_Z fixed)

Γ_Z [GeV]	$\Gamma_{nondiag}/\Gamma_{diag}$	$\Gamma_{nondiag}/\Gamma_{diag}$ $[\Gamma_Z/m_Z]$
2.4952	10.31 %	3.77
1.0	4.73 %	4.32
0.5	2.50 %	4.56
0.2	1.03 %	4.71
0.1	0.53 %	4.79
0.05	0.27 %	4.89

Table: Dependence of the rate ratios of nondiagonal and diagonal contributions on the Z–width for the decay $H \to Z^*(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$.

Power of the width suppression

Use the δ–function representation

$$\lim_{\Gamma_Z \to 0} \frac{1}{(q^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} = \frac{\pi}{m_Z \Gamma_Z}\, \delta(q^2 - m_Z^2)$$

to analyze the diagonal contribution in the vicinity of $q^2 = m_Z^2$ (keep M_Z fixed)

$$\begin{aligned}
\lim_{\Gamma_Z \to 0} \int dq^2 \frac{F(q^2)}{(q^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} &= \frac{\pi}{m_Z \Gamma_Z} \int dq^2\, \delta(q^2 - m_Z^2)\, F(q^2) \\
&= \frac{\pi}{m_Z \Gamma_Z}\, F(m_Z^2)
\end{aligned}$$

where the function $F(q^2)$ is regular at $q^2 = m_Z^2$. A similar analysis leads to $\lim_{\Gamma_Z \to 0} \Gamma_{\mathrm{interference}} = const.$ **One finds**

$$\begin{aligned}
\lim_{\Gamma_Z \to 0} \frac{\Gamma_{\mathrm{interference}}}{\Gamma_{\mathrm{diagonal}}} &= const. \cdot m_Z \Gamma_Z \\
&= const. \cdot m_Z^2\, [\Gamma_Z / m_Z]
\end{aligned}$$

Factorization of phase space

Consider the cascade decay process $H \to Z(\to \tau^+\tau^-) + Z^*(\to \tau^+\tau^-)$ **with momenta**

$$Z(p) \to \tau^+(p_1)\tau^-(p_3) \qquad H(p_H) \to Z(p)Z^*(q) \qquad Z(p) \to \tau^+(p_2)\tau^-(p_4)$$

Factorize the four-dimensional phase space

$$\Phi_4 = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4-p_H)\,,$$

using the identity e.g.

$$1 = \int dp^2 \frac{d^3p}{2E_p} \delta^{(4)}(p - p_1 - p_3)\,.$$

leading to

$$\int \frac{dp^2}{2\pi} \int \frac{dq^2}{2\pi} \Phi_2(p_H; p, q) \Phi_2(p; p_1, p_3) \Phi_2(q; p_2, p_4)\,.$$

The cascade decay $H \to Z(\to e^+e^-) + Z^*(\to \tau^+\tau^-)$

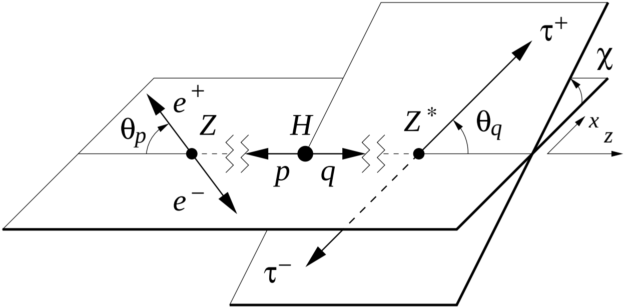


Figure: Definition of the momenta p and q, the polar angles θ_p and θ_q, and the azimuthal angle χ in the cascade decay $H \to Z(\to e^+e^-) + Z^*(\to \tau^+\tau^-)$

Two-dimensional phase spaces

$Z(p) \to \tau^+(p_1)\tau^-(p_3)$ **:**

$$
\begin{aligned}
\Phi_2(p;p_1,p_3) &= \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(p_1+p_3-p) = \\
&= \frac{1}{8(2\pi)}\sqrt{1-4m^2/p^2}\int d\cos\theta_p,
\end{aligned}
$$

$Z(q) \to \tau^+(p_2)\tau^-(p_4)$**:**

$$
\begin{aligned}
\Phi_2(q;p_2,p_4) &= \int \frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(p_2+p_4-q) = \\
&= \frac{1}{8(2\pi)^2}\sqrt{1-4m^2/q^2}\int d\chi\int d\cos\theta_q\,,
\end{aligned}
$$

$H(p_H) \to Z(p)Z^*(q)$:

$$
\Phi_2 := \Phi_2(p_H;p,q) = \int \frac{d^3p}{(2\pi)^3 2E_p}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^{(4)}(p+q-p_H) = \frac{\sqrt{\lambda(p_H^2,p^2,q^2)}}{4p_H^2(2\pi)}.
$$

Narrow Width Approximation (NWA)

To illustrate the use of the Narrow Width Approximation we take a less involved process, namely $t \to b + W^+(\to \ell^+ \, \nu_\ell)$. As we shall see the phrase "Narrow Width Approximation" is not quite correct. One should really say "Zero Width Limit".

Consider the three body decay decay rate formula

$$\Gamma_3 = \frac{1}{2m_t} \int \frac{1}{(2\pi)^3} \frac{d^3\vec{p}_b}{2\,E_b} \int \frac{1}{(2\pi)^3} \frac{d^3\vec{p}_{l^+}}{2\,E_{l^+}} \int \frac{1}{(2\pi)^3} \frac{d^3\vec{p}_{\nu_l}}{2\,E_{\nu_l}} |M_3|^2 (2\pi)^4 \delta^{(4)}(p_t - p_b - p_{\ell^+} - p_{\nu_l})$$

Use the identity

$$1 = \int dq^2 \int \frac{d^3q}{2\,E_W} \delta^{(4)}(q - p_{\ell^+} - p_{\nu_l}))$$

to factorize the phase space integrals.

Proof of the identity. The invariant identity can be seen to be true in the W^+ rest frame where $q^2 = E_W^2 \, (= m_W^2)$, c.f.

$$1 = \int \frac{dE_W^2}{2E_W} \delta(E_W - E_{\ell^+} - E_{\nu_l}) \int d^3q \delta^{(3)}(\vec{q} - \vec{p}_{\ell^+} - \vec{p}_{\nu_l})$$

Factorization of phase space

Use the identity to factorize the three-body phase space into two two-body phase spaces.

$$R_3 = 2m_W \int \frac{dq^2}{(2\pi)} \overbrace{\frac{1}{2m_t}\left\{\int \frac{1}{(2\pi)^3}\frac{d^3 p_b}{2\,E_b}\int \frac{1}{(2\pi)^3}\frac{d^3 q}{2\,E_W}(2\pi)^4\delta^{(4)}(p_t - p_b - q)\right\}}^{R_2(t\to b+W^+)}$$
$$\underbrace{\frac{1}{2m_W}\left\{\int \frac{1}{(2\pi)^3}\frac{d^3 p_{\ell^+}}{2\,E_{\ell^+}}\int \frac{1}{(2\pi)^3}\frac{d^3 p_{\nu_l}}{2\,E_{\nu_l}}(2\pi)^4\delta^{(4)}(q - p_{\ell^+} - p_{\nu_l}))\right\}}_{R_2(W^+\to l^+ +\nu_l)} .$$

$$\Gamma_3 = 2m_W \int \frac{dq^2}{(2\pi)}\left\{\frac{1}{2m_t}\int \frac{1}{(2\pi)^3}\frac{d^3\vec{p}_b}{2\,E_b}\int \frac{1}{(2\pi)^3}\frac{d^3\vec{q}}{2\,E_W}(2\pi)^4\delta^4(p_t - p_b - q)\right\}$$
$$\left\{\frac{1}{2m_t}\int \frac{1}{(2\pi)^3}\frac{d^3\vec{p}_{\ell^+}}{2\,E_{\ell^+}}\int \frac{1}{(2\pi)^3}\frac{d^3\vec{p}_{\nu_l}}{2\,E_{\nu_l}}\delta^4(q - p_{\ell^+} - p_{\nu_l}))\right\}\frac{1}{2}|M_3|^2$$

Phase space nicely factorizes. But how about the factorization of $|M_3|^2$?

Factorization of squared matrix element

Matrix element squared also factorizes *after angular integration*:

$$
\begin{aligned}
\int d\Omega \left| H^{\mu}(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}) L^{\nu} \right|^2 & \\
&= \int d\Omega \left| \sum_m H^{\mu} \varepsilon^*_\mu(m) \varepsilon_\nu(m) L^{\nu} \right|^2 \\
&\longrightarrow \frac{1}{3} \left| \sum_m H^{\mu} \varepsilon^*_\mu(m) \right|^2 \left| \sum_n L^{\nu} \varepsilon_\nu(n) \right|^2
\end{aligned}
$$

Factor $1/3$ **is important. It provides for the statistical factor** $1/(2s_W + 1)$ **in the** W^+ **width formula.**
Narrow–width approximation consists in the replacement of the Breit-Wigner line shape by a δ**–function,** *cif.*
Result as expected from physical intuition.

Finite width correction

Finite width correction can be calculated by keeping the original Breit-Wigner form of the propagator and integrating over q^2, *cif*.

$$\int_0^{m_t^2} dq^2\, \delta(q^2 - m_W^2) \;\rightarrow \int_0^{m_t^2} dq^2\, \frac{m_W \Gamma_W}{\pi} \frac{1}{(q^2 - m_W^2)^2 + m_W^2 \Gamma_W^2}$$

Numerically one finds a -1.56% correction which is of the order of $\Gamma_W / m_W = 2.64\%$.

The diagonal contribution $\sim |A|^2$

Remember that for e.g. $H \to Z^*(\to \mu^+\mu^-) + Z^*(\to \tau^+\tau^-)$ one only has the diagonal contribution.

- **The width formula for $H \to Z^*(p^2) + Z^*(q^2)$ (Grau, Pancheri, Phillips 1990)**

$$\Gamma(H \to \text{all}\, Z^*\, Z^*) = \int_0^{m_H^2} \frac{dp^2\, m_Z \Gamma_Z}{\pi[(p^2 - m_Z^2)^2 + (m_Z\Gamma_Z)^2]} \times \int_0^{(m_H - p)^2} \frac{dq^2\, m_Z \Gamma_Z}{\pi[(q^2 - m_Z^2)^2 + (m_Z\Gamma_Z)^2]}\, \Gamma(H \to Z^* Z^*)$$

where

$$\Gamma(H \to Z^* Z^*) = \frac{1}{2}\,\frac{g_W^2}{8\pi \cos^2\theta_W}\,\frac{|\vec{p}|}{m_H^2 m_Z^2}\left(p^2 q^2\right)\underbrace{(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2})(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2})}_{\frac{(2p^2q^2 + pq\, pq)}{p^2 q^2}}$$

Choice of gauge

- **unitary gauge**

$$\begin{aligned}\Gamma(H \to Z^* Z^*) &\sim \sum_m \varepsilon^{\alpha}(m,p^2)\varepsilon^{*\,\beta}(m,p^2)\sum_n \varepsilon^{\beta}(n,q^2)\varepsilon^{*\,\alpha}(n,q^2)\\ &= (-g^{\mu\nu} + \frac{p^\mu p^\nu}{m_Z^2})(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_Z^2})\\ &= \left(4 - \frac{p^2}{m_Z^2} - \frac{q^2}{m_Z^2} + \frac{pq\,pq}{m_Z^4}\right)\end{aligned}$$

- **Spin 1 (Lorenz/Landau) gauge**

$$\begin{aligned}\Gamma(H \to Z^* Z^*) &\sim (-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2})(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2})\\ &= \left(2 + \frac{pq\,pq}{p^2 q^2}\right)\end{aligned}$$

- **Feynman gauge**

$$\begin{aligned}\Gamma(H \to Z^* Z^*) &\sim (-g^{\mu\nu})(-g_{\mu\nu})\\ &= 4\end{aligned}$$

Correct result

The result is obviously gauge variant.

The problem is that the concept of an external off-shell gauge boson is not a gauge invariant concept. One must attach fermion pairs to the off-shell gauge boson to get a gauge invariant result. In addition one must use the unitary gauge to get a gauge invariant result.

The unitary gauge

Consider the gauge boson propagator in the general R_ξ gauge and rewrite it into a convenient form.

$$
\begin{aligned}
D^{\mu\nu} &= \frac{i}{q^2 - m_Z^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu (1-\xi_W)}{q^2 - \xi_Z m_Z^2}\right) && (1)\\
&= \frac{i}{q^2 - m_Z^2}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}\right) - i\,\frac{q^\mu q^\nu}{m_Z^2}\,\frac{1}{q^2 - \xi_Z m_Z^2}. && (2)
\end{aligned}
$$

The first term is the unitary propagator. The second gauge-dependent term is cancelled by the contribution of the neutral Goldstone ϕ^0.

Again one needs to attach fermion pairs to the gauge boson Z and to the neutral Goldstone boson ϕ^0 to see the cancellation. Explicit examples of this cancellation can be found in the book of Peskin-Schroeder (fermion-fermion scattering ps. 734-736) and in Körner [1402.2787] for the decay $t \to b + W^+(\to \ell^+ \nu_\ell)$.

Feynman diagrams for $t \to b + W^+(\to \ell^+ \nu_\ell)$ in the R_ξ gauge

To demonstrate the cancellation of the gauge parameter R_ξ we take the simpler decay $t \to b + W^+(\to \ell^+ \nu_\ell)$. In the R_ξ gauge there are two Feynman diagrams that contribute to $t \to b + W^+(\to \ell^+ \nu_\ell)$

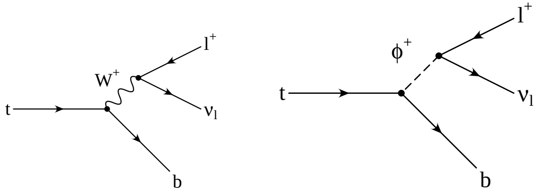


Figure: Feynman diagrams for the decay $t \to b + W^+(\to \ell^+ \nu_\ell)$ in the R_ξ gauge

where ϕ^+ is the charged Goldstone boson. In our case one would have from W^+-exchange

$$q^\nu \bar{u}_\nu \gamma_\nu (1-\gamma_5) v_\ell = +m_\ell \bar{u}_\nu (1+\gamma_5) v_\ell ,$$

$$q^\mu \bar{u}_b \gamma_\mu (1-\gamma_5) u_t = m_t \bar{u}_b (1+\gamma_5) u_t - m_b \bar{u}_b (1-\gamma_5) u_t .$$

Using the Feynman rules for charged Goldstone boson ϕ^+-exchange one can see that the gauge parameter dependent terms exactly cancel.

Attaching (massless) fermion pairs to the off-shell gauge bosons

Split the unitary gauge propagator into a spin 1 ($P_1^{\nu\beta}(q)$) and a spin 0 piece ($P_0^{\nu\beta}(q)$):

$$-g^{\nu\beta} + \frac{q^\nu q^\beta}{m_V^2} = \underbrace{\left(-g^{\nu\beta} + \frac{q^\nu q^\beta}{q^2}\right)}_{\text{spin 1}} - \underbrace{\frac{q^\nu q^\beta}{q^2}\left(1 - \frac{q^2}{m_V^2}\right)}_{\text{spin 0}},$$

For massless external fermions the spin 0 piece gives zero contribution. When zero mass fermions are attached to off-shell gauge bosons one can use the spin 1 gauge.

The correct result is

$$\Gamma(H \to Z^* Z^*) \quad \sim \quad \left(\frac{p^2 q^2}{m_Z^4}\right) \Gamma(\text{spin 1 gauge})$$

Where do the factors p^2 and q^2 come from? They come from attaching a fermion pair to the off-shell gauge bosons. To keep things simple take $m_\ell = 0$.

$$\int d\Omega_p L^{\mu\nu}(p) = \frac{4\pi}{3}\, p^2\, P_1^{\mu\nu}(p)) \qquad \int d\Omega_q L^{\mu\nu}(q) = \frac{4\pi}{3}\, q^2\, P_1^{\mu\nu}(q)$$

Attaching (massless) fermion pairs to the off-shell gauge bosons

Factorization for $H \to Z^*(\to f_i \bar{f}_i) + Z^*(\to f_j \bar{f}_j)$ $(i \neq j)$

$$\frac{d\Gamma_{ij}}{dp^2 dq^2}(p^2, q^2) = 2 \cdot B_i\, B_j \frac{1}{\pi} \frac{m_Z \Gamma}{(p^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \frac{1}{\pi} \frac{m_Z \Gamma}{(q^2 - m_Z^2)^2 + m_Z^2 \Gamma_Z^2}$$

$$\cdot \left(\frac{p^2 q^2}{m_Z^4} \right) \Gamma(H \to Z^* + Z^*)_{\text{spin 1 gauge}}$$

Sum over the channels (factorization!)

$$\sum_{i,j} B_i\, B_j \approx \frac{1}{2} \Big(\sum_i B_i \Big) \Big(\sum_j B_j \Big) = \frac{1}{2}$$

$$\begin{array}{lll} i) & i \neq j & B_i\, B_j = B_j\, B_i \\ ii) & i = j & B_i\, B_i \approx \frac{1}{2} B_i\, B_j \end{array}$$

Angular decay distribution 1

Covariant expression:

$$W(\theta_p, \theta_q, \chi) = g_{\alpha\alpha'}\, P_1^{\alpha\mu}(p)\, P_{0\oplus 1}^{\alpha'\mu'}(q)\, L_{\mu\nu}^{(p)}(p)\, L_{\mu'\nu'}^{(q)}(q)\, P_1^{\nu\beta}(p)\, P_{0\oplus 1}^{\nu'\beta'}(q)\, g_{\beta\beta'}^{*}$$

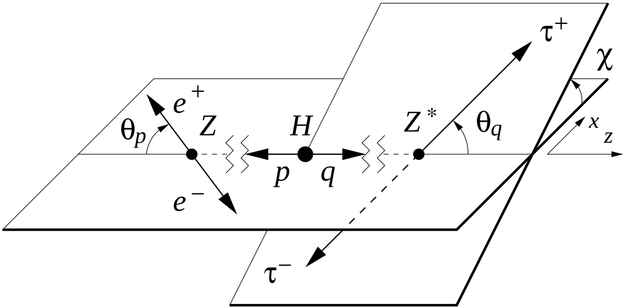


Figure: Definition of the momenta p and q, the polar angles θ_p and θ_q, and the azimuthal angle χ in the cascade decay $H \to Z(\to e^+e^-) + Z^*(\to \tau^+\tau^-)$

Angular decay distribution 2

Two routes to proceed:

- **Define momenta in three frames. Boost momenta to Higgs rest frame. Do the contractions. (Cabibbo, Maksymowicz 1965, Buchalla et al. 2014)**
- **Helicity method (Jacob, Wick 1959)**

Transform covariant distribution to a helicity distribution with the help of the completeness relations for polarization vectors. To make life a bit simpler we treat the on-shell (p)– off-shell (q) case.

- **Off-shell spin 1+spin 0 propagator (unitary gauge)**

$$P_{0\oplus 1}^{\mu'\alpha'}(q) = -g^{\mu'\alpha'} + \frac{q^{\mu'}q^{\alpha'}}{m_V^2} = -\sum_{\lambda_{V*}=t,\pm 1,0} \varepsilon^{\mu'}(\lambda_{V^*})\varepsilon^{*\,\alpha'}(\lambda_{V^*})\,\hat{g}_{\lambda_{V*}\lambda_{V*}}\,.$$

- **On-shell spin 1 propagator ($p^2 = m_Z^2$)**

$$P_1^{\alpha\mu}(p) = -g^{\alpha\mu} + \frac{p^\alpha p^\mu}{p^2} = \sum_{\lambda_V=\pm 1,0} \bar{\varepsilon}^{\alpha}(\lambda_V)\bar{\varepsilon}^{*\,\mu}(\lambda_V)$$

Angular decay distribution 3

Helicity representation of angular decay distribution

$$W(\theta_p, \theta_q, \chi) = \sum_{\substack{\lambda_V, \lambda'_V \\ J, J' \lambda_{V*}, \lambda'_{V*}}} (-F_S)^{2-J-J'} L^{(p)}_{\lambda_V \, \lambda'_V}(\cos\theta_p) H_{\lambda_V, \lambda_{V*}} H^*_{\lambda'_V, \lambda'_{V*}} L^{(q)}_{\lambda_{V*} \, \lambda'_{V*}}(\cos\theta_q, \chi)$$

with $J, J' = 0, 1 \quad \lambda_{V^*}, \lambda'_{V^*} = t, \pm 1, 0, \lambda_V, \lambda'_V = \pm 1, 0$

Angular decay distribution 4; a sample result

Normalized angular decay distribution (see also Cheng, Sinha et al.)
($\int_{\text{angles}} \widetilde{W}(\theta_p,\theta_q,\chi) = 1$)

$$\widetilde{W}(\theta_p,\theta_q,\chi) = \frac{1}{8\pi}\left(1 + \sum_{i=1}^{7} \widetilde{\mathcal{F}}_i h_i(\theta_p,\theta_q,\chi)\right)$$

i	$\widetilde{\mathcal{F}}_i^Z$ **($m_\ell = 0$)**	$\widetilde{\mathcal{F}}_i^Z$ **($m_\ell = m_\tau$)**	$h_i(\theta_p,\theta_q,\chi)$
1	-0.9115	-0.6257	$P_2(\cos\theta_q)$
2	-0.9115	-0.9391	$P_2(\cos\theta_p)$
3	$+0.9557$	$+0.6561$	$P_2(\cos\theta_p)P_2(\cos\theta_q)$
4	$+0.0030$	$+0.0023$	$\cos\theta_p\cos\theta_q$
5	$+0.0167$	$+0.0132$	$\sin\theta_p\sin\theta_q\cos\chi$
6	$+0.1875$	$+0.1287$	$\sin 2\theta_p\sin 2\theta_q\cos\chi$
7	$+0.0332$	$+0.0228$	$\sin^2\theta_p\sin^2\theta_q\cos 2\chi$

Table: Numerical results for the normalized coefficient functions $\widetilde{\mathcal{F}}_i(q^2)$ at $q^2 = 50\,\mathrm{GeV}^2$. Legendre polynomial $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$.

Legendre polynomial

It is quite convenient and is now a common praxis to expand the angular decay distribution $\widetilde{W}(\theta_p, \theta_q, \chi)$ in terms of Legendre polynomial because of their orthogonality properties. In fact, one has

$$\int_{-1}^{+1} P_m(x)P_n(x) = 0 \quad m \neq n \qquad \int_{-1}^{+1} P_m(x)P_m(x) = \frac{2}{2m+1}$$

The first three Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \end{aligned}$$

Note, in particular, $\int P_2(x)dx = 0$.

Helicity composition of the gauge bosons

On-shell – off-shell case

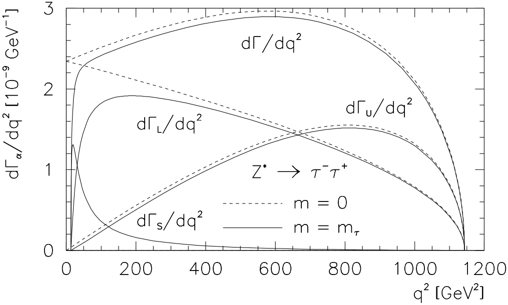


Figure: Differential rates $d\Gamma^Z_\alpha/dq^2$ (indices $\alpha = U, L, S$ for the decay $H \to Z(\to e^+e^-) + Z^*(\to \ell^+\ell^-)$ with $m_\ell = 0$ and $m_\ell = m_\tau$.

- **L refers to** (ZZ^*) **double density matrix element** ρ_{LL}
- **U** – // – ρ_{TT}
- **S** – // – ρ_{LS}

On-shell – off-shell vs. Off-shell – off-shell decays

	Γ^Z $[GeV]$	Γ^Z_U/Γ^Z	Γ^Z_L/Γ^Z	Γ^Z_S/Γ^Z
$H \to Z(\to e^+e^-) + Z^*(\to \ell^+\ell^-)$				
$(m_\ell = m_\mu)$	$1.01 \times 10^{-7}\,\mathrm{GeV}$	0.41	0.59	0
$(m_\ell = m_\tau)$	$0.97 \times 10^{-7}\,\mathrm{GeV}$	0.41	0.55	0.04
$H \to Z^*(\to e^+e^-) + Z^*(\to \ell^+\ell^-)$				
$(m_\ell = m_\mu)$	$1.22 \times 10^{-7}\,\mathrm{GeV}$	0.39	0.61	0
$(m_\ell = m_\tau)$	$1.20 \times 10^{-7}\,\mathrm{GeV}$	0.39	0.59	0.02

Table: Total and normalized partial decay rates for the four-body decays $H \to Z(\to e^+e^-) + Z^*(\to \ell^+\ell^-)$ and $H \to Z^*(\to e^+e^-) + Z^*(\to \ell^+\ell^-)$.

Opening Pandora's box

- **Using the two Feynman diagrams A and B we find**

$$\Gamma(H \to 4\tau; m_\tau \neq 0) \quad < \quad \Gamma(H \to 4\tau; m_\tau = 0)$$

  This is expected since mass effects reduce the available phase space.

- **MadGraph instead finds**

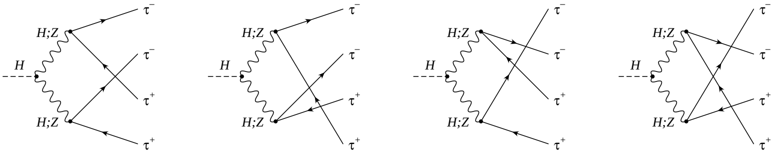
$$\Gamma(H \to 4\tau; m_\tau \neq 0) \quad > \quad \Gamma(H \to 4\tau; m_\tau = 0)$$

  counter to naive intuition

- **The solution to the problem is that, with $m_\tau \neq 0$, there are altogether 28 instead of 2 contributing Feynman diagrams.**
- **And MadGraph is so kind to display them for you.**
  Here they are:

More Feynman diagrams

Four diagrams including our diagrams A and B

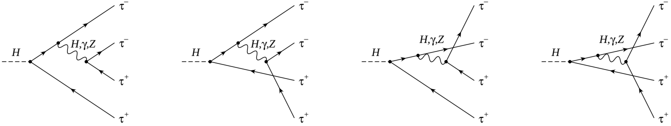


(1a) **(1b)** **(1c)** **(1d)**

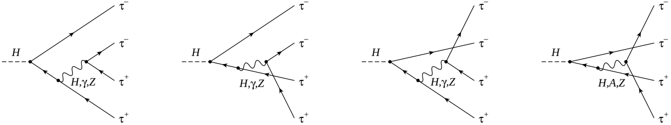
Diagrams $\{(1c);(1d)\}$ are topologically equivalent to $\{(1a);(1b)\}$, as before

More Feynman diagrams cont'd

Diagrams that come in for $m_\tau \neq 0$ through the Yukawa coupling $g_{H\tau\tau} \propto m_\tau$



(2a) **(2b)** **(2c)** **(2d)**

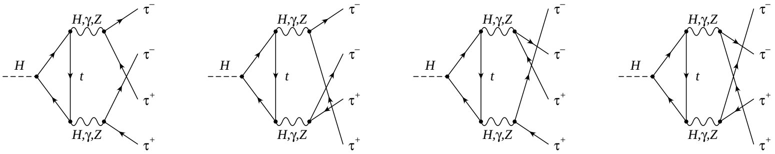


(3a) **(3b)** **(3c)** **(3d)**

All diagrams are topologically nonequivalent. Experience from having calculated $e^+e^- \to q\bar{q}q\bar{q}$ in 1979/80 came in handy (Ali et al.).

More Feynman diagrams cont'd

Attaching lepton pairs to the process $H \to \gamma\gamma$



(4a) **(4b)** **(4c)** **(4d)**

MadGraph results (without diagrams 4):

$$
\begin{array}{llll}
m_\tau = 0 & \Gamma = 1.33 \cdot 10^{-7}\,\mathrm{GeV} & & \\
m_\tau \neq 0 & \Gamma = 1.43 \cdot 10^{-7}\,\mathrm{GeV} & \text{28x28 terms} & \\
m_\tau \neq 0 & \text{but} \quad g_{H\tau\tau} = 0 & \Gamma = 1.28 \cdot 10^{-7}\,\mathrm{GeV} &
\end{array}
$$

Most important of the extra contributions are diagrams that allow for Z–pole contributions. We are presently calculating contributions of $|2a + 3a|^2$, $|2b + 3b|^2$, $|2c + 3c|^2$ and $|2d + 3d|^2$.

Some final remarks

- **There is a great deal of interesting physics in the process $H \to \tau^+\tau^+\tau^+\tau^+$. Gauge invariance, identical particle effects, lepton mass effects, topology of Feynman diagrams, momentum mismatch, no external off-shell gauge bosons, ...**
- **Experimentalists are getting better and better in the identification of tauons. Taus from the process $H \to \tau^+\tau^-\tau^+\tau^-$ are used to train tau finding algorithms for the gold plated process $H \to \tau^+\tau^-$.**
- **My office mate and friend Jian Wang calculated many of the rates in minutes using MadGraph. We took two years. There is essential agreement but some differences in detail. My congratulations for the MadGraph team for having done a tantalizingly good job.**
- **Many thanks to my colleagues Stefan Groote, Lauri Kaldamäe and Jian Wang for collaboration and PRISMA for support.**