LKF Transformation for QCD and Quark Propagator

M. Jamil Aslam

In Collaboration with

Adnan Bashir and L. X. Gutierrez-Guerrero

Phys. Rev. D 93 (2016) 076001.

SF → HQ, Dubna, July 18 - 30, 2016

Greens Function transforms in a specific manner under the variation of a gauge: (LKF-transformations in QED)

Landau-Khalatnikov-Fradkin transformations (LKFT)	Ward-Fradkin-Green-Takahashi identities (WFGTI)
Non perturbative in nature Govern the behavior of a single Green function when variation of covariant gauge parameter is performed.	Non perturbative in nature Relate different Greens functions under local gauge transformations.

WFGTI holds in one gauge, then if the Green functions involved are transformed to another gauge under LKFT, the WFGTI in that other gauge will be satisfied.

LKF transformations (Field Dependent gauge transformations) for the photon propagator are

$$D_{\mu\nu}(x,\Delta) = D_{\mu\nu}(x,0) + \partial_\mu \partial_\nu \Delta(x)$$

and for the fermion propagator, these become

$$S(x,\Delta) = S(x,0)e^{-i[\Delta_d(0)-\Delta_d(x)]}$$

Related to particular choice of gauge fixing

$$\Delta_d(x) = -i\xi e^2 \mu^{4-d} \int_0^\infty \frac{d^d p}{(2\pi)^d} \frac{e^{-ip.x}}{p^4}$$

$$S(x;\xi) = S(x;0)e^{-(\frac{\alpha\xi}{2})x}$$

Ward-Takahashi Identity and Bare Vertex

Relate Fermion propagator with gauge boson vertex

$$q_\mu \Gamma^\mu(k,p) = S_F^{-1}(k) - S_F^{-1}(p)$$

Replace with bare vertex γ^μ

$$\not{q} = \frac{\not{k}}{F(k^2)} - \frac{\not{p}}{F(p^2)} - \frac{\mathcal{M}(k^2)}{F(k^2)} + \frac{\mathcal{M}(p^2)}{F(p^2)}$$

It can not be expected to be satisfied in all gauges. In Landau Gauge $F(k^2) = F(p^2) = 1$

above equation gives

$$\mathcal{M}(p^2) = \mathcal{M}(k^2)$$

This equation does not hold true except for k^2 and $p^2 \to 0$, where this mass function is constant.

WTI and Longitudinal Vertex

A Straight forward conclusion from WTI is that

$$\Gamma^{\mu}(k,p) = \frac{S_F^{-1}(k) - S_F^{-1}(p)}{k^2 - p^2}(k+p)^{\mu}$$

Kinematical singularity

$$S_F(p) = \frac{F(p^2)}{\not p - \mathcal{M}(p^2)}$$

$$\Gamma^{\mu}(k,p) = \frac{1}{k^2 - p^2}\left[\frac{\not k}{F(k^2)} - \frac{\not p}{F(p^2)} - \frac{\mathcal{M}(k^2)}{F(k^2)} + \frac{\mathcal{M}(p^2)}{F(p^2)}\right](k+p)^{\mu}$$

Let $k^2 \to p^2$ without demanding $k \to p$

$$\frac{1}{k^2 - p^2}\left[\frac{\not k}{F(k^2)} - \frac{\not p}{F(p^2)}\right]$$

Ball and Chiu noticed that we can decompose the full non-perturbative vertex into two comp.; Longitudinal and Transverse

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p)$$

$$q_\mu \Gamma^\mu_T(k,p) = 0$$

By definition, this is unspecified by WTI.

BC made a crucial assumption, that the vertex should be free from kinematical singularity. This led to the Unique form of the longitudinal part of the vertex.

This can be taken care of if we start with $k \to p$ i.e., WTI goes to WI

$$\frac{\partial S_F^{-1}(p)}{\partial p_\mu} = \Gamma^\mu(p,p)$$

$$\Gamma^\mu_L(k,p) = \frac{1}{2}\left[\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right]\gamma^\mu + \frac{1}{2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right]\frac{(k+p)^\mu(\not{k}+\not{p})}{k^2-p^2} - \left[\frac{\mathcal{M}(k^2)}{F(k^2)} - \frac{\mathcal{M}(p^2)}{F(p^2)}\right]\frac{(k+p)^\mu}{k^2-p^2} \quad .$$

Free from Kinematical Singularity.

The Transverse Vertex

Again BC has made an important contribution:

3- Four Vectors → $\gamma^\mu, \quad k^\mu, \quad p^\mu$

Four Lorentz Scalars → $1, \quad \not{k}, \quad \not{p}, \quad \not{k}\,\not{p}$

12 Independent vectors →

$$\begin{array}{cccc} \gamma^\mu, & \not{k}\gamma^\mu, & \not{p}\gamma^\mu, & \not{k}\,\not{p}\,\gamma^\mu \\ k^\mu, & \not{k}k^\mu, & \not{p}k^\mu, & \not{k}\,\not{p}\,k^\mu \\ p^\mu, & \not{k}p^\mu, & \not{p}p^\mu, & \not{k}\,\not{p}\,p^\mu \end{array}$$

Coefficient identically zero

3-basis vectors $\gamma^\mu, \ (k+p)^\mu \quad (\not{k} + \not{p})(k+p)^\mu$

The remaining 8 has to satisfy → $q_\mu T_i^\mu = 0$

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau_i(k^2, p^2, q^2) T_i^\mu(k,p)$$

In Perturbation theory WFGTI and LKFT are satisfied at every order of approximation.

This is fine for the QED.

The gauge theory whose non-perturbative understanding is relevant to the fundamental interactions in the SM is QCD.

Slavnov-Tayolr Identities (STI) relate the Greens functions in QCD. Play a key role in the gauge renormalizability of QCD.

Generalized LKFT for the QCD were missing in literature.

↓

Purpose is to derive LKFT for the quark propagator.

↓

First we derive them in a general SU(N) gauge theory.

↓

Setting the color factors we can specialize them to QCD and even to QED.

Local SU(N) transformations for the quark field

$$\psi_i(x) \to \psi'_i(x) = \mathrm{e}^{i[g_s\varphi_a(x)T_a]}\psi_i(x)$$

The quark Green Function (propagator in Coordinate space)

$$iS^F_{ij}(x,x') \equiv iS^F_{ij} = \langle \mathrm{T}\{\psi_i(x)\bar{\psi}_j(x')\}\rangle$$

Under Local SU(N) transformations

$$iS^F_{ij}(x,x') \to \langle \mathrm{T}\{\mathrm{e}^{i[g_s\varphi_a(x)T_a]}\psi_i(x)\bar{\psi}_j(x')\mathrm{e}^{-i[g_s\varphi_b(x')T_b]}\rangle$$

$$iS^F_{ij}(x,x') \to iS^{0F}_{ij}(x,x')\langle \mathrm{T}\{\mathrm{e}^{i[g_s\varphi_a(x)T_a]}\mathrm{e}^{-i[g_s\varphi_b(x')T_b]}\rangle$$

Green Function for quark in Landau Gauge

In Abelian approximation of QCD

$$iS_{ij}^{F}(x,x') = iS_{ij}^{0F}(x,x')e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]}$$

This matches with the QED.

Notation for Green Function associated with scalar field

$$i\Delta_F(x-x')\delta_{ab} = \langle 0|\mathrm{T}\{\varphi_a(x)\varphi_b(x')\}|0\rangle$$

$$\langle 0|\mathrm{T}\{e^{i[g_s\varphi_a(x)T_a]}e^{-i[g_s\varphi_b(x')T_b]}\}|0\rangle$$

$$= \left\langle 0|\mathrm{T}\left\{\left[1 + ig_s\varphi_a(x)T_a + \frac{(ig_s)^2}{2!}(\varphi_a(x)T_a)^2 + \frac{(ig_s)^3}{3!}(\varphi_a(x)T_a)^3 + \frac{(ig_s)^4}{4!}(\varphi_a(x)T_a)^4 + \cdots\right]\right.\right.$$

$$\left.\left.\times\left[1 - ig_s\varphi_b(x')T_b + \frac{(-ig_s)^2}{2!}(\varphi_b(x')T_b)^2 + \frac{(-ig_s)^3}{3!}(\varphi_b(x')T_b)^3 + \frac{(-ig_s)^4}{4!}(\varphi_b(x')T_b)^4 + \cdots\right]\right\}|0\right\rangle$$

We will calculate terms at different orders of g_s .

At 4th order of the strong coupling g_s

$$iS^{F}_{ij}(x,x') = iS^{0F}_{ij}(x,x')\Big[e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]} - \frac{g_s^4 C_A C_F}{24}\{(i\Delta_F(x-x')-i\Delta_F(0))(3i\Delta_F(x-x')-i\Delta_F(0))\} + \mathcal{O}(g_s^6)\Big]$$

Desired term as it leaves the quark condensate of QCD invariant.

Next to leading log term in the two loop perturbative expansion of the massless quark propagator.

It will be interesting to see what happens at if we make an expansion to the next order in the strong coupling.

At 6th order of the strong coupling g_s

$$iS^F_{ij}(x,x') = iS^{0F}_{ij}(x,x')\Bigg[e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]}$$
$$-\frac{g_s^4 C_A C_F}{(2!)(3!2!1!)}\{[i\Delta_F(x-x')-i\Delta_F(0)][3i\Delta_F(x-x')-i\Delta_F(0)]\}[1+g_s^2 C_F(i\Delta_F(x-x')-i\Delta_F(0))]$$
$$+\frac{g_s^6 C_F C_A^2}{(1!)(4!3!2!1!)}[i\Delta_F(x-x')-i\Delta_F(0)][8(i\Delta_F(x-x'))^2-7(i\Delta_F(x-x'))(i\Delta_F(0))+(i\Delta_F(0))^2]+\mathcal{O}(g_s^8)\Bigg]$$

This expression suggest that a possible new next log series at $\mathcal{O}(g_s^4)$

$$iS^F_{ij}(x,x') = iS^{0F}_{ij}(x,x')\Bigg[e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]}$$
$$-\frac{g_s^4 C_A C_F}{(2!)(3!2!1!)}\{[i\Delta_F(x-x')-i\Delta_F(0)][3i\Delta_F(x-x')-i\Delta_F(0)]\}e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]}$$
$$+\frac{g_s^6 C_F C_A^2}{(1!)(4!3!2!1!)}[i\Delta_F(x-x')-i\Delta_F(0)][8(i\Delta_F(x-x'))^2-7(i\Delta_F(x-x'))(i\Delta_F(0))+(i\Delta_F(0))^2]+\mathcal{O}(g_s^8)\Bigg]$$

Key remarks regarding LKFT in QCD

For QED: $C_F = 1,\ g_s = e,\ C_A = 0$. We can get back the LKFT for the electron propagator.

We find closed expression for the perturbative series in the color factor in fundamental representation, i.e. C_F^n . This is not the case for the color factor in adjoint representation C_A^n .

Recall the definition of chiral condensate:

$$\begin{aligned}\langle\bar{\psi}\psi\rangle_\xi &= -\mathrm{Tr}[S^F(x,x')]_{x'=x} \\ &= -\mathrm{Tr}[S^{0F}(x,x')]_{x'=x} = \langle\bar{\psi}\psi\rangle_0\end{aligned}$$

This is manifestly gauge invariant quantity in any SU(N) theory. To the best of our knowledge this is first field theoretical calculation in QCD which indicates that the chiral condensate is gauge invariant.

To the order we have carried out the calculations, we have observed the following factors:

$$C_A \Rightarrow \frac{1}{3!2!1!}[3i\Delta_F(x-x') - i\Delta_F(0)],$$

$$C_A^2 \Rightarrow \frac{1}{4!3!2!1!}[8\{i\Delta_F(x-x')\}^2 + \{i\Delta_F(0)\}^2 - 7\{i\Delta_F(x-x')\}\{i\Delta_F(0)\}].$$

We verified that the first of this secures at $\mathcal{O}(g_s^6)$ and $\mathcal{O}(g_s^8)$, and we conjecture it to become a coefficient of $e^{g_s^2 C_F[i\Delta_F(x-x')-i\Delta_F(0)]}$.

The local gauge transformation for the quark propagator has verifiable consequences to all orders in the perturbation theory e.g., it implies Multiplicative Renormalizability of the massless quark propagator.

Implications of LKFT

Multiplicative Renormalizability: We rescale the fields, masses and couplings of theory to make the GF finite.

$$F(p^2/\Lambda^2) = 1 + \alpha A_1 \ln \frac{p^2}{\Lambda^2} + \alpha^2 A_2 \ln^2 \frac{p^2}{\Lambda^2} + \ldots$$

Fermion wave function. UV cut-off

$$F(p^2/\Lambda^2) \;=\; 1 + \frac{1}{1!}\alpha A_1 \ln \frac{p^2}{\Lambda^2} + \frac{1}{2!}\left(\alpha A_1 \ln \frac{p^2}{\Lambda^2}\right)^2 + \frac{1}{3!}\left(\alpha A_1 \ln \frac{p^2}{\Lambda^2}\right)^3 + \ldots$$

$$= \exp\left(\alpha A_1 \ln \frac{p^2}{\Lambda^2}\right) = \left(\frac{p^2}{\Lambda^2}\right)^{\alpha A_1}$$

$$Z_2^{-1}(\mu^2/\Lambda^2) = \exp\left(\alpha A_1 \ln \frac{\Lambda^2}{\mu^2}\right) = \left(\frac{\Lambda^2}{\mu^2}\right)^{\alpha A_1}$$

Renormalized Wave function can be expressed as

$$F_R(p^2/\mu^2) \;=\; F_R(1) \exp\left(\alpha A_1 \ln \frac{p^2}{\mu^2}\right) \;=\; F_R(1)\left(\frac{p^2}{\mu^2}\right)^{\alpha A_1}$$

In leading Log. approximation in PT:

$$F_R(p^2/\mu^2) = 1 + \frac{\alpha\xi}{4\pi} \ln \frac{p^2}{\mu^2} + \cdots$$

$$F_R(p^2/\mu^2) = \left[\frac{p^2}{\mu^2}\right]^{\alpha\xi/4\pi}$$

MR and choice of vertices(e-photon case)

Not all the vertices are expected to satisfy MR (Brown and Dorey)

Bare Vertex

$$\frac{1}{F(p^2)} = 1 + \frac{\alpha\xi}{4\pi} \int_0^{\Lambda^2} \frac{dk^2}{k^2} F(k^2) \left[\frac{k^4}{p^4}\theta(p^2 - k^2) + \theta(k^2 - p^2)\right]$$

$$F(p^2) = A(p^2)^\nu$$

Taking

$$\frac{1}{F(p^2)} = 1 - \frac{\alpha\xi}{4\pi\nu} \left[\frac{2F(p^2)}{\nu + 2} - F(\Lambda^2)\right]$$

Does not satisfy MR.

Ball-Chiu vertex

$$F(p^2) = 1 - \frac{\alpha\xi}{4\pi\nu} F(\Lambda^2) + \frac{3\alpha}{16\pi} F(p^2) \left[\frac{5}{2} + 2\pi \cot \pi\nu - \frac{1}{\nu} - \frac{2}{\nu + 1} - \frac{1}{\nu + 2} + \ln\frac{\Lambda^2}{p^2}\right]$$

This does not lead to have MR.

MR and choice of vertices: Contd

Curtis-Pennington Vertex

$$\Gamma_T^\mu(k,p) = \tau_6(k^2,p^2)T_6^\mu(k,p)$$

$$\begin{aligned}\frac{1}{F(p^2)} &= 1 + \frac{\alpha\xi}{4\pi}\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{F(k^2)}{F(p^2)} - \frac{\alpha}{4\pi}\int_0^{\Lambda^2}\frac{dk^2}{k^2}F(k^2) \\ &\quad \left\{\frac{k^4}{p^4}\left[\frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)+\frac{3}{2}(k^2-p^2)\tau_6(k^2,p^2)\right]\theta(p^2-k^2)\right. \\ &\quad \left.\left[\frac{3}{4}\frac{k^2+p^2}{k^2-p^2}\left(\frac{1}{F(k^2)}-\frac{1}{F(p^2)}\right)+\frac{3}{2}(k^2-p^2)\tau_6(k^2,p^2)\right]\theta(k^2-p^2)\right\} \quad .\end{aligned}$$

$$F(p^2) = \left[\frac{p^2}{\Lambda^2}\right]^{(\alpha\xi/4\pi)}$$

This satisfies the MR.

Step-1: We know what the Fermion propagator is in one gauge in the momentum space. We carry out the FT to find the corresponding expression in coordinate space.

Step-2: Using LKFT rule we obtain the expression of the Fermion propagator in arbitrary gauge in the coordinate space.

Step-3: We then FT the result back to the momentum space.

Quark Propagator

$$S^F(p:\xi) = \frac{F(p;\xi)}{ip\cdot\gamma - \mathcal{M}(p;\xi)},$$

$$S^F(x:\xi) = x\cdot\gamma X(x;\xi) + Y(x;\xi)$$

Let us start with tree level quark propagator

$$F(p;0) = 1, \qquad \mathcal{M}(p;0) = 0$$

After transforming the free quark propagator in coordinate space and using LKFT, we have

$$i\Delta_F(x) = -\frac{\xi}{(4\pi)^2}\left[\frac{1}{\epsilon} + \gamma + 2\ln(\mu x) + \mathcal{O}(\epsilon)\right].$$

We restricted ourself to the one loop here and $C_A=0$.

$$i\Delta_F(x_{\text{min}}) - i\Delta_F(x) = \lambda \ln\left(\frac{x^2}{x_{\text{min}}}\right)$$

Cut-off

$$\lambda = \xi/(4\pi)^2$$

$$\nu = C_F\alpha_s\xi/(4\pi)$$

$$S^F(x;\xi) = -\frac{x\cdot\gamma}{2\pi^2 x^4}\left(\frac{x^2}{x_{\text{min}}}\right)^{-\nu}$$

Inverse Fourier Transforms Yield

$$S^F(p;\xi) = \int d^4x e^{ip\cdot x} S^F(x;\xi).$$

$$F(p;\xi) = \frac{1}{2^{2\nu}} \frac{\Gamma(1-\nu)}{\Gamma(2+\nu)} (p^2 x_{\rm min}^2)^\nu.$$

$$\frac{F_R(p^2/\mu^2;\xi)}{F_R(k^2/\mu^2;\xi)} = \frac{F(p^2/\Lambda^2;\xi)}{F(k^2/\Lambda^2;\xi)} \qquad F_R(k^2/\mu^2;\xi)|_{k^2=\mu^2} = 1$$

LKFT ensures the MR of quark propagator and it yields

$$\boxed{F_R(p^2/\mu^2;\xi) = \left(\frac{p^2}{\mu^2}\right)^\nu.}$$

General structure of LKFT and MR suggest the following form of exponents

$$\nu = f_0 C_F \alpha_s + f_1^a C_F^2 \alpha_s^2 + f_1^b C_F C_A \alpha_s^2 + \cdots \qquad f_0 = \frac{\xi}{4\pi}.$$

Quark Gluon Vertex and LKFT

Any non-perturbative construction of the quark-gluon vertex must ensure the MR of the quark propagator.

Vertex	Structure	a_2, a_3, a_6, a_8	MR	ν
Bare	γ_μ	—	No	
BC	$\Gamma_\mu^{BC} = \sum_{i=1}^{4} \lambda_i L_\mu$	—	No	
CP	$\Gamma_\mu^{CP} = \Gamma_\mu^{BC} + \tau_6 T_\mu^6$	$a_6 = \frac{1}{2}$	Yes	$C_F \alpha \xi/(4\pi)$
B	$\Gamma_\mu^{B} = \Gamma_\mu^{BC} + \sum_{i=2,3,6,8} \tau_i T_\mu^i$	$a_6 = -\frac{1}{2}$, $a_2 + 2(a_3 + a_8) = -2$	Yes	Numerical
QC	$\Gamma_\mu^{QC} = \Gamma_\mu^{BC} + \sum_{i=2,3,6,8} \tau_i T_\mu^i$	$a_2 = a_6 = 0$, $a_3 = 1/2$, $a_8 = -1$	Yes	Numerical
Our	$\Gamma_\mu = \Gamma_\mu^{BC} + \sum_{i=2,3,6,8} \tau_i T_\mu^i$	$a_6 = +\frac{1}{2}$, $a_2 + 2(a_3 + a_8) = 0$	Yes	$C_F \alpha \xi/(4\pi)$

Thanks to Adnan Bashir for introducing me to this subject.

Special thanks to the organizers for their invitation and different kind of supports during the conference.