Dispersion relations & analyticity

References:

- "The Analytic S-matrix", CUP,
  Eden, Landshoff, Olive, Polkinhorne
- "Analytic properties of Feynman integrals"
  I. Todorov
- "Quantum Field Theory" McGraw-Hill
  Itzykson & Zuber
- "The theory of quantized fields" Vol I, II CUP
  S. Weinberg

homework lecture I (R. Zwicky)

Q: spectral fct $\rho(s) > 0 \Leftrightarrow$ unitary states

What happens if there are negative normed states

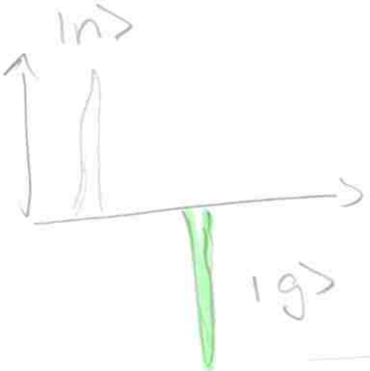
A: negative normed states $|g\rangle$, $g =$ "ghost"

$$\langle g | g' \rangle = -\delta_{gg'} \; ; \quad \langle n | n' \rangle = \delta_{nn'}$$

$$\mathbb{1} = \sum_n |n\rangle\langle n| - \sum_g |g\rangle\langle g|$$

since $\mathbb{1}\,|g'\rangle = (-1)^2 |g'\rangle = |g'\rangle$ ✓

$\mathbb{1}\,|n'\rangle = |n'\rangle$



$\rho(s)$ is not positive definite anymore!

Q: Given
$$\boxed{\Gamma(p^2) = i \int_x e^{ip\cdot x} \langle T \phi(x) \phi^\dagger(0) \rangle = \int_0^\infty \frac{ds\, \rho(s)}{s - p^2 - i\varepsilon}}$$

Källén - Lehmann representation

Show that: $\mathrm{Im}\,\Gamma(p^2) = \pi \rho(p^2)$

$$= \underbrace{\frac{1}{2i}\left[\Gamma(p^2 + i\varepsilon) - \Gamma(p^2 - i\varepsilon)\right]}_{\equiv \, \mathrm{disc}\, \Gamma(p^2)}$$

A: $\dfrac{1}{s-p^2 \mp i\varepsilon} = P\dfrac{1}{s-p^2} \pm i\pi\,\delta(s-p^2)$ (1)

principal part

$\triangleright\ \mathrm{Im}\,\dfrac{1}{s-p^2\mp i\varepsilon} = \pm\pi\,\delta(s-p^2)$ (2)

• from (1) $\triangleright$ $\boxed{\mathrm{Im}\,\Gamma(p^2) \overset{(2)}{=} \pi\,\rho(p^2)}$

since $\rho \in \mathbb{R}$

• $\boxed{\mathrm{Im}\,\Gamma(p^2) = \dfrac{1}{2i}\Big[\underbrace{\Gamma(p^2+i\varepsilon) - \Gamma(p^2-i\varepsilon)}_{\mathrm{disc}\,\Gamma(p^2)}\Big]}$

in principle we can stop here.

$$= \frac{1}{2i}\left[\int_0^\infty \frac{\rho(s)\,ds}{s-p^2-i\varepsilon} - \int \frac{\rho(s)\,ds}{s-p^2+i\varepsilon}\right]$$

$$\overset{(1)}{=} \frac{2i\pi}{2i}\,\rho(p^2) = \pi\,\rho(p^2)$$

which is consistent!

<u>Note</u>: slightly sloppy with $i\varepsilon$ .. should start from real part and analytically continue

$\Gamma(p^2-i\varepsilon)$

use this ε over the one in $\dfrac{1}{s-p^2-i\varepsilon}$ $\to 0$