Moduli spaces

1. $g_{a\bar b} \to CY_3$ - metric . $R_{\mu\nu} = 0$
   $d\omega = 0$, Ω - (3,0)-form } deform these

work with $h_{a\bar b} \Rightarrow \delta h_{ab}, \delta h_{\bar a\bar b}, \delta h_{a\bar b}$;

1). $\delta\omega = \delta h_{a\bar b}\, dz^a \wedge d\bar z^b$; $d\delta\omega = 0$ - closed

for $\cancel{\delta\omega = d\xi = \partial\xi + \bar\partial\xi = \partial_a \xi_b\, dz^a\wedge dz^b + \partial_a \xi_{\bar b}\, dz^a\wedge d\bar z^b}$
$\cancel{+ \bar\partial_{\bar a} \xi_b\, d\bar z^a \wedge dz^b + \partial_{\bar a}\xi_{\bar b}\, d\bar z^a \wedge d\bar z^b}$

in real basis

$$\delta\omega = d\xi = \partial_\mu \xi_\nu\, dx^\mu \wedge dx^\nu \; ; \quad \omega'_{\mu\nu} = \omega_{\mu\nu} + \partial_\mu \xi_\nu$$

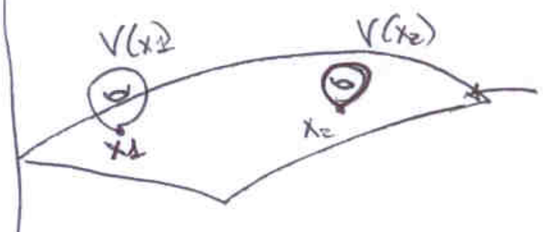
coordinate transf

$$x^{\mu\prime} = x^\mu + \xi^\mu$$

Consider all such $\delta\omega$: $d\delta\omega = 0$, $\delta\omega \neq d\xi$ } $\in H^{(1,1)}(M)$

$\Omega^{(1,1)}$

$\boxed{\delta\omega = \delta v^i \omega_i}$ $\{\omega_i\}$ = bas $H^{(1,1)}$



Σ_4

- Volume of the ext. space $V(x)$ - is a field

$\Rightarrow$ $R_1(x)$, $R_2(x)$ - 2 fields on Σ_4

moduli fields

2). $\delta h_{ab}, \delta h_{\bar a\bar b}$ - turn into forms.

$$\delta h_{ab} = \frac{1}{\|\Omega\|^2} \Omega_{acd} h^{c\bar c} h^{d\bar d} \bar\chi_{\alpha\, b\bar c\bar d}\, \delta\bar z^\alpha$$

$$\delta h_{\bar a\bar b} = \frac{1}{\|\Omega\|^2} \bar\Omega_{\bar a\bar c\bar d} h^{c\bar c} h^{d\bar d} \chi_{\alpha\, \bar b cd}\, \delta z^\alpha$$

$\{\bar\chi_\alpha\}$ = bas $H^{(1,2)}$

$\{\chi_\alpha\}$ = bas $H^{(2,1)}$

$R_{\mu\nu} = 0 \Rightarrow$ δh_{ab}, $\delta h_{\bar a\bar b}$ } closed

$$\|\Omega\|^2 = \frac{1}{3!}\Omega_{abc}\Omega^{abc} = \text{const}$$

2. Geometry of the space of Kähler structure moduli

$\downarrow$

metric on this space ($H^{(1,1)}$ of CY).

define $\delta t = (\underbrace{b^i}_{\text{Kalb-Ramond}} + i\underbrace{\delta\sigma^i}_{\text{Kähler structure}})\,\omega_i = \delta t^i\,\omega_i$ $\leftarrow$ hypermultiplet

$$\|\delta t\|^2 = \frac{1}{V}\int \delta t \wedge *\overline{\delta t} = \delta t^i\,\overline{\delta t}^j \underbrace{\boxed{\frac{1}{V}\int \omega_i \wedge *\omega_j}}_{G_{ij}}$$

Strominger : $$*\sigma = -\omega\wedge\sigma + \frac{3}{2}\frac{\kappa(\sigma,\omega,\omega)}{\kappa(\omega,\omega,\omega)}\,\omega\wedge\omega$$ for CY_3

$$\kappa(\sigma_1,\sigma_2,\sigma_3) = \int \sigma_1\wedge\sigma_2\wedge\sigma_3 \qquad \sigma\in\Omega^{(1,1)}$$

$$\boxed{G_{ij} = -\frac{3!\,\kappa(\omega_i,\omega_j,\omega)}{\kappa(\omega,\omega,\omega)} + 9\,\frac{\kappa(\omega_i,\omega,\omega)\,\kappa(\omega_j,\omega,\omega)}{\kappa(\omega,\omega,\omega)^2}}$$

since $\omega = \sigma^i\omega_i$ (property of CY)

$$G_{ij} = \frac{\partial}{\partial\sigma^i}\frac{\partial}{\partial\sigma^j}\log(\kappa(\omega,\omega,\omega)) = \frac{\partial}{\partial t^i}\frac{\partial}{\partial\bar t^j}\log\kappa(\omega,\omega,\omega)$$

3. Complex structure moduli

$$J^{*} = J_0 + \epsilon ; \quad J^2 = (J_0+\epsilon)^2 = -1 \Rightarrow (\epsilon_h{}^{a}{}_{\bar b} , \epsilon_{ah}{}^{\bar a}{}_{b})$$

$$\epsilon = \epsilon_h{}^{a}{}_{\bar b} \partial_a \otimes d\bar z^{b} + \epsilon_{ah}{}^{\bar a}{}_{b} \partial_{\bar a} \otimes dz^{b}$$

reality cond.: $\overline{J(X)} = J(\bar X) \Rightarrow \bar\epsilon_h = \epsilon_{ah}$

integrability: $N(J)=0 \quad \Rightarrow \bar\partial \epsilon_h = 0 \Rightarrow \epsilon_h = \epsilon_h(z)$

exact transf.: $J' = J + \bar\partial \sigma_h + \partial \sigma_{ah}$;

$$\bar\partial \epsilon_h = 0 , \ \epsilon_h \neq \bar\partial \sigma_h ; \qquad \epsilon_h \in H^{(0,1)}(TM)$$

$$\epsilon_h = \underbrace{\epsilon_h{}^{a}{}_{\bar b} \partial_a}_{\text{1-form on } TM} \otimes d\bar z^{b}$$

Theorem: for CY_n-manifolds $H^{(0,1)}(TM) = H^{(0,1)}(\Lambda^{n-1} T^{*}M) =$

$$= H^{(n-1,1)}(M)$$

$$\epsilon_h \to \epsilon^{a}{}_{\bar b} \Omega_{acd} \, d\bar z^{b} \wedge dz^{c} \wedge dz^{d} \in H^{n-1,1}(M)$$

$$\hookrightarrow \delta h_{ab} = \dots ; \quad \delta h_{\bar a \bar b} = \dots$$

$$\|\delta z^{\alpha}\|^2 := \frac{1}{4V} \int_{CY_3} h^{a\bar a} h^{b\bar b} \delta h_{ab} \delta h_{\bar a \bar b} \sqrt{g} \, d^6 x =$$

$$= -\frac{2i}{V \, \cancel{\|\Omega\|^2}} \delta z^{\alpha} \delta \bar z^{\bar\beta} \int_{CY_3} \chi_\alpha \wedge \bar\chi_{\bar\beta} = 2 \delta z^{\alpha} \delta \bar z^{\bar\beta} G_{\alpha\bar\beta}$$

$$G_{\alpha\bar\beta} = \frac{\int \chi_\alpha \wedge \bar\chi_{\bar\beta}}{\int \Omega \wedge \bar\Omega} ; \qquad \boxed{\frac{1}{3!} \omega\wedge\omega\wedge\omega = i \, \Omega \wedge \bar\Omega}$$

$$V = \frac{1}{n!} \int \underbrace{\omega \wedge \dots \wedge \omega}_{n} \simeq -i \int \Omega \wedge \bar\Omega$$

observation by Kodaira: (X. de la Ossa, Candelas)

$$\delta\Omega = k_\alpha \Omega \, \delta z^\alpha + \chi_\alpha \delta z^\alpha ; \quad \delta\Omega \in H^{(3,0)} \oplus H^{(2,1)}$$

$$\hookrightarrow \frac{\partial \Omega}{\partial z^\alpha} = k_\alpha \Omega + \chi_\alpha ; \Rightarrow$$

$$G_{\alpha\bar\beta} = -\frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial \bar z^{\bar\beta}} \log\left(i \int \Omega \wedge \bar\Omega \right);$$

$$K = -\log\left(i \int \Omega \wedge \bar\Omega \right)$$

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4. Hodge diamond

$h^{p,q} = \dim H_{(p,q)}(M)$

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$$

$\boxed{\text{for all}}$ $h^{0,0} = 1$ $\varphi^{(0)} = \text{const}$

$h^{3,3} = 1$ $\omega^{(3,3)} = dV$

$\boxed{CY}$ $h^{3,0} = 1 = h^{0,3}$

$\Omega \qquad \bar\Omega$

$h^{1,0} = 0, \; h^{2,0} = 0$

$h^{3,1} = 0, \; h^{3,2} = 0$

for CY:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$h^{p,q} = h^{q,p} =$

$= h^{n-p,n-q}$

$\omega \wedge \omega \sim * \omega$

if $\omega \wedge \omega = d\varphi \Rightarrow$

then $\underbrace{\omega\wedge\omega\wedge\omega}_{d\,Vol} = d(\varphi \wedge \omega)$

Mirror symmetry:

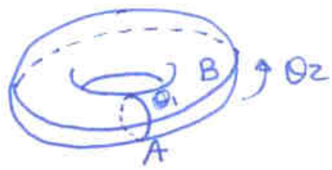
$h^{p,q} \to h^{q,p}$

3-cycles: $1, h^{2,1}, h^{1,2}, 1 \Rightarrow 2(h^{1,2}+1)$

real basis: $\{\alpha_A, \beta^B\}$; $\int_{CY_3} \alpha_A \wedge \beta^B = \delta_A^B$

cycles: $\int_{A^A} \alpha_B = \delta_B^A$; $\int_{B_B} \beta^A = -\delta_B^A$

EG: $\mathbb{T}^2$



$b^0 = 1, b^1 = 2, b^2 = 1$
↑ 1-cycle

coord: (θ_1, θ_2)

forms. $\alpha = d\theta_1$; $\beta = d\theta_2$; $\int_{\mathbb{T}^2} d\theta_1 \wedge d\theta_2 = 1$

$\int_A d\theta_1 = 1$; $\int_B d\theta_2 = 1$; $\int_B d\theta_1 = 0$

The cycles are defined such that:

$Z^A = \int_{A^A} \Omega$; $G_A = \int_{B^A} \Omega$; $z^\alpha = \frac{Z^\alpha}{Z^0}$

the deformed form

$\Omega = Z^A \alpha_A - G_A \beta^A$; $2 G_A = \frac{\partial G}{\partial Z^A}$; $G = G_A Z^A$

Hodge star: $* \alpha_A = A_A{}^B \alpha_B + B_{AB} \beta^B$

$* \beta^A = C^{AB} \alpha_B + D^A{}_B \beta^B$

$** = -1$: $A^2 + BC = -1$; $CB + D^2 = -1$ | $[B, C] = 0$

from $\int$'s : $A = -D$, $C^{Tr} = C$, $B^{Tr} = B$

$A = \text{Re}\, M \cdot (\text{Im}\, M)^{-1}$

$B = -(\text{Im}\, M) - \text{Re}\, M (\text{Im}\, M)^{-1} \text{Re}\, M$

$C = (\text{Im}\, M)^{-1}$

gauge couplings

$M_{AB} = \bar{G}_{AB} + 2i \frac{(\text{Im}\, G)_{AC} \cdot Z^C \, (\text{Im}\, G)_{BD} Z^D}{Z^C (\text{Im}\, G)_{CD} Z^D}$

$$\det g = \frac{1}{n!} g_{m_1 n_1} g_{m_2 n_2} \cdots g_{m_n n_n} \, \varepsilon^{m_1 \ldots m_n} \varepsilon^{n_1 \ldots n_n}$$

$$\Rightarrow \quad n! \, \underbrace{g_{m_1 n_1} g_{m_2 n_2} \cdots g_{m_n n_n}} = \epsilon_{m_1 m_2 \ldots m_n} \epsilon_{n_1 n_2 \ldots n_n}$$

$$\epsilon_{m_1 m_2 \ldots m_n} = \sqrt{g}\, \varepsilon_{m_1 \ldots m_n}$$

$$\boxed{-(\rho,\sigma)\, d\mathrm{Vol} =: \rho \wedge * \sigma}$$

$$* \sigma = \frac{1}{16 i} \epsilon_{mnk} \epsilon_{\bar m \bar n \bar k} \sigma^{k \bar k} dz^m \wedge dz^n \wedge dz^{\bar m} \wedge dz^{\bar n} =$$

$$= \frac{3!}{16 i} h_{m \bar m} \underbrace{h_{n \bar n} h_{k \bar k}} \sigma^{k \bar k} (\; .. \wedge .. \wedge \quad \wedge \;) =$$

$$= \frac{3!}{16 i} \cdot \frac{1}{3} \left(h_{m \bar m} h_{n \bar n} h_{k \bar k} \sigma^{k \bar k} - 2 h_{m \bar m} \sigma_{n \bar n} \right) (\wedge \; \wedge) =$$

$$= \cancel{\frac{1}{8 i}} = - \omega \wedge \sigma - \frac{1}{2} (\omega, \sigma)\, \omega \wedge \omega$$

$$\sigma = \omega: \quad * \omega = - \omega \wedge \omega - \frac{1}{2} (\omega, \omega)\, \omega \wedge \omega$$

$$(\omega, \omega) = \omega_{m \bar m} \omega_{n \bar n} h^{m \bar n} h^{n \bar m} = -\delta^m_n = -3$$

$$\omega = \frac{i}{2} h_{m \bar m} dz^m \wedge dz^{\bar m} \qquad \boxed{* \omega = \frac{1}{2} \omega \wedge \omega}$$

$$(\omega, \sigma) = \mathrm{const} \Rightarrow (\omega, \sigma) = \frac{1}{V} \int (\omega, \sigma)\, dV =$$

$$= - \frac{1}{V} \int \sigma \wedge * \omega = - \frac{1}{2V} \int \sigma \wedge \omega \wedge \omega = -3 \frac{\int \sigma \wedge \omega \wedge \omega}{\int \omega \wedge \omega \wedge \omega}$$

$$* \sigma = - \omega \wedge \sigma + \frac{3}{2} \frac{\int \sigma \wedge \omega \wedge \omega}{\int \omega \wedge \omega \wedge \omega} \omega \wedge \omega$$

$$V = \frac{1}{3!} \int \omega \wedge \omega \wedge \omega = - i \int \Omega \wedge \bar\Omega$$