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Geometry for string compactifications

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Abstract

These notes contain a basic introduction to the geometry of fibre bundles in application to string compactifications. We start with a pedagogical definition of fibre bundles and connection and continue with manifolds and spin geometry. This review is supposed to fill the gap between the basic knowledge of differential geometry and the level needed for understanding papers on string compactifications. This should not be considered as a conventional review paper as we prefer to give more strict and mathematical explanation of the problem rather than trying to cover a wide field.

Based on the lectures given on the Faculty of Mathematics, HSE; at DIAS, JINR; and in Nesin Mathematical Village, Izmir.

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1 Introduction

Manifolds with covariantly constant spinors often appear in the problem of string compactifications as those that preserve certain amount of supersymmetry. Since supersymmetric string theories are consistently formulated in 10 dimensions the problem of finding a way of reducing them to the conventional 4 dimensions is of great importance. Although there are many great books on differential geometry such as [1] or [2] which cover the topic discussed further in more details, we decided to present this text to the community because it contains a brief technical introduction to the field of dimensional reductions. The authors believe that this material will be useful for those who is starting learning string compactifications and will help to understand the ideas on a more technical level. With this aim we try to keep the presentation pedagogical with inclusion of necessary calculation details. At the end we consider few examples which relate the beautiful world of differential geometry to the physical problems. Since this text does not give an exhaustive physical introduction to the topic the reader is referred to the last section which contains a list of recommended literature with brief comments on the content.

Usually, people talk about compactifications of supergravity as a low energy limit of string theory with possible inclusion of stringy effects as α' corrections to equations of motion and Bianchi identities. It is already known that the most obvious and the most simple ways to compactify the theory do not work. For example, one could try to choose a six-torus $\mathbb{T}^6$ as a compact manifold. This would give a theory with maximal amount of supersymmetry $\mathcal{N} = 8$ and trivial interactions. The obtained theory is too constrained by the supersymmetry and is too simple to capture any phenomenology.

The next attempt is to take a manifold that preserves less supersymmetries than the torus, for example a Calabi-Yau manifold, that leads to $\mathcal{N} = 2$ theory in 4 dimensions. In further sections we will see that purely geometrical properties of the internal manifold X, such as the holonomy group and the number of constant spinors, are tightly related to the physical properties of the resulting theory.

Indeed, lets look at the supersymmetry transformations in Type IIA supergravity for example

$$\delta \psi_M = \nabla_M \varepsilon + H_M \varepsilon \tag{1.1}$$

One notices that in order to have a solution that preserves at least some of the supersymmetries and does not contain fermionic fields all their transformations should vanish. For example, if we have $\psi_M = 0$ for our solution and $\delta \psi_M \neq 0$ then this solution can not be supersymmetric since a SUSY transformation will lead to $\psi_M \neq 0$.

The condition of vanishing supersymmetry transformation can be nicely written as a parallel spinor condition on a compact manifold. Having a covariantly constant (in other words, parallel) spinor is a very strong requirement from the geometric point if view. The restriction is so rigid that compact manifolds with parallel spinors can all be classified. According to the classification by Bär [3], the topology and geometry of a manifold depends on the number of parallel spinors defined on it.

To decompose supersymmetry parameters ε, that are ten-dimensional spinors, the internal manifold must have a non-vanishing spinor η

$$\varepsilon = \xi \otimes \eta. \tag{1.2}$$

Warm-up example: compactification on a Minkowski background without fluxes (from Grana)

2 Differential geometry

2.1 Fibre bundles

A fibre bundle is a triple (E, π, M) where E and M are manifolds and π is a map

$$\pi : E \to M. \tag{2.1}$$

This map is surjective meaning that it covers the whole manifold M or in other words $\forall p \in M$, $\exists x \in E$ such that $\pi(x) = p$. The manifold M is called the base of the bundle while E is called the total space. The crucial defining feature of a fibre bundle is that images of all points in M are isomorphic to some space F called a typical fibre

$$\pi^{-1}[p] \cong F, \quad \forall p \in M. \tag{2.2}$$

If the space F is a vector space then the corresponding fibre bundle is called vector bundle.

In what follows we will deal with the special class of fibre bundles called locally trivial bundles. Define an atlas $\mathcal{A}$ that is a set of patches $\{U_\alpha\}$ covering the manifold M

$$\bigcup_\alpha U_\alpha \cong M. \tag{2.3}$$

Then the fibre (E, π, M) is locally trivial if there is defined a trivialisation map

$$\begin{tikzcd} \pi^{-1}[U_\alpha] \arrow[r, "f_\alpha"] \arrow[d, "\pi"] & U_\alpha \times F \arrow[dl, "pr"] \\ U_\alpha & \end{tikzcd} \tag{2.4}$$

for all patches from $\{U_\alpha\}$. Here the map pr denotes the canonical projection defined as $pr(p, a) = p$, $\forall a \in F$ and $\forall p \in M$. The diagram above means that locally the fibre bundle can always be represented as a direct product of the base M and the fibre F. If this trivialisation map is the same for all U_α then the fibre bundle is trivial and $E \cong M \times F$.

Consider an intersection of two patches U_α and U_β and the patch of the total space over it $\pi^{-1}[U_\alpha \cap U_\beta]$. Then the following commutative diagram can be written

$$\begin{tikzcd} U_\alpha \cap U_\beta \times F \arrow[dr, "pr"'] & \pi^{-1}[U_\alpha \cap U_\beta] \arrow[l, "f_\alpha"'] \arrow[r, "f_\beta"] \arrow[d, "\pi"] & U_\alpha \cap U_\beta \times F \arrow[dl, "pr"] \\ & U_\alpha \cap U_\beta & \end{tikzcd} \tag{2.5}$$

Here the maps f_α and f_β denote the trivialisation maps over the patches U_α and U_β. Although we here restrict them only to the intersection of the patches these maps are in general not the same.

Hence, we see that the map called the gluing co-cycle

$$\phi_{\alpha\beta} := f_\alpha \circ f_\beta^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F \tag{2.6}$$

allows us to glue locally trivial patches of the total space. Then one can consider the fibre bundle E as consisting of trivial pieces of the form $U_\alpha \times F$ glued together by the co-cycles $\phi_{\alpha\beta}$. Since each patch is a direct product these maps act only on fibres leaving the base points untouched

$$\phi_{\alpha\beta}(p, a) = \big(p, g_{\alpha\beta}(p)(a)\big), \quad g_{\alpha\beta}(p) \in \mathrm{End}(F). \tag{2.7}$$

Here the transition functions $g_{\alpha\beta}(p)$ defined at each point p of the base are elements of the endomorphism group of the fibre F. If the fibre is a vector space of dimension n then the transition functions will be elements of $GL(n)$. This is a good place to turn to specific examples.

2.2 A simple example: cylinder and the Möbius strip

The simplest example of a vector bundle is a bundle over a circle $\mathbb{S}^1$ with a line $\mathbb{R}^1$ as a typical fibre. Locally this bundle looks like $\mathbb{R}^1 \times \mathbb{R}^1$ and it is known that there are precisely two distinct types of gluing one can figure out. The first one gives us a cylinder, the other leads to Möbius strip.

Lets start with introducing a proper an atlas covering the circle $\mathbb{S}^1$. For a circle one should use minimum two patches to cover it that can be chosen in the following way

$$\begin{aligned} U_1 &= \{0 < \theta < 2\pi\}, \\ U_2 &= \{0 < \theta < \pi,\ \pi < \theta \leq 2\pi\}, \end{aligned} \tag{2.8}$$

where θ is a coordinate. The patches of this covering can be obtained by removing one point of the circle: $\theta = 0$ for U_1 and $\theta = \pi$ for U_2 as illustrated on the Fig. 1. The intersection of the sets U_1 and U_2 is then a unification of two

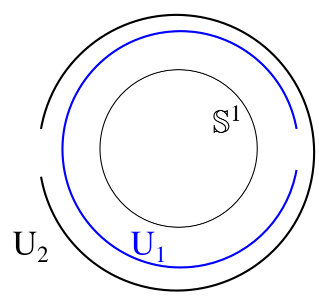

Figure 1: Two patches U_1 and U_2 is enough to cover the circle $\mathbb{S}^1$.

sets

$$U_1 \cap U_2 = \{0 < \theta < \pi\} \cup \{\pi < \theta < 2\pi\}. \tag{2.9}$$

Over each of these sets our bundle is just a direct product of the form $U_1 \cap U_2 \times \mathbb{R}^1 \cong \mathbb{R}^1 \times \mathbb{R}^1$.

The transition functions that glue the patches of the bundle together are elements of $\mathrm{End}(\mathbb{R}^1) = \mathbb{R}^1$. Since a rescaling of the fibre is not relevant for the topology there are only two distinct classes of transition functions for this bundle

$$g^{\pm}_{12} = \begin{cases} +1, & \{0 < \theta < \pi\} \\ \pm 1, & \{\pi < \theta < 2\pi\}. \end{cases} \tag{2.10}$$

This can be easily visualised as follows. Image that the patches $U_1 \times \mathbb{R}^1$ and $U_2 \times \mathbb{R}^1$ are just paper strips. Then the transition function g^{+}_{12} glues these strips on both intersections in the same way. This gives us a cylinder. On the contrary, the function g^{-}_{12} glues the stripes on the intersection $\{0 < \theta < \pi\}$ in the trivial way, but in the region $\{\pi < \theta < 2\pi\}$ one of the stripes should be twisted. This is exactly Möbius strip. The twisting means just flipping the fibre by multiplying by -1.

In other words, after we glue together the stripes in one region we get one longed strip. Then how we glue the remained two ends of this stripe is crucial.

2.3 Tangent and cotangent bundles

A tangent space T_pM to a manifold M at a point $p \in M$ is defined as a space of vectors that are tangent to all curves on M that pass through the point p. A curve in differential geometry is usually understood as a map from a unit line segment $\mathbb{I} = [0, 1]$ to the manifold

$$u : \mathbb{I} \to M. \tag{2.11}$$

In a chosen basis this map is just given by the coordinates of points of the curve

$$u(t) = \big(x^1(t), \dots, x^n(t)\big) \in M. \tag{2.12}$$

Components of a vector tangent to the path u are given by the usual formula

$$\{X^i\} = \dot{u}(t) = \big(\dot{x}^1(t), \dots, \dot{x}^n(t)\big) \in T_{u(t)}M. \tag{2.13}$$

It is often convenient to use a coordinate-free notation for the tangent vector that is by definition given by

$$X := \dot{u} = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} = X^i \partial_i. \tag{2.14}$$

Here we introduced the canonical natural basis of vector fields $\{\partial_i\}$. Each element ∂_i of the basis is itself a vector field with components $v^j = \delta^j_i$. Hence, this is the usual definition of a basis vector but in strange notations. It is convenient to understand vector fields as differential operators on M (this is actually their proper definition).

In the same way the canonical basis for 1-forms can be introduced. A form on the manifold M can be written as $\omega = \omega_i dx^i$ if $\{x^i\}$ are the coordinates on M. The basis $\{dx^i\}$ of 1-forms is conjugate to the basis of vector fields in the following sense

$$dx^i(\partial_j) = \delta^i_j. \tag{2.15}$$ orth

This implies that the action of a 1-form ω on a vector field X is just

$$\omega(X) = \omega_i X^j dx^i(\partial_j) = \omega_i X^i. \tag{2.16}$$

Now we can define the tangent bundle over the manifold M that is constructed as the usual fibre bundle with $F_p = T_pM$ for $\forall p \in M$.

$$s : M \to TM \tag{2.17}$$

that should satisfy $\pi \circ s = \mathrm{id}$, where id is the identity map $\mathrm{id} : p \mapsto p$, $\forall p \in M$. The section s gives an element of the fibre F for each point of the base M. In the case, when F is a vector space V a section s defines a vector field on the manifold M.

The typical fibre of the cotangent bundle T^*M over the manifold M is given by the space of 1-forms T_p^*M at each $p \in M$ that is a canonical conjugate of the tangent space T_pM. The conjugation is given by the isomorphism of vectors and covectors

$$X^i \Leftrightarrow \omega_i. \tag{2.18}$$

A section of the cotangent bundle is a differential 1-form on the manifold.

2.4 Horizontal spaces and connection 1-form

In this section we will briefly introduce the notion of connection on a fibre bundle, covariant derivative and parallel (horizontal) transport. For a more detailed explanation one can follow Nakahara:2003nw [1].

Consider a fibre bundle (E, π, M) with $\dim E = m + n$, where $m = \dim M$ is the dimension of the base and $n = \dim F$ is the dimension of the fibre F. There always exist n linearly independent 1-forms on E

$$\exists\, \theta^1, \ldots, \theta^n \in T^*E. \tag{2.19}$$

Then one can construct the so-called annulator of these forms $H_p = \mathrm{Ann}(\theta_p^1, \ldots, \theta_p^n)$ at each point $p \in E$ that is a vector space $H_p \subset TE$ whose elements are all annihilated by the forms θ^i at this point

$$\forall X_p \in H_p, \quad \theta_p^i(X_p) = 0. \tag{2.20}$$

Basically, the tangent space TE_p at the point $p \in E$ was split into 2 parts, the horizontal subspace H_p and the vertical subspace V_p

$$TE_p = H_p \oplus V_p. \tag{2.21}$$

split

Dimension of the horizontal space H_p is that of the base M and $\dim V_p \equiv \dim F = n$. If it is possible to define a distribution of the horizontal spaces H_p globally, i.e. for each $p \in E$, then this is equivalent to having a **connection** on the base M.

To show this lets introduce coordinates (a^i, x^μ) on the bundle E. The natural basis of 1-forms on E is then written as $\{da^i, dx^\mu\}$. Here small Latin indices run from 1 to n and correspond to fibre directions and small Greek indices run from 1 to m and correspond to base directions.

In this basis the vertical forms θ^i can be written as

$$\theta^i = f_j^i(a, x)da^j + g_\mu^i(a, x)dx^\mu, \tag{2.22}$$

where the functions f_j^i and g_μ^i are just components of the forms. Since these forms are vertical they themselves provide a basis for 1-form on **the fibre** F implying that we can always choose $f_j^i = \delta_j^i$.

If we choose the functions g^i_μ to be linear with respect to the coordinates a^i, i.e. $g^i_\mu(a, x) = \Gamma^i_{j\mu}(x)a^j$ then the vertical forms become

$$\theta^i = da^i + \Gamma^i_{j\mu}a^j dx^\mu. \tag{2.23}$$

In this forms one can see something very familiar. Indeed, the 1-forms $\omega^i_j = \Gamma^i_{j\mu}dx^\mu$ are called the **connection forms**.

2.5 Covariant derivative

In this chapter we introduce covariant derivative of a section of a fibre bundle. To remind, a section of a fibre bundle E with base manifold M and a typical fibre F is a map

$$s : M \to E. \tag{2.24}$$

Hence, a section gives an element of the fibre for each chosen point on M. If we are talking about a vector bundle over M then sections of this bundles are nothing but vector fields on M, that are often denoted by X.

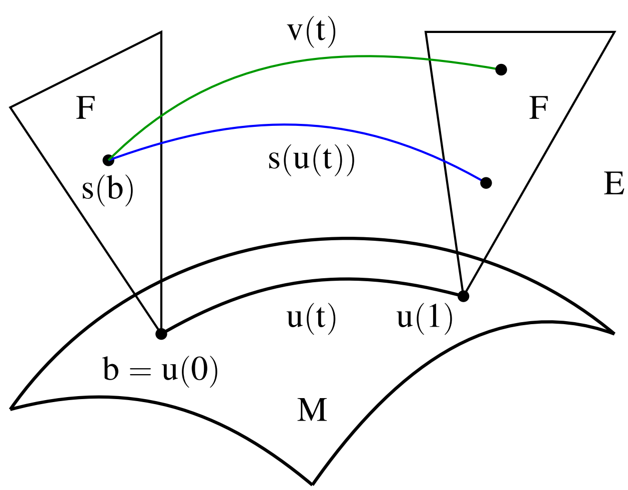

Figure 2: Curves

Following our definition of a horizontal space we can define a horizontal section s by the condition

$$\theta^i(s) = 0. \tag{2.25}$$

If we are talking about a vector bundle whose sections are vector fields X on M, then we have horizontal vector fields.

Consider a section of a vector bundle $s : M \to E$ and a curve $u : \mathbb{I} \to M$ on the base (see Figure 2.). Consider a horizontal curve $v : \mathbb{I} \to E$ that covers the curve $u(t)$

$$\pi(v(t)) = u(t), \quad \forall t \in [0, 1]. \tag{2.26}$$

A curve is called horizontal if its tangent vector is annihilated by all the vertical forms

$$\theta^i(\dot{v}) = 0. \tag{2.27}$$

The curve $v(t)$ is called the **horizontal lift** of u. Obviously, in general the section $s(u(t))$ is not horizontal since we do not impose any conditions on it. However, it covers the curve $u(t)$ as well

$$\pi(s(u(t))) = u(t) \tag{2.28}$$

simply because $\pi \circ s = \mathrm{id}$.

A simple way to understand this is to think about vector bundles. Then the section $s(u(t))$ is just a vector field over the curve u. I.e. at each point of $u(t) \in M$ we have a vector. The (horizontal) section $v(t)$ is again a vector field. It is **important** to understand that $v(t) \notin H_{u(t)}$, i.e. the vector $v(t)$ on M at the point $u(t)$ does not belong to the horizontal space defined earlier. Only the tangent vector to this curve belongs to $H_{u(t)}$. Basically, at each point we have a vector $v(t)$ and we define the velocity of $\dot{v}(t)$ of the curve. This velocity shows how fast the vector is changing when we moving along the curve.

Now the covariant derivative $\nabla_X s|_b$ of the section $s(u)$ along the vector field $X = \dot{u}$ at the point $b = u(0)$ is defined as a difference between the section and the horizontal curve covering the curve $u(t)$

$$\nabla_X s|_b = \lim_{t\to 0} \frac{s(u(t)) - v(t)}{t}. \tag{2.29}$$

def_cov_der

Lets show that this reproduces exactly the well known formula for the covariant derivative in the coordinate form. The curves are then given by

$$\begin{aligned} u(t) &= (x^1(t), \dots, x^m(t)), \\ v(t) &= (x^1(t), \dots, x^m(t), a^1(t), \dots, a^n(t)), \\ s(u(t)) &= (x^1(t), \dots, x^m(t), s^1(t), \dots, s^n(t)), \end{aligned} \tag{2.30}$$

where x^μ are the coordinates on the base M. The tangent vector $\dot{v}$ has the following form

$$\dot{v}(t) = \dot{a}^i \partial_i + \dot{x}^\mu \partial_\mu. \tag{2.31}$$

Since this vector is horizontal the following is true

$$\theta^i(\dot{v}) = \dot{a}^i + \Gamma^i_{j\mu} a^j \dot{x}^\mu = 0, \tag{2.32}$$

hor

where we used the orthogonality condition (orth 2.15). Then the covariant derivative reads

$$
\begin{aligned}
\nabla_X s^i &= \lim_{t\to 0}\frac{s^i(x(t)) - a^i(t)}{t} \equiv \\
&\equiv \lim_{t\to 0}\frac{s^i(x(t)) - s^i(x(0))}{t} - \lim_{t\to 0}\frac{a^i(t) - s^i(x(0))}{t} = \\
&= \lim_{t\to 0}\frac{s^i(x(t)) - s^i(x(0))}{t} - \lim_{t\to 0}\frac{a^i(t) - a^i(0)}{t} = \\
&= \dot{s}^i(0) - \dot{a}^i(0) = \dot{s}^i + \Gamma^i_{j\mu} a^j(0)\dot{x}^\mu = \\
&= \dot{s}^i(0) + \Gamma^i_{j\mu} s^j(0)\dot{x}^\mu .
\end{aligned}
\tag{2.33}
$$

Here in the second line we just added and subtracted $s^i(x(0))$ in the numerator, in the third line we used that $s^i(x(0)) \equiv a^i(0)$ by the construction (see Fig. 2.). Finally in fourth line we used the equation (2.32).

In more familiar notations the last formula of the equation above can be written as

$$
\nabla_X s^i = X^\mu \nabla_\mu s^i = X^\mu \left(\partial_\mu s^i + \Gamma^i_{j\mu} s^j\right). \tag{2.34}
$$

Finally, if we are talking about the tangent bundle TM then the section s^i is just a vector field and the fibre indices i are exactly the same as the base indices μ. Then we can replace s^i by some vector say Y^α and write

$$
\nabla_\mu Y^\alpha = \partial_\mu Y^\alpha + \Gamma^\alpha_{\mu\nu} Y^\nu, \tag{2.35}
$$

that is exactly the covariant derivative of a vector field.

2.6 Parallel transport and holonomy groups

Recall the definition (2.29) of the covariant derivative $\nabla_X s$ of a section $s : M \to E$ of a fibre bundle E along a vector field $X = \dot{\gamma}$ being a tangent vector to a curve $\gamma : \mathbb{R} \to M$

$$
\nabla_X s|_b = \lim_{\Delta t\to 0}\frac{s(u(t+\Delta t)) - v(t+\Delta t)}{\Delta t}. \tag{2.36}
$$

For infinitesimally small shifts of the parameter Δt one can then write

$$
v(t+\Delta t) = s(u(t+\Delta t)) - \nabla_X s(t)\Delta t. \tag{2.37}
$$

It is important to mention here, that a section is defined on the manifold M globally, hence we have a value $s(p)$ for each point $p \in M$ on the manifold. Hence, $s(u(t+\Delta t))$ is a value of the section at the point $u(t+\Delta t)$ defined by the curve $u(t)$.

However, if we would like to compare the value $s(u(t+\Delta t))$ of the section to the value of the same section $s(u(t))$, we encounter a difficulty since these values are taken in different point. One should define a way of shifting the value $s(u(t))$ to the point $u(t+\Delta t)$ in a way, that does not depend on the section. This is done by introducing a notion of parallel trandport, that in our notations is just $v(t+\Delta t)$.

So, one **defines** a parallel transport of a section $s : M \to E$ from the point $u(t) \in M$ to the point $u(t+\Delta t) \in M$ to be a value $v(t+\Delta t)$ of the horizontal lift $v : \mathbb{R} \to M$ of the curve $u : \mathbb{R} \to M$ at the point $u(t+\Delta t)$, with the value at $u(t)$ fixed by $v(t) = s(u(t))$.

Using the definiton of the covariant derivative above one writes

$$v(t+\Delta t) = s(u(t+\Delta t)) - \dot{s}(u(t))\Delta t - \Gamma_X[s](t)\Delta t = s(u(t)) - \Gamma_X[s](t)\Delta t, \qquad (2.38)$$ transport

where we define $\Gamma_X[s] = X^\mu \Gamma_\mu{}^i{}_j s^j e_j$ for $\{e_j\} = basF$. Hence, one comes up with a geometric understanding of the notion of covariant derivative, that is: an amount to which a parallel transport of a section differs from the section at this point. In addition one notes that a horizontal curve can be defined as such a curve that is invariant under parallel transport.

Finally, one is able to define a notion of a geodesic curve on the manifold M. Consider a tangent bundle TM and a curve $\gamma : \mathbb{R} \to M$ on the manifold. Then $\dot{\gamma}$ is a curve on TM. The curve γ is said to be **geodesic** if its lift $\dot{\gamma}$ to TM is horizontal (with respect to a connection Γ), i.e. $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. In a local frame this implies

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_{\dot{\gamma}}\dot{x}^\mu \partial_\mu = \left(\ddot{x}^\mu + \Gamma_{\nu\rho}{}^\mu \dot{x}^\nu \dot{x}^\rho\right)\partial_\mu = 0, \qquad (2.39)$$

that is the well known geodesic equation.

The notion of parallel transport allows to define the concept of holonomy group (algebra), that is very useful for applications. The holonomy group $Hol_p(\nabla)$ of a connection ∇ at the point $p \in M$ of the manifold is defined to be a set of all transformations of sections at the point p under parallel transport along a closed loop $\gamma : \mathbb{S}^1 \to M$ starting and ending at the point p. Obviously, for any fibre bundle the holonomy group is a subgroup of all allowed transformations of the typical fibre F, i.e. $Hol_p(\nabla) < Hom(F)$. Hence, for a vector bundle with a fibre V of dimension $\dim V = n$ the holonomy groupat a point is $Hol_p(\nabla) < GL(n)$.

There exists a theorem, that is not included here, stating that the holonomy group $Hol_p(\nabla)$ actually does not depend on the chosen point $p \in M$ (under some conditions, that are always satisfied for physical applications).

Before going to explicit examples in the next section, we show that to calculate holonomy group of a connection one actually does not need to perform all possible parallel transport along infinitesimal curves and look at their group structure. It appears, that such parallel transport is actually given by Cartan tensor (curvature of the connection Γ).

2.7 Geometric meaning of curvature tensor

We would like now to show, that holonomy of a section s around a closed loop is related to $[\nabla_{X_1}, \nabla_{X_2}]s$ and hence to curvature. Here, $X_{1,2} = \dot{\gamma}_{1,2} \in TM$ are vector field on the manifold tangent to curves $\gamma_{1,2}(t)$ (see Fig. 3). One has to choose the closed curve along which the parallel transport is performed to be partially smooth, i.e. consisting of finite number of smooth pieces, since otherwise the fields $X_{1,2}$ will be singular (have a vortex somewhere).

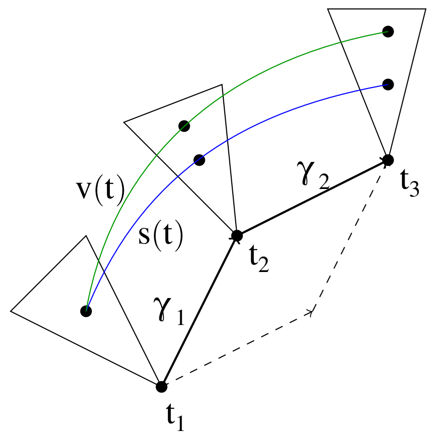

Figure 3: Illustration of the parallel transport along a closed loop (see the text). loop

Consider a section $s : M \to E$ of a fiber bundle E and its parallel transport along the curves $\gamma_{1,2}$ defined by the horizontal curve $v : \mathbb{R} \to E$. According to (2.38) we have

$$v(t_2) = s(t_1) - \Gamma_{X_1}[s](t_1)dt = s(t_1) - \Gamma_\mu[s](t_1)dx^\mu, \tag{2.40}$$

where we used the definition of a vector tangent to a curve $X^\mu = (dx^\mu/dt)\partial_\mu$ and denoted $\Gamma_\mu[s] = \Gamma_\mu{}^i{}_j s^j e_i$ for $\{e_i\} = basF$.

Doing the same for the second point $\gamma_2(t_3)$ we write

$$v(t_3) = v(t_2) - \Gamma_{X_2}[v](t_2) = v(t_2) - \Gamma_\mu[v](t_2)d\tilde{x}^\mu, \tag{2.41}$$

where $d\tilde{x}^\mu$ is the shift along the curve γ_2 from t_2 to t_3.

Note, that since $v(t)$ is a horizontal curve, a parallel transport shifts the curve into itself. I.e. a parallel transport of $v(t_2)$ to the point t_3 is just $v(t_3)$. And hence one could have written

$$v(t_3) = v(t_2) + \dot{v}(t_2)dt. \tag{2.42}$$

However, because of the horizontality condition $\nabla_{X_2} v = 0$ this expression is exactly the same as the previous one. Obviously, it is consistent to just formally apply the rules of parallel transport to the curve $v(t)$.

Substituting $v(t_2)$ into this expression we have

$$\begin{aligned} v(t_3) &= v(t_2) - \Gamma_\mu[v](t_2)d\tilde{x}^\mu = s(t_1) - \Gamma_\mu[s](t_1)dx^\mu - \Gamma_\mu[v](t_2)d\tilde{x}^\mu \\ &= s(t_1) - \Gamma_\mu[s](t_1)dx^\mu - \left(\Gamma_\mu + \dot{\Gamma}_\mu dt\right)\left[s - \Gamma_\nu[s]dx^\nu\right](t_1)d\tilde{x}^\mu. \end{aligned} \tag{2.43}$$

The second term in the last line needs a bit of explanation. The second bracket here just originates from substitution of $v(t_2)$ as in the first line. The first bracket contains $\dot{G}_\mu$ that takes into account that befor the function $\Gamma_\mu{}^i{}_j$ was defined at the point $\gamma_1(t_2)$ rather than $\gamma_1(t_1)$. This is cured by the usual expansion of Γ_μ as a function around t_2.

Opening the brackets and writing everything together we have

$$\begin{aligned} v(t_3) &= s(t_1) - \Gamma_\mu[s](t_1)(dx^\mu + d\tilde{x}^\mu) - \dot{\Gamma}_\mu[s](t_1)dt d\tilde{x}^\mu - \Gamma_\mu[\Gamma_\nu[s]](t_1)dx^\nu d\tilde{x}^\mu = \\ &= s(t_1) - \Gamma_\mu[s](t_1)(dx^\mu + d\tilde{x}^\mu) - \Big(\partial_\nu\Gamma_\mu + \Gamma_\mu\Gamma_\nu\Big)[s](t_1)dx^\nu d\tilde{x}^\mu, \end{aligned} \tag{2.44}$$

where in the second line we replaced $\dot{G}_\mu(t_1)dt = \partial_\nu\Gamma_\mu dx^\nu$ since the derivative is taken along the curve γ_1 that corresponds to the shift dx^μ. The term $\Gamma_\mu\Gamma_\nu[s](t_1)$ in the local frame is just $\Gamma_\mu{}^i{}_k\Gamma_\nu{}^k{}_j s^j(t_1)e_i$.

To close the loop one should preform the same steps going further along the curves denoted by dashed lines on the figure. In this case one would have to perform multiple Taylor expansions to relate Gamma's at say t_4 to Gamma's at t_1. To avoid this procedure we will do a trick and instead go along the dashed lines in the opposite direction starting at $\gamma_1(t_1)$. Obviously. this will provide us the same resilt up to change $dx^\mu \leftrightarrow d\tilde{x}^\mu$, i.e.

$$v'(t_3) = s(t_1) - \Gamma_\mu[s](t_1)(d\tilde{x}^\mu + dx^\mu) - \Big(\partial_\nu\Gamma_\mu + \Gamma_\mu\Gamma_\nu\Big)[s](t_1)d\tilde{x}^\nu dx^\mu. \tag{2.45}$$

We see, that the result of parallel transport crucially depends on the curve chosen and not on the final point only. This failure of integrability is described by curvature tensor

$$\begin{aligned} v'(t_3) - v(t_3) &= -R_{\mu\nu}[s](t_1)d\tilde{x}^\mu dx^\nu; \\ R_{\mu\nu}{}^i{}_j &= 2\partial_{[\mu}\Gamma_{\nu]}{}^i{}_j - 2\Gamma_{[\mu}{}^i{}_k\Gamma_{\nu]}{}^k{}_j. \end{aligned} \tag{2.46}$$

Finally, it is straightforward to check that the Riemann curvature tensor can be calculated by taking commutator of covariant derivatives

$$[\nabla_\mu, \nabla_\nu]s^i = R_{\mu\nu}{}^i{}_j s^j - T_{\mu\nu}{}^\rho\nabla_\rho s^i, \tag{2.47}$$

where $T_{\mu\nu}{}^\rho = \Gamma_{\mu\nu}{}^\rho - \Gamma_{\nu\mu}{}^\rho$ is torsion of the connection $\Gamma_{\mu\nu}{}^\rho$ in the tangent bundle induced by the connection $\Gamma_\mu{}^i{}_j$ in the bundle E. Geometric meaning of torsion will be considered in the next section.

In the coordinate free form these tensors can be written as

$$\begin{aligned} R(X,Y)s &= [\nabla_X, \nabla_Y]s + \nabla_{[X,Y]}s, \\ T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned} \tag{2.48}$$

Here $X, Y \in \mathrm{Vect}(M)$ are vector field on the base M and $[X,Y] = L^\nabla_X Y$ is a covariant Lie derivative of Y along X

$$[X,Y]^\mu = X^\nu\nabla_\nu Y^\mu - Y^\nu\nabla_\nu X^\mu. \tag{2.49}$$

Note, that the above procedure can be understood as going around a closed loop starting from t_3 and hence it defines action of the algebra of the holonomy group (the loop is infinitesimally small). This implies $R_{\mu\nu} \in \mathrm{hol}(\nabla)$ and hence the curvature tensor can be written as

$$R = \tau \otimes h, \tag{2.50}$$

where $\tau \in \Omega^2(M)$ is a 2-form on M and $h \in \mathrm{hol}(\nabla)$ is an element of the holonomy algebra. This basically means that we can represent the curvature as

$$R_{\mu\nu}{}^i{}_j = \tau_{\mu\nu} \otimes h^i_j. \tag{2.51}$$

As a basic example one can take the Schwarzschild metric in d dimensions, calculate its Riemann curvature and prove that its components are proportional to generators of SO(d). And indeed a Riemannian manifold has holonomy group SO(d).

This is a consequence of a very important theorem whose proof we will not present here [1]. Consider a tangent bundle TM over a manifold M with connection ∇. Holonomy group $H = \mathrm{Hol}(\nabla)$ is a subgroup of the structure group $H \leq GL(T_xM)$ for any $x \in M$. The theorem states, that if $S \in \Gamma(TM)$ is a constant tensor on M, i.e. it satisfies $\nabla S = 0$, then it is invariant under the holonomy group. In other words

$$\nabla S = 0 \iff L(S) = 0, \quad L \in \mathrm{hol}(\nabla), \tag{2.52}$$

where $\mathrm{hol}(\nabla)$ is the Lie algebra of the holonomy group. This allows to determine the holonomy group of a bundle by looking at its parallel sections.

Example. Let take a Riemann manifold that is a manifold endowed with a metric g. Consider a connection ∇^{LC} that is compatible with the metric, i.e.

$$\nabla^{LC} g = 0. \tag{2.53}$$

Such a connection is called Levi-Civita connection. From the theorem above we derive that the holonomy group is the one that fixes the metric g. Written in components for an element $\Lambda \in \mathrm{Hol}(\nabla^{LC})$ this condition reads

$$\Lambda^a{}_c \Lambda^b{}_d g_{ab} = g_{cd}. \tag{2.54}$$

This gives $\Lambda \in SO(d)$, where $d = \dim M$. Hence, we see that holonomy group of a Riemann manifold is indeed SO(d).

2.8 Geometric meaning of torsion tensor

Let us now turn to the torsion tensor and consider vectors fields X and Y at a point $p \in M$ that has coordinates $\{x^\mu\}$ in a local frame. The question

Joyce:2000ct
[1]The proof can be found in [2] as Lemma 2.5.1 and proposition 2.5.2

is if it is possible to form a parallelogram on M from integral curves of these vector fields. To do so, one should be able to say which sides of the figure are parallel and which sides are equal to each other. The first concept can be easily related to the notion of parallel transport, while the second needs more careful consideration.

Let us start with the point $p \in M$ with coordinates $\{\xi^\mu\}$ and to vectors X_p and Y_p at this point, whose integral curves intersect. One draws the first side pr of the parallelogram by going say along the integral curve of the field X (green on Fig. 4). As usual the curve is parametrised by a real number t, and we define the point r as a shift along the vector X by the amount of dt. i.e. r has coordinates

$$x^\mu + X^\mu(p)dt. \tag{2.55}$$

Doing the same with the field Y we construct another side pq of the parallelogram and arrive to the point q with coordinates

$$x^\mu + Y^\mu(p)dt', \tag{2.56}$$

where in general $dt' \neq dt$.

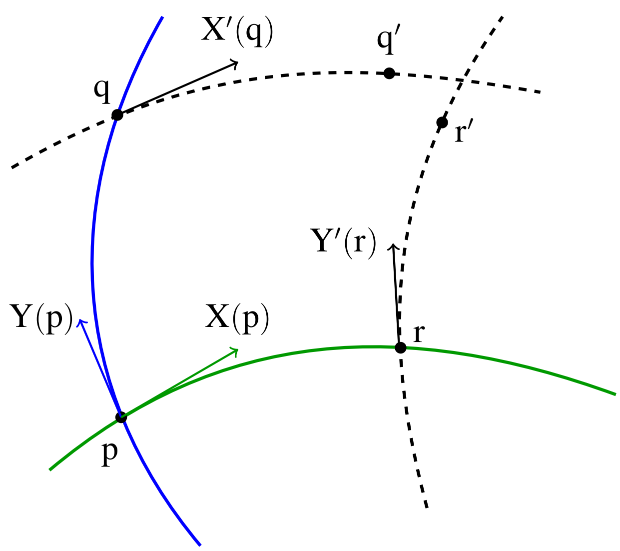

Figure 4: Failure of parallel transported vectors to form a parallelogram

torsion

Let us now construct the side qq′ parallel to pr, that is given by a segment of the integral curve of the vector X'_q **parallely transported** from the point p. Note, that in general $X'_q \neq X_q$, but we have instead $X'^\mu(q) = X^\mu(p) - \Gamma_{\nu\rho}{}^\mu X^\nu Y^\rho dt'$ since we transport X along Y to the distance determined by dt′. This gives the following coordinates of q′

$$x^\mu + Y^\mu(p)dt' + X^\mu(q)dt'' = x^\mu + Y^\mu(p)dt' + X^\mu(p)dt - \Gamma_{\nu\rho}{}^\mu X^\nu(p)Y^\rho(p)dt'dt \tag{2.57}$$

It is important here, that the side qq′ should be equal to pr that implies $dt'' = dt$.

To construct the last side rr' we use the parallel transport of the vector Y to the point r that is $Y'^{\mu}(r) = Y^{\mu}(p) - \Gamma_{\rho\nu}{}^{\mu} Y^{r}(p) X^{\nu}(p) dt$ and the coordinate of r' reads

$$x^{\mu} + X^{\mu}(p)dt + Y^{\mu}(r)dt' = x^{\mu} + X^{\mu}(p)dt + Y^{\mu}(p)dt' - \Gamma_{\rho\nu}{}^{\mu} Y^{\rho}(p) X^{\nu}(p) dt dt'. \quad (2.58)$$

Finally, comparing this to the coordinates of q' we see, that they do not match and the difference Δ^{μ} is given by

$$\begin{aligned} \Delta^{\mu} &= \Big(\Gamma_{\nu\rho}{}^{\mu} X^{\nu}(p) Y^{\rho}(p) - \Gamma_{\rho\nu}{}^{\mu} Y^{\rho}(p) X^{\nu}(p)\Big) dt dt' \\ &= (\Gamma_{\nu\rho}{}^{\mu} - \Gamma_{\rho\nu}{}^{\mu}) X^{\nu}(p) Y^{\rho}(p) dt dt' = T_{\nu\rho}{}^{\mu} X^{\nu}(p) Y^{\rho}(p) dt dt', \end{aligned} \quad (2.59)$$

where $T_{\mu\nu}{}^{\rho}$ is the torsion tensor.

Hence, one concludes that non-zero torsion of a connection on a tangent bundle of a manifold M does not allow to build parallelograms. Note, that in general one is able to build any closed figure, but for it to have parallel and mutually equal sides the connection shoud be torsionless.

2.9 Berger's classification

In general case one can consider different types of constant tensors on a manifolds and obtain different types of manifolds. Following the classification by Berger we may write a table

	Holonomy group	Manifold type
1.	$SO(n)$	generic Riemann manifold
2.	$U(n)$	Kähler metrics on 2n-dimensional manifold
3.	$SU(n)$	Calabi-Yau manifold
4.	$Sp(n)$	4n-dimensional hyper-Kähler manifold
5.	G_2	G_2 manifold
6.	$Spin(7)$	$Spin(7)$ manifold

Table 1: Berger's classification of manifolds according to their holonomy group.

3 Complex geometry

3.1 Complex structure

A complex manifold is defined in the similar way as a real differential manifold. Without entering into unnecessary mathematical details we can say that complex manifold is a manifold with complex coordinates defined on it and with transition functions being holomorphic. This is self-consistent definition that does not need the notion of real even-dimensional manifold. However, in application one usually starts with a real manifold and then defines a structure of the complex manifold on it. Evidently, this can be done not for any real manifold and in this section we describe the procedure following mainly the reference [4].

First of all for a physicist it is pretty straightforward that to be able to carry a complex structure a real manifold should be even dimensional. We will start with more simple case of a linear space V of dimension $\dim V = 2n$. A complex structure is such endomorphism $J \in \mathrm{End}(V)$ that squares to minus the identity transformation $J^2 = -1$:

$$\begin{aligned} J &: V \to V, \\ J &: v \longmapsto v', \qquad \forall v, v' \in V, \\ J \circ J &: v \longmapsto -v, \end{aligned} \tag{3.1}$$

where small circle denotes composition of endomorphisms. For a given basis the matrix representation of the operator J takes the familiar form

$$J_0 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{3.2}$$

where I_n is $n \times n$ identity matrix. We call this the canonical complex structure and denote by J_0.

Since a complex vector space is defined over the field of complex numbers while a real vector space is defined over the reals we are not allowed to just multiply vectors of the 2n-dimensional space V by complex numbers. This is where the complex structure J becomes useful: we define multiplication of a vector $X \in V$ by a complex number $a + ib$ in the following way

$$(a + ib)X := aX + bJX. \tag{3.3}$$

Hence the tensor J basically plays the role of the imaginary unit i. This allows to go backwards and define a complex structure on a **complex** vector space as $JZ := iZ$. Note that here the vector Z is an element of the complex space and this can be multiplied by the imaginary unit.

The complex structure J can be used to define a naturally consistent basis. Indeed, for any set of vectors $\{e_1, \dots, e_n\}$ on V of dimension $\dim V = 2n$ we can define

$$\{e_1, \dots, e_n, Je_1, \dots, Je_n\} = \mathrm{bas} V \tag{3.4}$$

since all elements are linearly independent as $\det J = 1$.

The canonical complex structure J_0 is not the only tensor with the property $J^2 = -1$ that can be defined on a vector space. In general, it can be rotated by a matrix $S \in GL(2n, \mathbb{R})$

$$J = SJ_0S^{-1}. \tag{3.5}$$

It is easy to check that such defined J introduces a proper complex structure, i.e. $J^2 = -1$. However, matrices $S_0 \in GL(n, \mathbb{C})$ commute with J_0 and hence the space of complex structures on $\mathbb{R}^{2n}$ is equivalent to $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$.

This result is proven to be very important for investigating complex structure moduli on manifolds. As a simple example consider the linear space $\mathbb{R}^2$ that for us will play the role of a torus $\mathbb{T}^2$, that is often used in string compactifications. A general matrix on $\mathbb{R}^2$ that satisfies $J^2 = -1$ can be written as

$$J = \begin{bmatrix} \sqrt{-1-bc} & b \\ c & -\sqrt{-1-bc} \end{bmatrix} \tag{3.6}$$

Now fix a vector $e_1 \in \mathbb{R}^2$ and consider the basis generated by J

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = Je_1 = \begin{bmatrix} \sqrt{1-bc} \\ c \end{bmatrix}. \tag{3.7}$$

A general vector $X = x^1 e_1 + z^2 e_2$ on the vector space $\mathbb{R}^2$ then takes the following form

$$X = \begin{bmatrix} x^1 \\ 0 \end{bmatrix} + \begin{bmatrix} x^2\sqrt{-1-bc} \\ x^2 c \end{bmatrix}. \tag{3.8}$$

Now we define the complex coordinate on $\mathbb{R}^2$ in the usual way, namely $\Re z = X^1 = x^1 + x^2\sqrt{-1-bc}$ and $\Im z = X^2 = x^2 c$. In more sensible notations this can be written as

$$z = x^1 + \tau x^2, \tag{3.9}$$ tau

where $\tau = \sqrt{-1-bc} + ic \in \mathbb{C}$ is what is usually called **complex structure modulus**.

The fact that our definition of the complex coordinate z on the torus is different from what one would expect, namely $z = x^1 + ix^2$, is a consequence of the general form of the complex structure J. For $J = J_0$, i.e. if we set $b = 1$ and $c = -1$ one would have the canonical definition.

Now we say that complex structure on a 2-dimensional vector space (let's say a 2-torus) can be parametrised by a complex structure modulus $\tau \in \mathbb{C}$. Hence, the complex structure moduli space of the 2-torus is two-dimensional in agreement with the general picture.

3.2 (Almost) complex structure and Hermitian manifolds

Kähler manifolds are defined as a special case of Hermitian manifolds which in turn are a complex analogue of Riemann manifolds. To avoid the common confusion one should distinguish between the Hermitian metric g (or better, inner product, compatible with J) and the Hermitian structure on the tangent space H (or alternatively the Hermitian inner product). One defines the inner product g compatible with J on a complex manifold M in the usual way

$$\begin{aligned} &g : TM \otimes TM \to \mathbb{R}, \\ &g(JX, JY) = g(X, Y). \end{aligned} \tag{3.10}$$ def_hmetr

In turn the Hermitian structure H on M is defined as a Hermitian inner product on each fibre T_pM of the tangent bundle TM

$$
\begin{aligned}
&H : T_pM \times T_pM \to \mathbb{C},\\
&H(a_1X_1 + a_2X_2, Y) = a_1H(X_1, Y) + a_2H(X_2, Y),\\
&H(Y, X) = \overline{H(X, Y)},\\
&H(JX, Y) = iH(X, Y),\\
&\text{for all } X, X_1, X_2, Y \in T_pM.
\end{aligned}
\tag{3.11}
$$

def_hstr

Such defined Hermitian structure leads to what is also often called a Hermitian metric h leading to the confusion

$$h(X, Y)|_p = H(X_p, Y_p). \tag{3.12}$$

In addition, there exists a relation between these objects $g(X, Y) = \Re h(X, Y)$. In what follows we will refer to g as the Hermitian metric, while h will be called the Hermitian inner product on TM (in constract the form H is the Hermitian inner product on T_pM).

To see, how the products h and H and the Hermitian metric g can be expressed in the coordinate form, let us define the canonical basis on TM in the spirit of the previous section

$$\text{basTM} = \{e_a, \tilde{e}_a\} = \{e_a, Je_a\}, \quad a = 1, .., \dim_{\mathbb{C}}M. \tag{3.13}$$

Hence, any vector field in this basis can be written as $X = x^a e_a + y^a \tilde{e}_a$, or in complex coordinates we may write $z^a = x^a + iy^a$. The tangent space TM is then split into its holomorphic and anti-holomorphic part spanned by vectors

$$
\begin{aligned}
T^hM &= \left\{\frac{\partial}{\partial z^a}\right\} = \left\{\frac{\partial}{\partial x^a} - i\frac{\partial}{\partial y^a}\right\},\\
T^{ah}M &= \left\{\frac{\partial}{\partial \bar{z}^a}\right\} = \left\{\frac{\partial}{\partial x^a} + i\frac{\partial}{\partial y^a}\right\}.
\end{aligned}
\tag{3.14}
$$

The spaces T^hM and $T^{ah}M$ are eigenspaces of the operator J with eigenvalues $\pm i$. Equivalently for cotangent space T^*M we have $\{dz^a\}$ and $\{d\bar{z}^a\}$ as bases of its holomorphic and anti-holomorphic parts.

Indeed, for the canonical basis defined above one writes the action of J as follows

$$
\begin{aligned}
&J(e_a) = \tilde{e_a}, \quad J(\tilde{e}_a) = -e_a,\\
&\text{equivalently in the coordinate notation}\\
&J\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^a}, \quad J\left(\frac{\partial}{\partial y^a}\right) = -\frac{\partial}{\partial x^a}.
\end{aligned}
\tag{3.15}
$$

action_J

Now it is straightforward to see that for the holomorphic coordinates z^a and $\bar{z}^a$ one has

$$J(\partial_a) = i\partial_a, \quad J(\bar{\partial}_a) = -i\bar{\partial}_a. \tag{3.16}$$

The vector $X = x^a e_a + y^a \tilde{e}_a$ can be written as $X = z^a \partial_a + \bar{z}^a \bar{\partial}_a$, that using the equations above gives

$$J(X) = i(z^a \partial_a - \bar{z}^a \bar{\partial}_a) = (iz^a)\partial_a + \overline{iz^a}\bar{\partial}_a. \tag{3.17}$$

Hence, the action of the complex structure J on the holomorphic components of the vector X is just a multiplication by i.

Let us now look first at what we call the Hermitian metric g on the complex manifold M. Using the definition one obtains for components in the holomorphic basis

$$g_{ab} = g(\partial_a, \partial_b) = g(J(\partial_a), J(\partial_b)) = -g_{ab}, \quad \Longrightarrow \quad g_{ab} = 0. \tag{3.18}$$

In the same manner one has $g_{\bar{a}\bar{b}} = 0$. In addition, since g is a usual metric on a complex manifold, it is said to be symmetric and real, that gives

$$\begin{aligned} \overline{g(X,Y)} = g(X,Y) \quad &\Longrightarrow \quad \overline{g_{a\bar{b}}} = g_{\bar{a}b}, \\ g(X,Y) = g(Y,X) \quad &\Leftrightarrow \quad g_{a\bar{b}} = g_{\bar{b}a}. \end{aligned} \tag{3.19}$$

Gathering all this together we write the metric g in the following form

$$\begin{aligned} g &= g_{a\bar{b}} dz^a \otimes dz^{\bar{b}} + g_{\bar{a}b} dz^{\bar{a}} \otimes dz^b \\ &= g_{a\bar{b}}(dz^a \otimes dz^{\bar{b}} + dz^{\bar{b}} \otimes dz^a) \end{aligned} \tag{3.20}$$

Now let us turn to the Hermitian structure h. In the canonical basis $\{e_1, \tilde{e}_a\}$ the only independent component is $h(e_a, e_b)$. Indeed, using the properties in the definition of Hermitian inner product one obtains

$$\begin{aligned} &h(\tilde{e}_a, \tilde{e}_b) = ih(e_a, \tilde{e}_b) = -ih(Je_a, e_b) = h(e_a, e_b), \\ &h(\tilde{e}_a, e_b) = ih(e_a, e_b), \\ &h(e_a, \tilde{e}_b) = -h(Je_a, e_b) = -ih(e_a, e_b). \end{aligned} \tag{3.21}$$

These relations imply that the only non-zero component of the form h in the holomorphic basis is $h_{a\bar{b}} = h(\partial_a, \partial_b)$. To illustrate the idea, let us check only one of the vanishing components

$$\begin{aligned} h(\bar{\partial}, \bar{\partial}) &= h(e,e) + h(e, i\tilde{e}) + h(i\tilde{e}, e) + h(i\tilde{e}, i\tilde{e}) \\ &= h(e,e) - ih(e,\tilde{e}) + ih(\tilde{e}, e) + h(\tilde{e},\tilde{e}) \\ &= 2h(e,e) - 2h(e,e) = 0, \end{aligned} \tag{3.22}$$

where the indices of the basis vectors e_a and $\tilde{e}_a$ were omitted. The other identities go in the same way.

Note, that the component $h(\partial_a, \partial_b)$ was denoted by $h_{a\bar{b}}$ rather than h_{ab}, that is due to sesquilinearity condition. Indeed, taking into account the above identities we have for the form action on two vectors X and Y

$$\begin{aligned} h(X,Y) &= h(X^a \partial_a + \bar{X}^a \bar{\partial}_a, Y^b \partial_b + \bar{Y}^{\bar{b}} \bar{\partial}_{\bar{b}}) = X^a \bar{Y}^{\bar{b}} h(\partial_a, \partial_b) \\ &= h_{a\bar{b}} X^a \bar{Y}^{\bar{b}}. \end{aligned} \tag{3.23}$$

It is important to mention, that barred and unbarred indices here run the same set of values, however we use barred indices to denote components of complex conjugate components. Since here one obtains $\overline{Y^b}$ still having the ∂_b inside, there appears a summation of a barred and unbarred index. To avoid all these subtleties it is suggestive to write the inner product as

$$h = h_{a\bar{b}} dz^a \otimes d\bar{z}^{\bar{b}}. \tag{3.24}$$

Note, that because of the third line in (3.11) the matrix $h_{a\bar{b}}$ is Hermitian, i.e. $h_{a\bar{b}} = \overline{h_{\bar{b}a}}$. Often such defined tensor field h is called the metric on a Hermitian manifold, that sometimes proves to be convenient. In what follows, we always distinguish between h and g, although the former will be referred to as a metric.

Starting from the inner product h one is able to define the Hermitian metric g and a 2-form ω, that plays a crucial role in complex geometry

$$\begin{aligned} g &= \frac{1}{4}(h + \bar{h}), \\ \omega &= \frac{i}{4}(h - \bar{h}). \end{aligned} \tag{3.25}$$

Coefficients here are chosen to make contact with those widely used in the literature. In the natural basis the above expressions can be written as follows

$$\begin{aligned} g &= \frac{1}{2} h_{ab} dz^a \odot d\bar{z}^b \\ &= \frac{1}{2} \Re h_{ab} (dx^a \odot dx^b + dy^a \odot dy^b) - \Im h_{ab} dxa \odot dy^b, \\ \omega &= \frac{i}{2} h_{ab} dz^a \wedge d\bar{z}^b \\ &= \Re h_{ab} dx^a \wedge dy^b + \frac{1}{2} \Im h_{ab} (dx^a \wedge dx^b + dy^a \wedge dy^b), \end{aligned} \tag{3.26}$$

where $\alpha \odot \beta = \alpha \otimes \beta + \beta \otimes \alpha$. Simple calculation shows, that such defined $2n \times 2n$ real matrix g and $n \times n$ hermitian matrix h satisfy the following relation

$$\sqrt{\det g} = \det h. \tag{3.27}$$

This will be of crucial importance for definition of Hodge dual and a volume element on complex space. Finally, it is straightforward to check, that the metric g and the 2-form ω satisfy the following equations

$$\begin{aligned} g(JX, JY) &= g(X, Y), \\ \omega(X, Y) &= g(JX, Y), \end{aligned} \tag{3.28}$$

that is precisely the usual definition of Hermitian metric and the associated 2-form.

3.3 Differential forms on complex manifolds

One should note, that components of a $(1,1)$-form

$$\omega = \frac{1}{2}\omega_{m\bar{m}} dz^m \wedge d\bar{z}^{\bar{m}}, \tag{3.29}$$

defined on the complex manifold are not given by an antisymmetric tensor $\omega_{m\bar{m}} \neq -\omega_{\bar{m}m}$ in contrast to differential forms on real manifolds.

3.4 Kähler manifolds

A Kähler manifold is defined as such Hermitian manifold with Hermitian metric h whose imaginary part defines a 2-form ω that is closed

$$\text{Kähler:} \quad d\omega = 0. \tag{3.30}$$

The corresponding metric h is called then Kähler. In the coordinate form the condition above gives precisely the Cauchi-Riemann equations

$$\begin{aligned} \frac{\partial h_{ab}}{\partial z^c} &= \frac{\partial h_{cb}}{\partial z^a}, \\ \frac{\partial h_{ab}}{\partial \bar{z}^c} &= \frac{\partial h_{ac}}{\partial \bar{z}^b}. \end{aligned} \tag{3.31}$$

This implies that a Kähler metric can be written as derivative of a complex function K called Kähler potential

$$h_{ab} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}. \tag{3.32}$$

The scalar function K is defined up to a shift by holmorphic and antiholmorphic functions called Kähler transformation

$$K'(z,\bar{z}) = K(z,\bar{z}) + f_1(z) + f_2(\bar{z}). \tag{3.33}$$

The potentials K' and K define the same Kähler metrics.

Connection on a Kä hler manifold is defined as a complex linear extension of the usual connection. I.e. we have

$$\nabla_{X+iY}(U+iV) = \nabla_X U + i\nabla_Y U + i\nabla_X V - \nabla_Y V, \tag{3.34}$$

for vectors $X, Y, U, V \in T_pM$ on the complex manifold M. This is further restricted by the condition to be compatible with the complex structure J and to be a Levi-Civita connection

$$\begin{aligned} &\nabla_X JY = J\nabla_X Y = \nabla_{JX} Y, && \text{compatibility with } J; \\ &\nabla_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) && \text{Levi-Civita condition.} \end{aligned} \tag{3.35}$$

The latter basically means that the connection preserves length of a vector. In addition one has

$$\overline{\nabla_X Y} = \nabla_{\bar{X}} \bar{Y}, \tag{3.36}$$

that tells us how to do complex conjugation.

Christoffel symbols with respect to a chosen basis $\{e_a, e_{\bar{a}}\}$ for this connection are defined in the usual way

$$\nabla_{e_a} e_b =: \Gamma_{ab}{}^{c} e_c + \Gamma_{ab}{}^{\bar{c}} e_{\bar{c}}, \tag{3.37}$$

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and the same for three other combinations $\nabla_{e_a} e_{\bar{b}}$, $\nabla_{e_{\bar{a}}} e_b$ and $\nabla_{e_{\bar{a}}} e_{\bar{b}}$. We will work in the holomorphic basis $\{\partial_a, \partial_{\bar{b}}\}$, whose properties under the action of J immediately imply

$$\begin{aligned} \nabla_{\partial_a} \partial_{\bar{b}} &= 0, \\ \nabla_{\partial_{\bar{a}}} \partial_b &= 0. \end{aligned} \tag{3.38}$$

A certain component of the Christoffel symbol can be singled out from the definition (3.37) by taken the inner product with a basis vector. Hence, we write say for $\Gamma_{ab}{}^{\bar{c}}$

$$\begin{array}{c} g(\nabla_{\partial_a} \partial_b, \partial_c) = \Gamma_{ab}{}^{\bar{c}} g_{c\bar{c}} \\ || \\ g(J\nabla_{\partial_a} \partial_b, J\partial_c) = g(\nabla_{J\partial_a} \partial_b, J\partial_c) = -g(\nabla_{\partial_a} \partial_b, \partial_c) = -\Gamma_{ab}{}^{\bar{c}} g_{c\bar{c}}, \end{array} \tag{3.39}$$

where in the vertical line we used compatibility of the metric and the complex structure (3.10). Acting in the similar way, one concludes that the only non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma_{ab}{}^{c} &= g^{c\bar{c}} \partial_a g_{a\bar{c}}, \\ \Gamma_{\bar{a}\bar{b}}{}^{\bar{c}} &= g^{c\bar{c}} \partial_{\bar{a}} g_{c\bar{b}}. \end{aligned} \tag{3.40}$$

Given the relation between the metric and the Kähler potential, the connection is indeed torsionless.

An important consequence of the above construction is that a holomorphic vector $X = X^a \partial_a$ remains holomorphic after parallel transport by the Levi-Civita connection. Indeed, for an infinitesimal translation one has

$$\nabla_a X = \Gamma_{ab}{}^{c} X^b \partial_c \quad \Longrightarrow \quad \delta_\xi X^a = \xi^b \Gamma_{bc}{}^{a} X^c \tag{3.41}$$

Since the connection preserves length of a vector, parallel transport along a closed loop gives just a $U(n)$ rotation, where $\dim_{\mathbb{C}} M = n$. To see this, let us denote a finite transformation of a holomorphic vector X^a after performing the transport by $X'^a = U^a_b X^b$. Then for its length we have

$$g(X', X') = g_{a\bar{b}} X'^a X'^{\bar{b}} = g_{a\bar{b}} U^a_c U^{\bar{b}}_{\bar{d}} X^c \bar{X}^{\bar{d}} = g_{a\bar{b}} X^a X^{\bar{b}}. \tag{3.42}$$

Since locally one is always able to diagonalise the metric this implies $U^\dagger U = 1$. Hence, the group of parallel transport along a closed loop starting and ending at a given point (that is the holonomy group $\mathrm{Hol}_p(\nabla)$ of the connection at this point) is indeed $U(N)$. Given the theorem, that the holonomy group does not depend on a choice of the point, this result is extended to the full manifold.

3.5 Hodge star operation

Due to existence of a 2-form directly related to Hermitian metric and Hermitian structure on the complex manifold, properties of the Hodge star operation lead to come peculiar identities. Start with the fact, that the volume form $d\mathit{Vol} = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n$ is just an n-th power of the 2-form

$$\mathit{Vol} = \frac{1}{n!} \int \underbrace{\omega \wedge \cdots \wedge \omega}_{n} \tag{3.43}$$

Indeed, using the fact that $dz^i \wedge d\bar{z}^i = 2i dy^i \wedge dx^i$ for any given i (no sum), one writes

$$\begin{aligned} \underbrace{\omega \wedge \cdots \wedge \omega}_{n} &= \frac{i^n}{2^n} h_{m_1 \bar{m}_1} \cdots h_{m_n \bar{m}_n} dz^{m_1} \wedge d\bar{z}^{\bar{m}_1} \wedge \cdots \wedge dz^{m_n} \wedge d\bar{z}^{\bar{m}_n} \\ &= \frac{i^n}{2^n} h_{m_1 \bar{m}_1} \cdots h_{m_n \bar{m}_n} \epsilon^{m_1 \ldots m_n} \epsilon^{\bar{m}_1 \ldots \bar{m}_n} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ &= (-1)^n n! (\det h) dy^1 \wedge dx^1 \wedge \cdots \wedge dy^n \wedge dx^n = n! d\mathit{Vol}, \end{aligned} \tag{3.44}$$

where $\epsilon^{m_1 \ldots m_n}$ is an alternating symbol (without determinant of the metric) and we used the simple relation, following from the properties of the wedge-product

$$dz^{m_1} \wedge \cdots \wedge dz^{m_n} = \epsilon^{m_1 \ldots m_n} dz^1 \wedge \cdots \wedge dz^n. \tag{3.45}$$

The above derivation works in any dimensions, however from now on in this section we would like to stick to a 3-fold, i.e. a manifold of complex dimension three. In this case we define the Hodge star by

$$* \, \omega = \frac{1}{2} \omega \wedge \omega. \tag{3.46}$$

hodge0

Note that the conventional definition of the Hodge star (up to a sign) is directly related to the above. Hence, for any (p, q)-forms ρ and σ we define

$$\rho \wedge * \sigma = -(\rho, \sigma) d\mathit{Vol}, \tag{3.47}$$

where (ρ, σ) is just contaction of all indices and the minus sign was used for consistency with the previous definition, that is commonly accepted in the literature.

Let us now prove the formula of a Hodge star of a closed $(1,1)$-form σ on a Kähler 3-fold first noticed by Strominger in [5] Strominger:1985ks

$$* \, \sigma = -\omega \wedge \sigma + \frac{3}{2} \frac{\kappa(\sigma, \omega, \omega)}{\kappa(\omega, \omega, \omega)} \omega \wedge \omega, \tag{3.48}$$

hodge

where $\kappa(\sigma_1, \sigma_2, \sigma_2) = \int \sigma_1 \wedge \sigma_2 \wedge \sigma_3$. This identity is of crucial use in the construction of Kähler moduli space (see Section 4 moduli) and to prove that we write for an arbitrary not necessarily closed (1,1)-form

$$
\begin{aligned}
*\sigma &= \frac{1}{16i}\varepsilon_{mnk}\varepsilon_{\bar m\bar n\bar k}\sigma^{k\bar k}dz^m\wedge dz^n\wedge d\bar z^{\bar m}\wedge d\bar z^{\bar n}\\
&= \frac{3!}{16i}h_{m[\bar m}h_{|n|\bar n}h_{|k|\bar k]}\sigma^{k\bar k}dz^m\wedge dz^n\wedge d\bar z^{\bar m}\wedge d\bar z^{\bar n}\\
&= \frac{1}{8i}\left(h_{m\bar m}h_{n\bar n}h_{k\bar k}\sigma^{k\bar k}-2h_{m\bar m}\sigma_{n\bar n}\right)dz^m\wedge dz^n\wedge d\bar z^{\bar m}\wedge d\bar z^{\bar n}\\
&= -\omega\wedge\sigma-\frac{1}{2}(\omega,\sigma)\omega\wedge\omega,
\end{aligned}
\tag{3.49}
$$

note the extra overall minus sign in the last line, that comes from proper arrangement of the differentials dz and $d\bar z$. The first line above is defined to be consistent with the non-coordinate expression (3.46) and

$$\sigma^{m\bar m}=h^{m\bar n}h^{n\bar m}\sigma_{n\bar n} \tag{3.50}$$

with $h^{m\bar m}$ being inverse of $h_{m\bar m}$. In the second line the following consequence of the definition of determinant was used

$$\varepsilon_{mnk}\varepsilon_{\bar m\bar n\bar k}=3!h_{m[\bar m}h_{|n|\bar n}h_{|k|\bar k]}, \tag{3.51}$$

where the antisymmetrization goes over $\{\bar m,\bar n,\bar k\}$ and $\varepsilon_{mnk}=\sqrt{h}\,\epsilon_{mnk}$ with ϵ_{mnk} being an alternating symbol.

Note, that the formula above is consistent with our definition of the Hodge star. Indeed, taking into account $(\omega,\omega)=-3$ we recover precisely the definition (3.46). The result obtained

$$*\,\sigma=-\omega\wedge\sigma-\frac{1}{2}(\omega,\sigma)\omega\wedge\omega \tag{3.52}$$

hodge1

is valid on any Hermitian 3-fold, while to make contact to the identity (3.48) one recalls the property of the Kähler form $d\omega=0$. Using this and the fact that σ is closed $d\sigma=0$ one concludes that $(\omega,\sigma)=$const and hence writes

$$
\begin{aligned}
(\omega,\sigma)&=\frac{1}{Vol}\int(\omega,\sigma)dVol=-\frac{1}{Vol}\int\sigma\wedge*\omega=-\frac{1}{2Vol}\int\sigma\wedge\omega\wedge\omega\\
&=-3\frac{\int\sigma\wedge\omega\wedge\omega}{\int\omega\wedge\omega\wedge\omega}=-3\frac{\kappa(\sigma\wedge\omega\wedge\omega)}{\kappa(\omega\wedge\omega\wedge\omega)}.
\end{aligned}
\tag{3.53}
$$

Substituting this back to (3.52) we recover precisely (3.48).

3.6 Examples of Kähler manifolds. Fubini-Study metric.

So far we have introduced Kähler manifolds as a class of complex manifolds with Hermitian metric h consistent with complex structure J. The most evident example of Kähler manifold is certainly the complex plane $\mathbb{C}^m$ where the metric and the complex structure are defined as

$$
\begin{aligned}
h &= \sum_{j=1}^{m} dz^j d\bar{z}^j, \\
\omega &= i \sum_{j=1}^{m} dz^j \wedge d\bar{z}^j.
\end{aligned}
\tag{3.54}
$$

Obviously the complex structure J satisfied the Kähler condition $dJ = 0$.

Another big class of manifolds with Kähler structure is the so=called Riemann surfaces that are 1-dimensional complex manifolds. The condition $d\omega = 0$ satisfies trivially since the dimension is 1.

Less trivial and a widely used class of Kähler manifolds is represented by complex projective spaces $\mathbb{CP}^n$ which are defined as a coset space $\mathbb{C}^{n+1}/_\sim$ with the equivalence relation $Z \sim \lambda Z$ for any $\lambda \in \mathbb{C}$. In other words a point of an n-dimensional projective space $\mathbb{CP}^n$ represent a straight line in $\mathbb{C}^{n+1}$.

Consider a complex n+1-dimensional space $\mathbb{C}^{n+1}$ with coordinates $\{\zeta^1, \ldots, \zeta^{n+1}\}$. On a patch where $\zeta^j \neq 0$ one can choose n coordinates $\{z^A = \zeta^A/\zeta^j\}$, where $A = 1, \ldots, j-1, j+1, \ldots, j$. These can be proven to be good local coordinates on the patch $V_j = U_j/_\sim$ of the projective space $\mathbb{CP}^n$. The set of $n+1$ patches V_j cover the whole projective space.

Such defined local coordinates can be used to construct a Kähler metric on the projective space. Let's fix $j = n+1$ for simplicity and denote the local coordinates as $\{z^a\}$ with $a = 1, \ldots, n$. Then the so-called **Fubini-Study** metric and complex structure are given by

$$
\begin{aligned}
h_{ab} &= \frac{1}{\left(1+|z|^2\right)^2}\Big(\delta_{ab}\left(1+|z|^2\right) - z_a \bar{z}_b\Big), \\
\omega &= -\frac{i}{\left(1+|z|^2\right)^2} z_a \bar{z}_b dz^a \wedge d\bar{z}^b,
\end{aligned}
\tag{3.55}
$$

where $|z|^2 = \sum_a z^a \bar{z}^a$. It is an easy exercise to show that the Käher form is indeed closed $d\omega = 0$.

3.7 Calabi-Yau manifolds

A Calabi-Yau n-fold is a compact Kähler manifold of **complex** dimension n that admits a nowhere vanishing $(n, 0)$ holomorphic form Ω. By definition a $(n, 0)$ form can be represented as

$$
\Omega = \Omega_{a_1 \ldots a_n} dz^{a_1} \wedge \ldots \wedge dz^{a_n}, \tag{3.56}
$$

i.e. it does not contain $d\bar{z}^a$. Holomorphicity of the form Ω means $\bar{\partial}\Omega = 0$. A holomorphic $(n, 0)$-form on an n-dimensional complex manifold is closed

$$d\Omega = (\partial + \bar{\partial})\Omega = 0. \tag{3.57}$$

Hence, a Calabi-Yau n-fold usually denoted as CY_n carries a closed holomorphic $(n, 0)$-form Ω that is not exact. In the next section we will show that the corresponding cohomology classes can be used to classify Calabi-Yau manifolds (CY moduli space).

Lets consider Riemann surfaces as a one-dimensional example of a Calabi-Yau manifold CY_1. A 2-torus, that is the only example of a Calabi-Yau 1-fold, can be understood in the spirit of Section 8.1 as an algebraic curve in 2-dimensional complex projective space $\mathbb{CP}^2$ (in this section we mainly follow [6])

$$y^2 = 4x^3 - ax - b \tag{3.58}$$

with local coordinates $y = z_2/z_0$ and $x = z_1/z_0$. These are proper coordinates in the patch $V_0 = U_0/_\sim$ of $\mathbb{CP}^2$, where U_0 is a patch of $\mathbb{C}^3$ with $z_0 \neq 0$. The 1-form Ω can be defined as follows

$$\Omega = \frac{dx}{2y(x)}, \tag{3.59}$$

where the factor 2 is just a matter of convenience. As was mentioned before any holomorphic 1-form on a Riemann surface is closed $d\Omega = 0$. Using the topological residue construction described in Section 8.2 one may write

$$\Omega = \frac{dx}{\partial f/\partial y} = -\frac{dy}{\partial f/\partial x}. \tag{3.60}$$

Since there is only one $(n, 0)$-form on Calabi-Yau n-fold it should be proportional to the Leray residue construction as it is an intrinsic form for the surface defined (locally) by a function f. Lets see that global properties of the form Ω restrict the function f to a particular form. Lets start with $\mathbb{CP}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$ and define the CY 1-fold by the following equation

$$F(z_0, z_1, z_2) = z_0^n + z_1^n + z_2^n = 0. \tag{3.61}$$

In the patch V_0 where $z \neq 0$ we choose local coordinates $x = z_1/z_0$ and $y = z_2/z_0$. The local function f then is defined as $F = z_0^n f(x, y)$ and becomes

$$f(x, y) = 1 + x^n + y^n = 0. \tag{3.62}$$

In this patch the residue can be written in the usual way

$$\Omega_0 = \frac{dx}{\partial f/\partial y} = \frac{1}{n}\frac{dx}{y^{n-1}}. \tag{3.63}$$

Lets now go to another local patch V_1 defined as $z_1 \neq 0$ with local coordinates $\tilde{x} = z_0/z_1$ and $\tilde{y} = z_2/z_1$. Performing coordinate transformation $x = \tilde{x}^{-1}$, $y = \tilde{y}/\tilde{x}$ we obtain on the intersection $V_0 \bigcap V_1$

$$\Omega_0|_{V_0 \bigcap V_1} = -\frac{1}{n}\frac{\tilde{x}^{n-3}dx}{\tilde{y}^{n-1}}. \tag{3.64}$$

This can be extended to the $(n, 0)$-form Ω_1 on the patch V_1 defined as

$$\Omega_1 = -\frac{1}{n}\frac{\tilde{x}^{n-3}dx}{\tilde{y}^{n-1}}. \tag{3.65}$$

The function f that locally defines the 1-fold in the patch has basically the same form $\tilde{f}(\tilde{x}, \tilde{y}) = 1 + \tilde{x}^n + \tilde{y}^n$ and comes from $F = z_1^n \tilde{f}(\tilde{x}, \tilde{y}) = 0$.

Hence in order for the form Ω_1 to be the Leray residue one has to fix $n = 3$ (the minus sign does not count since $g = -1$ is a holomorphic function on a compact manifold). Since the residue form is intrinsic and unique this implies that a Calabi-Yau 1-fold as a hypersurface in $\mathbb{CP}^2$ is always defined by a third-order polynomial.

It is straightforward now to repeat the same for a 2-fold hypersurface CY_2 in 3-dimensional projective space $\mathbb{CP}^3$. Using the same arguments as above one figures out that these should be defined by a 4-order polynomial

$$F(z_0, z_1, z_2, z_3) = \sum_a a_{ijkl} z_i z_j z_k z_l, \tag{3.66}$$

where a_{ijkl} are some constants. The $(2, 0)$-form in the local patch V_0 where $z_0 \neq 0$ then takes the following form (see constr (8.6))

$$\Omega_0 = \frac{dx_1 \wedge dx_2}{\partial f/\partial x_3} = -\frac{dx_1 \wedge dx_3}{\partial f/\partial x_2} = \frac{dx_2 \wedge dx_3}{\partial f/\partial x_1}, \tag{3.67}$$

where $x_\mu = z_\mu/z_0$. In mathematical literature such defined manifolds are called K3-surfaces. The completely symmetric 4-tensor a_{ijkl} has 35 independent parameters, 16 of which can be removed by action of the group $GL(4, \mathbb{C})$ on z_i. Hence, one is left only with $35 - 16 = 19$ parameters called moduli. These completely define the manifold.

Evidently, the same constructions can be repeated for the most interesting case of Calabi-Yau 3-folds, that are defined by 5-order polynomials. Here we have 126 parameters, 25 of which can be eliminated by $GL(5, \mathbb{C})$ leaving us with 101-dimensional moduli space.

Holonomy groups of Calabi-Yau and Kähler manifolds Bouchard:2007ik [7]

4 Geometry of Calabi-Yau moduli space

moduli

4.1 ??? Killing vectors, Killing prepotential

4.2 Infinitesimal deformations of complex and Kähler structure

In the study of supergravity compactifications of high importance is the notion of moduli fields, which from the point of view of the lower dimensional theory show up as scalar fields. Geometrically these fields appears as a set of parameters that completely define the internal manifold and all structures living on it. Indeed, restricting the internal manifold of our compactification model to be say a CY of certain topology, i.e. of certain Hodge numbers $h^{p,q}$, we are still allowed in principle to deform the manifold in such a way that preserves the given topology. In other words, there may be different CY manifolds for a given choice of Hodge numbers. Since these deformations are purely internal and may vary with position in the external space, they show up as scalar fields.

The most convenient way to describe moduli fields of Calabi-Yau manifolds is to consider deformations of the Hermitean metric δh, that includes deformation of the Riemannian metric (Kähler structure) itself and the complex structure. In general the deformations of the metric can be of the following three types

$$\delta h_{a\bar{b}}, \quad \delta h_{ab}, \quad \delta h_{\bar{a}\bar{b}}, \tag{4.1}$$

which are restricted by the Ricci-flatness and Kähler conditions. The latter simply states, that understood as components of a 2-form the deformation $\delta h_{a\bar{b}}$ belong to $H^{1,1}$

$$\delta\omega = \delta h_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}. \tag{4.2}$$

Indeed, to keep ω a Kähler form we want to consider closed deformation $\delta\omega = 0$, while all exact deformations $\delta\omega = d\delta\omega'$ represent just coordinate transformations and should be dropped.

Things are less straightforward for the remained components δh_{ab} and $\delta h_{\bar{a}\bar{b}}$ as one should consider Riemann tensor and expand it to first order in the deformation to ensure, that these indeed closed. Instead, using the fact that two of the triple (J, g, ω) unambiguously define the rest, we consider deformation of the complex structure $J^a{}_b$, that is an element of $\Lambda TM \otimes TM$, i.e. is a form on TM rather than M.

As a warm-up lets start with a 2-torus and show that its complex structure moduli space is parametrised by one complex number and hence is just $\mathbb{C}$. A 2-torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ can be defined as a complex plane factorized by the following action of the group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

$$z \sim z + m\lambda_1 + n\lambda_2, \quad m, n \in \mathbb{Z}, \tag{4.3}$$

where the complex numbers $\lambda_{1,2}$ define the 2-dimensional lattice. Collecting these number into one vector $\Lambda = [\lambda_1 \ \lambda_2]$ one immediately notices that two vectors Λ' and Λ define the same lattice if they a related by a matrix $U \in GL(2, \mathbb{Z})$

$$\Lambda' = \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = U\Lambda. \tag{4.4}$$

Indeed, going back to the equivalence relation this equation implies

$$z \sim z + m(a\lambda_1 + b\lambda_2) + n(c\lambda_2 + d\lambda_2) = z + (ma + nc)\lambda_1 + (mb + nd)\lambda_2. \tag{4.5}$$

Hence, if the entries of the matrix U are integer the above relation indeed defines the same torus. In addition we want to preserve orientation of the torus and its area that leads to further restriction giving $U \in SL(2, \mathbb{Z})$.

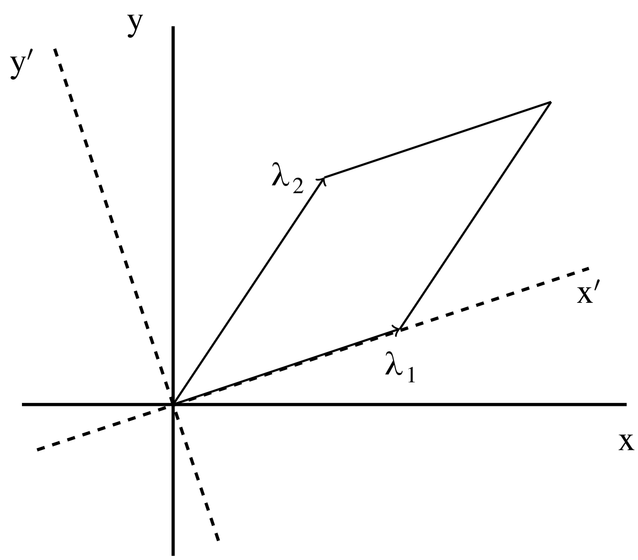

Figure 5: Elementary lattice on complex plane C defined by two complex numbers λ_1 and λ_2. We assume that $\mathrm{Arg}(\lambda_1) < \mathrm{Arg}(\lambda_2)$. The dashed lines represent rotated transformed coordinates $z = \lambda_1 z'$. parall

The lattice defined by the two complex numbers (λ_1, λ_2) looks like a parallelogram rotated with respect to the real axis (see Figure 5). It is more convenient to perform a coordinate transformation $z \to z'$ such that the point $(m, n) = (1, 0)$ belongs to the real axis

$$z \sim \lambda_1(z' + m + n\tau). \tag{4.6}$$

Under the $SL(2, \mathbb{Z})$ transformation the parameter τ transforms in the familiar way

$$\tau \to \frac{c + \tau d}{a + \tau b} \tag{4.7}$$

and is precisely the complex structure of (3.9). Indeed, turning from the complex coordinate $z' = x' + iy'$ to $z' = \hat{x} + \tau\hat{y}$ we end up with a square lattice for the torus. That is the complex structure modulus τ of a 2-torus shows the failure of the lattice to be square. Alternatively, this corresponds to different definitions of the complex structure J on the torus.

For the general discussion of the complex structure moduli lets consider the tensor J to be a small deformation from the canonical complex structure $J = J_0 + \varepsilon$. From the equation $J^2 = (J_0 + \varepsilon)^2 = -1$ one concludes that the deformation ε has only the holomorphic and anti-holomorphic components

$$\varepsilon = \varepsilon_h{}^a{}_b \partial_a \otimes d\bar{z}^b + \varepsilon_{ah}{}^a{}_b \bar{\partial}_a \otimes dz^b \tag{4.8}$$

Since J is a real tensor the anti-holomorphic part of ε should be a complex conjugate of ε_h. The linearised integrability condition $N(J) = 0$ implies the following equation

$$\bar{\partial}\varepsilon_h = 0. \tag{4.9}$$

Hence, we conclude that the tensor $\varepsilon_h = \varepsilon_h{}^a{}_b \partial_a \otimes d\bar{z}^b$ is closed as a (0,1)-form on the holomorphic tangent bundle TM.

To keep track fo only relevant deformation of the complex structure one has to remove trivial deformations. These are just coordinate transformations of the form $J' = M^{-1}JM$. Linearised this equation reads

$$J' = J + \bar{\partial} v_h + \partial v_{ah}. \tag{4.10}$$

This implies that trivial deformations of the form $\varepsilon_h^0 = \bar{\partial} v_h$ do not change the complex structure, instead these correspond to redefinition of coordinates on M. All together the above results imply that complex structure moduli are classified by the (0,1) cohomology classes $H^{0,1}(TM)$ of the manifold TM with respect to the differential operator $\bar{\partial}$.

Since it is more convenient to deal with cohomology groups of a manifold rather than of its tangent bundle one may use the fact that a Calabi-Yau n-fold has trivial tangent bundle to show

$$H^{(0,1)}(TM) = H^{(0,1)}(\Lambda^{n-1} T^* M) = H^{(n-1,1)}(M). \tag{4.11}$$

Hand-waving argument is, that we just lower the vector index of the tensor ε_h by the $(n, 0)$-form $\Omega_{a_1 \ldots a_n}$ ending up with a $(n-1, 1)$-form on M

$$\varepsilon_h{}^a{}_b \partial_a \otimes d\bar{z}^b \longrightarrow \varepsilon_h{}^a{}_b \Omega_{acd} d\bar{z}^b \wedge dz^c \wedge dz^d. \tag{4.12}$$

Hence, dimension of the complex structure moduli space of a Calabi-Yau n-fold is given by the Hodge number $h_{n-1,1}$. For the example of a 2-torus considered above we have $h_{1,1}$ that is 1, hence one complex structure parameter τ. In addition one would have one Kähler structure parameter usually denoted by U.

For the most interesting case of Calabi-Yau 3-fold we have $h_{1,1}$ and $h_{2,1}$ for the dimensions of Kähler structure and complex structure moduli spaces respectively. Introducing bases on the spaces of cohomology classes

$$\begin{aligned} &\{\omega^i\} = \mathrm{bas}H^{(1,1)}, \\ &\{\chi^\alpha\} = \mathrm{bas}H^{(2,1)}, \quad \{\bar{\chi}^\alpha\} = \mathrm{bas}H^{(1,2)}, \end{aligned} \tag{4.13}$$

where the forms χ are chosen to be complex for convenience as $h_{1,2} = h_{2,1}$. Deformations of the metric δh_{ab} and $\delta h_{\bar{a}\bar{b}}$ are related to the forms χ as follows

$$\begin{aligned} \delta h_{ab} &= \Omega_{abc} h^{b\bar{b}} h^{c\bar{c}} \bar{\chi}_{\alpha b\bar{b}\bar{c}} \delta \bar{z}^\alpha, \\ \delta h_{\bar{a}\bar{b}} &= \Omega_{\bar{a}\bar{b}\bar{c}} h^{b\bar{b}} h^{c\bar{c}} \chi_{\alpha \bar{b}bc} \delta z^\alpha. \end{aligned} \tag{4.14}$$

Given the obvious applications to compactifications of supergravity, it is suggestive to consider also the Kalb-Ramond field B, that is a 2-form in 10-dimensions. Equations of motion on its internal part b imply that the 2-form is harmonic and closed. Given that exact 2-forms give just trivial pure gauge configurations, the non-trivial part should be an element of the cohomology class H^2. Actually, one gets $b \in H^{(1,1)}$ as the Hodge numbers $h_{2,0} = 0 = h_{0,2}$ for a Calabi-Yau manifold. With this field included it is better to talk about moduli space of a field configuration, rather than just the Calabi-Yau manifold.

With all that in mind we consider the following deformations

$$\begin{aligned} &\delta t \equiv b + i\delta\omega = (b^i + i\delta v^i)\omega_i = \delta t^i \omega_i \quad \in H^{(1,1)}, \\ &\delta z = \delta z^\alpha \chi_\alpha \in H^{(2,1)}, \\ &\delta \bar{z} = \delta \bar{z}^\alpha \bar{\chi}_\alpha \in H^{(1,2)}. \end{aligned} \tag{4.15}$$

Let us now investigate metrics on the complex and Kähler structures moduli space separately.

4.3 Geometry of $H^{1,1}$

Metric on the moduli space of Kähler structure and perturbation of the Kalb-Ramond field is defined by the following measure

$$\begin{aligned} ||\delta t||^2 &= \frac{1}{V}\int \delta t \wedge *\overline{\delta t} = \delta t^i \overline{\delta t^j} \frac{1}{V}\int \omega_i \wedge *\omega_j = \\ &= \left(-\frac{3!\kappa(\omega_i,\omega_j,\omega)}{\kappa(\omega,\omega,\omega)} + 9\frac{\kappa(\omega_i,\omega,\omega)\kappa(\omega_j,\omega,\omega)}{\kappa(\omega,\omega,\omega)^2} \right) \delta t^i \overline{\delta t^j} \\ &= G_{i\bar{j}} \delta t^i \overline{\delta t^j}, \end{aligned} \tag{4.16}$$

where the property of the Hodge dual operation (hodge 3.48) has been used. In the last line the barred indices has been introduced just to emphasize that the coordinates t^i on the moduli space are complex.

Since the Kähler form can be represented as $\omega = v^i\omega_i$ in the chosen basis on $H^{(1,1)}$ the metric on the Kähler structure moduli space can be written as

$$G_{i\bar{\jmath}} = \frac{\partial}{\partial v^i}\frac{\partial}{\partial v^j}\log\kappa(\omega,\omega,\omega) = \frac{\partial}{\partial t^i}\frac{\partial}{\partial \bar{t}^{\bar{\jmath}}}\log\kappa(\omega,\omega,\omega), \tag{4.17}$$

that implies that it is Kähler. Here we are using the property of the CY moduli space that infinitesimal deformations can be integrated into the full ω. This is not true in general.

Note that the metric $G_{i\bar{\jmath}}$ depends only on v^i, that is expected as the Kalb-Ramond field is not a part of the deformations of the Calabi-Yau manifold.

For further application to supergravity compactifications it is convenient to introduce projective coordinates $t^I = (t^0, t^i)$ on the Kähler structure moduli space. Then introducing a function $\mathcal{F}$, the would be Kähler prepotential, we can write the quantity $\kappa(\omega,\omega,\omega,)$ as follows

$$\begin{aligned}\kappa(\omega,\omega,\omega) &= -\frac{3i}{4}\left(t^I\bar{\partial}_I\bar{\mathcal{F}} - \bar{t}^I\partial_I\mathcal{F}\right)\Big|_{t^0=1},\\ \mathcal{F} &= -\frac{1}{3!}\frac{\kappa_{ijk}t^it^jt^k}{t^0}\end{aligned} \tag{4.18}$$

with $\kappa_{ijk} \equiv \kappa(\omega_i,\omega_j,\omega_k)$ being the triple intersection numbers. To check that we write

$$\begin{aligned}\kappa(\omega,\omega,\omega) = \kappa_{ijk}v^iv^jv^k &= -\frac{i\kappa_{ijk}}{8}(t^i-\bar{t}^i)(t^j-\bar{t}^j)(t^k-\bar{t}^k)\\ &= -\frac{i\kappa_{ijk}}{8}(t^it^jt^k - 3\bar{t}^it^jt^k + 3t^i\bar{t}^j\bar{t}^k - \bar{t}^i\bar{t}^j\bar{t}^k)\\ &= -\frac{3i}{4}\left(f - \bar{f} - \bar{t}^i\partial_i f + t^i\bar{\partial}_i\bar{f}\right),\end{aligned} \tag{4.19}$$

where in the second line we've used the fact that κ_{ijk} is fully symmetric in its indices by construction and have defined the function

$$f = \frac{1}{3!}\kappa_{ijk}t^it^jt^k. \tag{4.20}$$

Now writing $f = -t^0\mathcal{F}$ we obtain the desired relation up to identification $t^0 = 1$.

With all this in hands the Kähler potential $K = -\log\kappa(\omega,\omega,\omega)$ can be written in terms of the prepotential as

$$e^{-K} = \frac{4}{3}\int\omega\wedge\omega\wedge\omega = -i\left(t^I\bar{\partial}_I\bar{\mathcal{F}} - \bar{t}^I\partial_I\mathcal{F}\right) \tag{4.21}$$

on the projective space with normal coordinates given by $t^I = (1, t^i)$. Note that Kähler symmetry $K(z,\bar{z}) \simeq K(z,\bar{z}) + g_1(z) + g_2(\bar{z})$ for any $g_{1,2}$ has been used to remove the additional prefactor.

4.4 ??? Geometry of $H^{2,1}$

5 Spin geometry

5.1 ??? Spinor bundles

A spinor bundle S over a manifold M is just a vector bundle whose sections are representations of the corresponding $\mathrm{Spin}(n)$, where $n = \dim M$. This group is a double cover of the orthogonal group $SO(n)$ that acts on tangent spaces of M.

The group $\mathrm{Spin}(n)$ is generated by the matrices $\gamma^{mn} = [\gamma^m, \gamma^n]$ where the gamma-matrices are defined as

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn}, \tag{5.1}$$

where η^{mn} is a (flat) metric on the tangent space T_pM for any $p \in M$.

Everything that was said about vector bundles is obviously true for spinor bundles. A connection on a spinor bundle is usually called spin-connection and is introduced as

$$\nabla^s \psi = (d + \omega)\psi. \tag{5.2}$$

Here $\omega = \omega_\mu dx^\alpha$ is the connection 1-form that takes values in the algebra $\mathrm{spin}(n)$ meaning $\omega_\mu = \omega_\alpha{}^{mn}\gamma_{mn}$. Here we distinguish between the curved indices μ, ν that label sections of TM in the natural basis $\{dx^\mu\}$ and the flat indices m, n that correspond to the orthonormal frame $\{e^a\}$ with

$$e^m = e^m{}_\mu dx^\alpha. \tag{5.3}$$

The components of the orthonormal frame $e^m{}_\mu$ written in the natural frame are exactly what we usually call a vielben.

5.2

5.3 Bär classification

Bar

Since the connection 1-form is an element of $\Lambda^1 M \otimes \mathrm{spin}(n)$ it implies that the holonomy group is a subgroup of $\mathrm{Spin}(n)$ in general. As in the case of vector bundle manifolds with the structure of spin bundle are classified according to the properties of constant section, i.e. parallel spinors, that are defined as

$$\nabla^s \psi = 0. \tag{5.4}$$

The classification is given by the important theorem by Bär that states the following. Consider a manifold M whose dimension is $\dim M \geq 3$ and a spinor bundle S over it. Denote the space of parallel spinors by Σ, i.e.

$$\Sigma = \{\sigma \in \Gamma(S), \nabla^s(\sigma) = 0\}, \quad N = \dim \Sigma. \tag{5.5}$$

Let's denote numbers of parallel spinors of positive and negative chiralities by N_+ and N_- respectively. Then one of the following holds

	Dimension	Holonomy group	Dimension of Σ
1.	$n = 4m,\ m \geq 1$	$SU(2m)$	$N_+ = 2,\ N_- = 0$
2.	$n = 4m,\ m \geq 2$	$Sp(m)$	$N_+ = m + 1,\ N_- = 0$
3.	$n = 4m + 2,\ m \geq 1$	$SU(2m + 1)$	$N_+ = 1,\ N_- = 1$
4.	$n = 7$	G_2	$N = 1$
6.	$n = 8$	$Spin(7)$	$N_+ = 1,\ N_- = 0$

Table 2: Bär classification of manifolds according to their holonomy group.

5.4 Explicit examples

Example 1. Consider a spinor bundle S over a manifold M of dimension 4 with a connection 1-form ω_μ and a parallel spinor ε

$$\nabla_\mu \varepsilon = (\partial_\mu + \omega_\mu)\varepsilon = 0. \tag{5.6}$$

We can choose the spinor to be constant then this equation reduces to

$$\omega_\mu{}^{ab}\gamma_{ab}\varepsilon = 0. \tag{5.7}$$

paral1

Components ω_μ of the connection 1-form are elements of $spin(1,3) = su(2) \oplus su(2)$. Under this splitting any spinor can be decomposed as

$$4 = (2,1) \oplus (1,2), \tag{5.8}$$

that is just the well-known decomposition of a Dirac spinor into two Weyl spinors

$$\psi = \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix}. \tag{5.9}$$

The connection 1-form then can be written in the block diagonal form

$$\omega_\mu = \begin{bmatrix} \omega_{\mu R} & 0 \\ 0 & \omega_{\mu L} \end{bmatrix}. \tag{5.10}$$

Hence, to solve the equation (5.7) we can choose a connection in $su(2)$, that is equivalent to setting $\omega_{\mu L} = 0$. Then the parallel spinor will be of the form

$$\varepsilon = \begin{bmatrix} 0 \\ \varepsilon_L \end{bmatrix}. \tag{5.11}$$

We know see that we have here two parallel spinors of positive chirality, i.e. $N_+ = 2$ that is invariant under the group $SU(2)$. According to the table above we constructed a 4-dimensional Calabi-Yau manifold with $SU(2)$-holonomy.

Example 2. Lets go now upper in dimension and consider the case relevant to string phenomenology, i.e. a 6-dimensional manifold M. Recall its special holonomy group $\mathrm{Hol}(g) = SO(6)$, that is just the holonomy group of the Levi-Civita connection. Sections of the spinor bundle over this manifold transform in the 4 of the double cover of $SO(6)$ that is $\mathrm{Spin}(6) = SU(4)$.

Now the question is to find such a subgroup of $SU(4)$ that preserves a spinor on M. One can consider action of the $SU(3)$ subgroup of $SU(4)$ on the 4 representation under which it is split as

$$4 = 3 \oplus 1, \tag{5.12}$$

where the 1 is the desired singlet. More deep investigation of the structure of the Clifford algebra reveals the fact that the real and imaginary components of this one-dimensional complex representation correspond to positive and negative chirality spinors of $SU(4)$.

Hence, we have here a 6-dimensional manifold with $SU(3)$ holonomy and $N_+ = 1, N_- = 1$. This is a six-dimensional Calabi-Yau manifold.

6 ??? Applications in string theory

6.1 Heterotic Calabi-Yau compactifications

6.2 Type IIA Calabi-Yau compactifications

6.3 Calabi-Yau orientifolding: N=1 models

6.4 Axions potential and cosmology

6.5 KKLT and LVS models

6.6 Towards non-geometric compactifications

7 Further reading

7.1 Geometry of fibre bundles

For a beginner reading in differential geometry one can take very nice book by Dubrovin, Novikov and Fomenko [8] and [9]. These books contain basic introduction to tensors, their transformations, Lie derivative, differential geometry and application to physics. At the same level of explanation on can find the famous book by Nakahara [1] that covers a lot of topics both on geometry and its application to physics.

For a more mathematical description of these and many other ideas one can follow the already mentioned book by Joyce [2] and the series of books by Postnikov [10] [11].

Geometry of spin bundles is beautifully covered in the book by Salamon [12]

7.2 Flux compactifications in string theory

The paper Ivanov:2009rh [13] contains a lot of links for various papers on flux compactifications and is highly recommended to look at.

8 Appendix

8.1 Weierstrass functions and a torus $\mathbb{T}^2$

wp

Double periodic functions on the complex plane $\mathbb{C}^2$ are called elliptic function. A special class of elliptic functions $\wp(z)$ which one can understand as the most simple ones where introduced by Weierstrass. These satisfy the following relations that will be the defining property of CY 1-folds

$$\wp(z)'^2 = 4\wp(z)^3 - g_2\wp(z)^2 - g_3, \tag{8.1}$$ wp_rel

where g_2 and g_3 are some constants.

Periodicity in 2-direction of Weierstrass functions can be understood as defining a 2-dimensional torus $\mathbb{T}^2$. There is a natural embedding of such defined 2-torus into a complex projective plane $\mathbb{CP}^2$ that is given by

$$w \mapsto [1 : \wp(w) : \wp'(w)]. \tag{8.2}$$ wp_embed

A point on $\mathbb{CP}^2$ is denoted as $[z_0 : z_1 : z_2]$, with the projective space defined as a complex plane $\mathbb{C}^3$ with coordinates $\{z_0, z_1, z_2\}$ factorized in the usual way.

The map above carries the natural group structure on the 2-torus $\mathbb{T}^2$ into the $\mathbb{CP}^2$. If we denote local coordinates on the projective space in the patch $z_0 \neq 0$ as $y = z_2/z_0$ and $x = z_1/z_0$, then the relation wp_rel (8.1) implies

$$y^2 = 4x^3 - g_2 x - g_3. \tag{8.3}$$

This defines an algebraic curve in $\mathbb{CP}^2$ that is isomorphic to the 2-torus by the isomorphism of Riemann surfaces wp_embed (8.2).

It is now straightforward to define an invariant differential form on the elliptic curve that is Rhoades:2003ac [14]

$$\Omega = dz = \frac{d\wp(z)}{\wp'(z)} = \frac{dx}{y(x)}. \tag{8.4}$$

This definition maps the form dz on the 2-torus onto the elliptic curve in the projective space $\mathbb{CP}^2$.

8.2 Topological residue

res

To consider more general complex hypersurfaces than elliptic curves defined by Weierstrass functions it is useful to go through the construction of Leray (or

Poincaré) topological residue. Consider a complex manifold M with a singular hypersurface defined by $f = 0$, where f is a holomorphic function with $df \neq 0$ along K.

We are interested in the $(n, 0)$ form on the n-dimensional space K. This can be constructed using a singular form $\omega \in \Omega^{n+1}(M\ K)$ that has a first order pole on K. This means that the form $f\omega$ can be smoothly extended to the whole M. Using the division property of df we can write Weber:2003ab, Nicolaescu:2003ac [15, 16]

$$\omega = \frac{df}{f} \wedge r + \theta, \tag{8.5}$$

where $e \in \Omega^n(M)$ and $\theta \in \Omega^{n+1}(M)$ are smooth forms on M. Such defined Leray residue $\mathrm{Res}\,\omega = r|_K$ of the form ω does not depend on f. It catches cohomology properties of the form ω and it is the intrinsic $(n, 0)$-form on K.

For explicit construction of the residue lets consider the manifold M to have a set of coordinates $(z^0, z^1, \ldots, z^n)$. Without loss of generality we choose a patch where $df/dz^0 \neq 0$ along the surface $f = 0$. Then we write

$$\begin{aligned} f\omega &:= g dz^0 \wedge \ldots \wedge dz^n; \\ \omega &= \frac{1}{f}\frac{1}{\frac{\partial f}{dz^0}}\frac{\partial f}{dz^0} dz^0 \wedge dz^1 \wedge \ldots \wedge dz^n \\ &= \frac{g}{\frac{\partial f}{dz^0}}\frac{df}{f} \wedge dz^1 \wedge \ldots \wedge dz^n. \end{aligned} \tag{8.6}$$ constr

Hence, the residue r of the form ω takes the following form

$$r = \frac{g}{\frac{\partial f}{dz^0}} dz^1 \wedge \ldots \wedge dz^n, \tag{8.7}$$

where g is a holomorphic function.

For particular examples of this construction see the section on Calabi-Yau manifolds since the residue form r is used as a natural $(n, 0)$-form on a CY_n n-fold defined as a hypersurface in $(n + 1)$-dimensional projective space $\mathbb{CP}^{n+1}$. Since there is only one $(n, 0)$-form on Calabi-Yau n-fold it should be proportional to the Leray residue construction described above.

8.3 de Ram cohomology

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