Introduction to Entanglement Entropy and Holography

Helmholtz International Summer School "Cosmology, Strings, and New Physics"

Lecture 2

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Plan of lectures

Lecture 1:

-Entanglement entropy (definitions and basic properties)
-EE and Renyi entropy (REE): methods of computations in free QFT's (spectral geometry and etc)
-logarithmic part of EE and conformal anomalies

Lecture 2:

- Holography
-Holographic EE (HEE)
- motivations for HEE
- HEE: how it works
- HEE and conformal anomalies
- ‘Proof’ of HEE
- Entanglement and Gravity

entanglement has to do with quantum gravity:

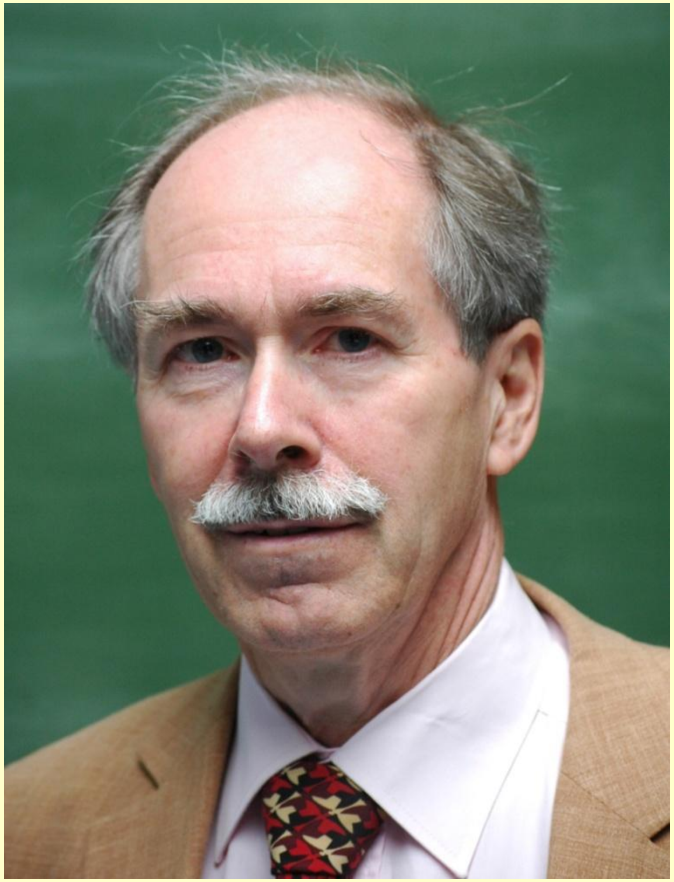
● possible source of the entropy of a black hole (states inside and outside the horizon);

● d=4 supersymmetric BH's are equivalent to 2, 3,… qubit systems

● entanglement entropy allows a *holographic interpretation* for CFT's with AdS duals

The World is a Hologram!

Black hole entropy indicates that gravity may have a holographic nature (1993)



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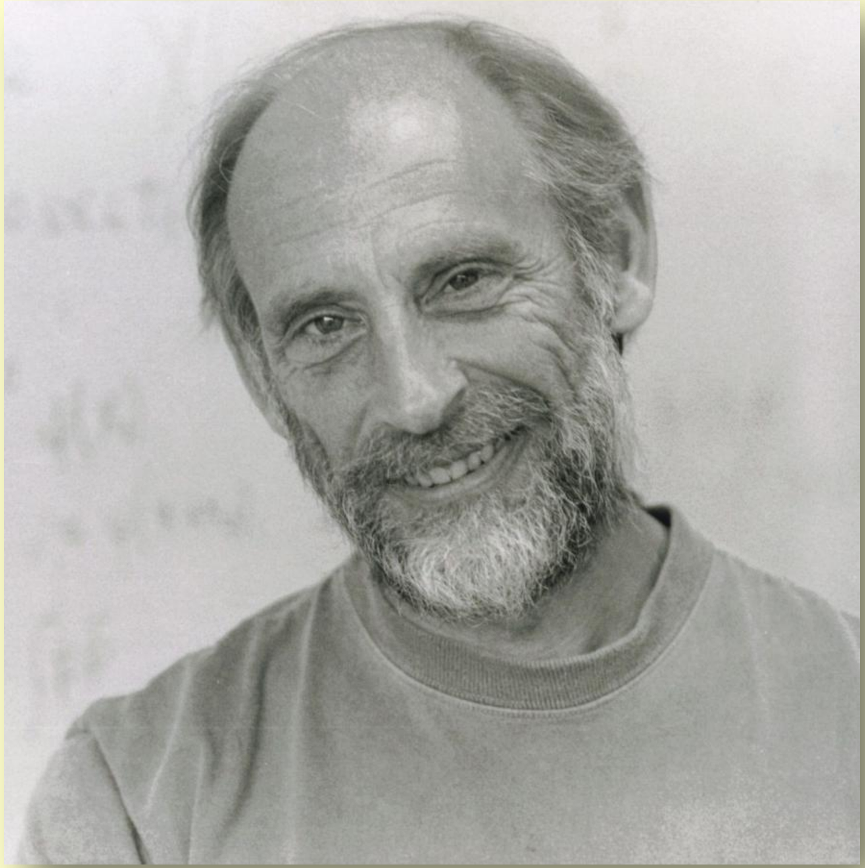


L.Susskind

Holography: in a quantum gravity the entropy of a region scales as the area of its boundary

$$S_{Thermal} \sim VT^3 = L^3 T^3$$

T cannot grow infinitely: gravitational collapse

$$T < T_{\max}:$$

$E_{\max} \sim VT_{\max}{}^4: \quad GE_{\max} < L$ - gravitational radius

$$S_{QG}(V) \sim S_{BH} \sim \frac{L^2}{G}$$

‘Holography in a nutshell’

$$I_{gravity,D+1} = \frac{1}{16\pi G_{D+1}} \int \sqrt{g}\, d^{D+1}x \,(R + 2\Lambda) + \text{b.t.}$$

$$I_{gravity,D+1} = F_D(T)\,/\,T$$

$F_D(T)$ – free energy of a CFT

$$F_{D=2}(T) \sim cT^2 L, \quad c = \frac{3l}{2G}, \quad \Lambda \sim \frac{1}{l^2}$$

$D = 4$: Type IIB string theory on $AdS_5 \times S_5$

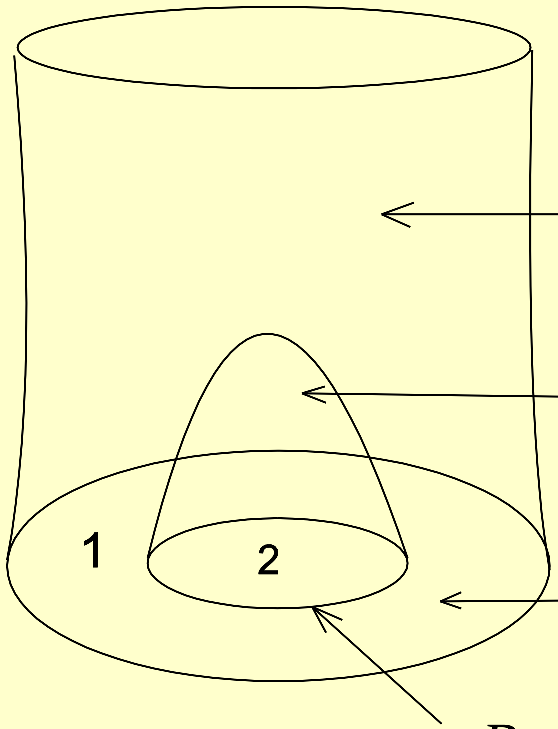
correponds to $\mathbb{N} = 4\ SU(N)$ SYM :

$\mathrm{g}_{YM}{}^2 \sim g_{string}, \quad l \sim \mathrm{g}_{YM}{}^2 N$, N is a 5-form flux on S^5

Toward a holographic description of Entanglement Entropy in CFT's

Holographic Formula for Entanglement Entropy (n=1)

Ryu and Takayanagi,
hep-th/0603001, 0605073



AdS_{d+1} (bulk space)

$\tilde{B}$ minimal (least area) surface in the bulk

4d space-time manifold (asymptotic boundary of AdS)

B separating surface

entropy of entanglement $$S = \frac{\tilde{A}}{4G^{(d+1)}}$$ is measured in terms of the area of $\tilde{B}$

$G^{(d+1)}$ is the gravity coupling in AdS

the Plateau Problem

(Joseph Plateau, 1801-1883)

It is a problem of finding a least area surface (minimal surface) for a given boundary

soap films:

$$k = h(p_1 - p_2)$$ - equilibrium equation

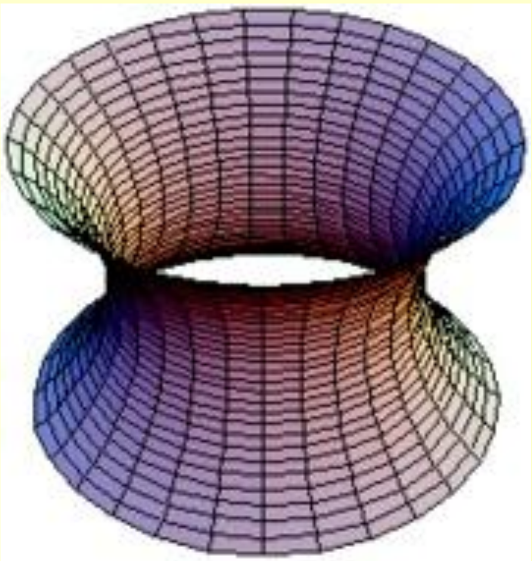
k - the mean curvature

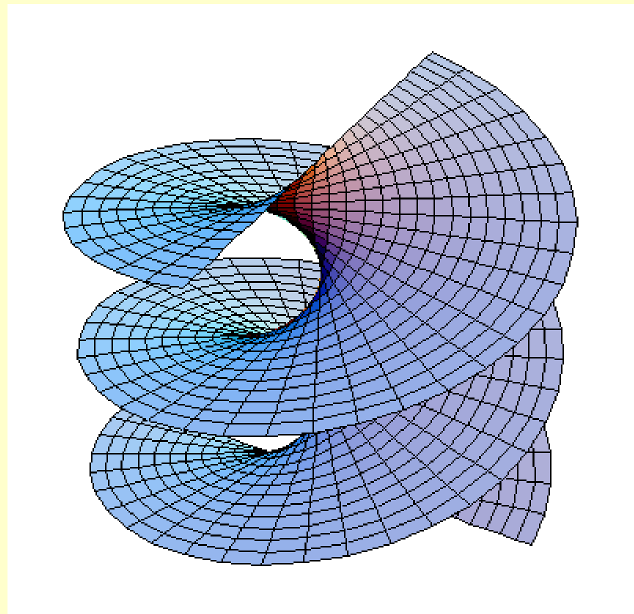
h^{-1} - surface tension

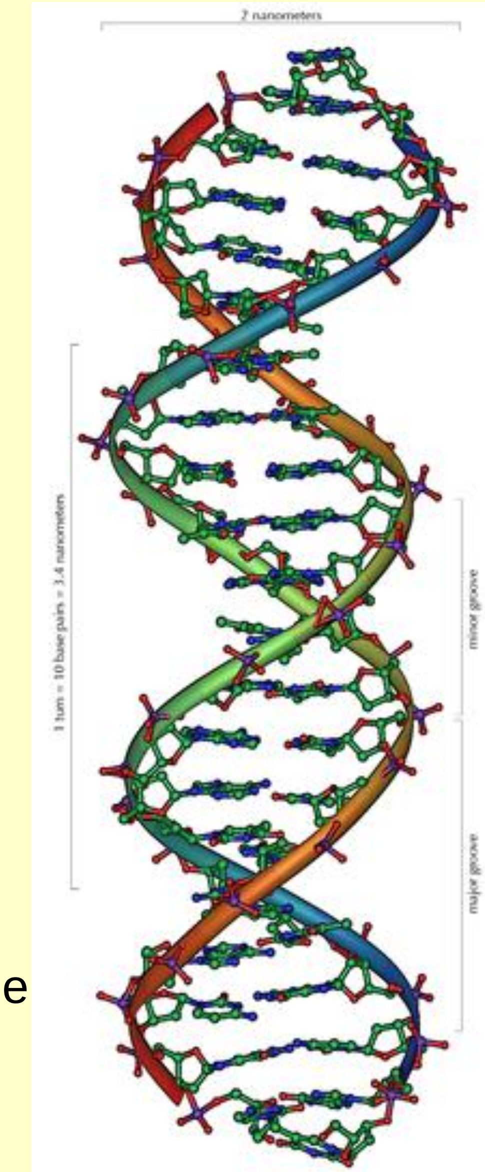
$p_1 - p_2$ -pressure difference across the film

the Plateau Problem

simple surfaces







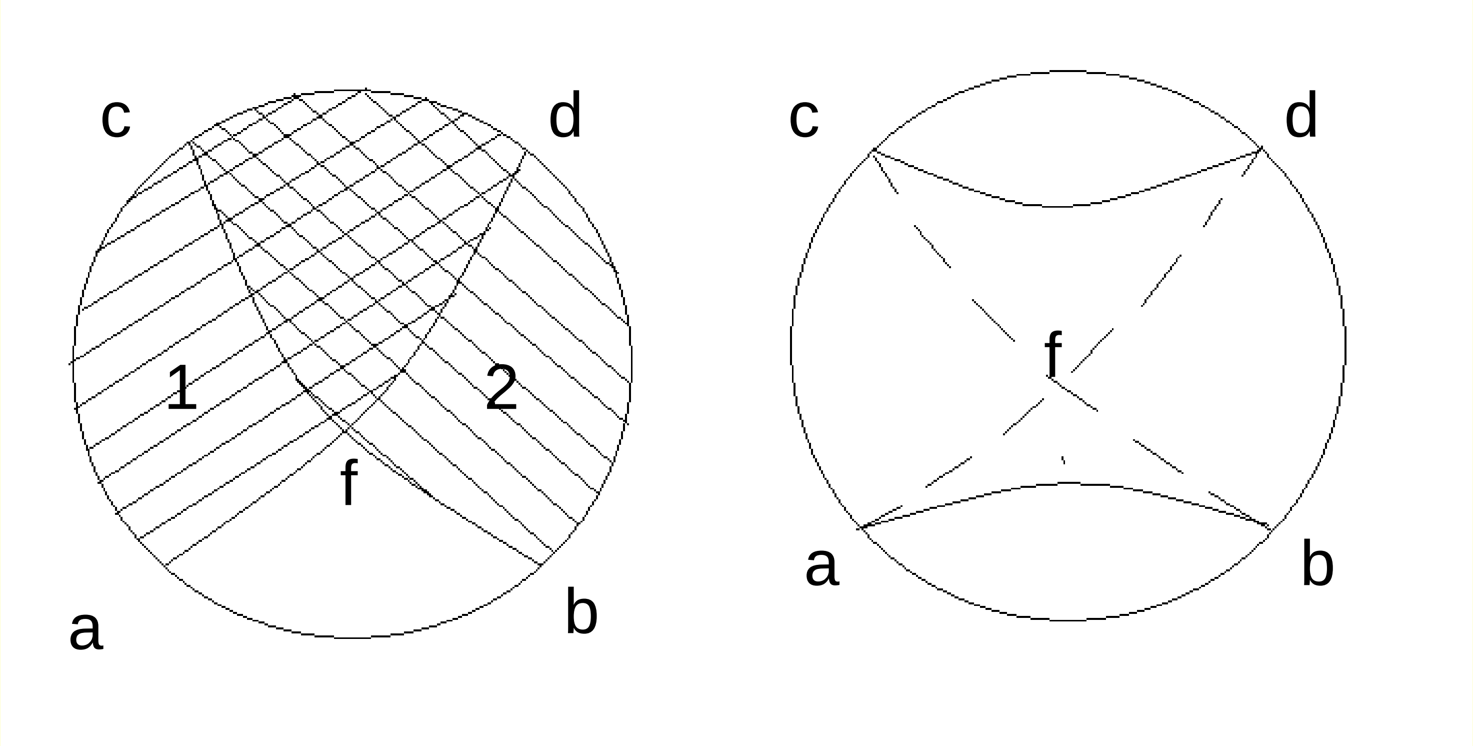
catenoid is a three-dimensional shape made by rotating a catenary curve (discovered by L.Euler in 1744)

helicoid is a ruled surface, meaning that it is a trace of a line

The structure of part of a DNA is double helix

The holographic formula at work

strong subadditivity: $S_1 + S_2 \geq S_{1\cup 2} + S_{1\cap 2}$



$$S_1 = A_{ad} \;, \quad S_2 = A_{bc}$$

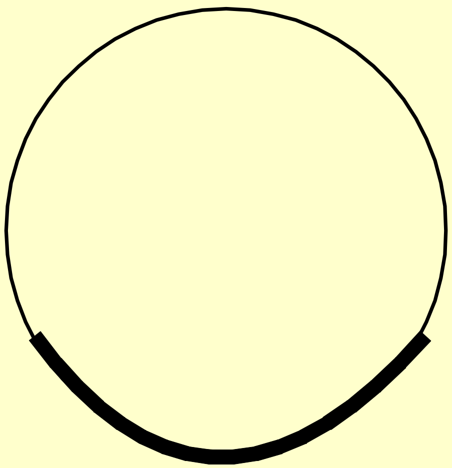
$$S_1 + S_2 = A_{ad} + A_{bc} = A_{af} + A_{fd} + A_{bf} + A_{fc} =$$

$$(A_{af} + A_{bf}) + (A_{fd} + A_{fc}) \geq A_{ab} + A_{dc} = S_{1\cup 2} + S_{1\cap 2}$$

entanglement in 2D CFT

L_1 is the length of Σ_1

$$S = \frac{c}{3}\ln\left(\frac{L}{\pi a}\sin\frac{\pi L_1}{L}\right)$$



Σ_1

ground state entanglement for a system on a circle

c – is a central charge

example in d=2: CFT on a circle

$$ds^2 = l^2\left(d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho d\varphi^2\right)$$

l - AdS radius

$$\Sigma_1 \to \frac{2\pi L_1}{L}$$

$$ds^2{}_{CFT} = ds^2\Big|_{\rho=\rho_0}$$

$$\cosh\frac{A}{l} = 1 + 2\sinh^2\rho_0 \sin^2\left(\frac{\pi L_1}{L}\right)$$

A is the length of the geodesic in AdS

$$e^{\rho_0} = \frac{L}{a}$$ - UV cutoff

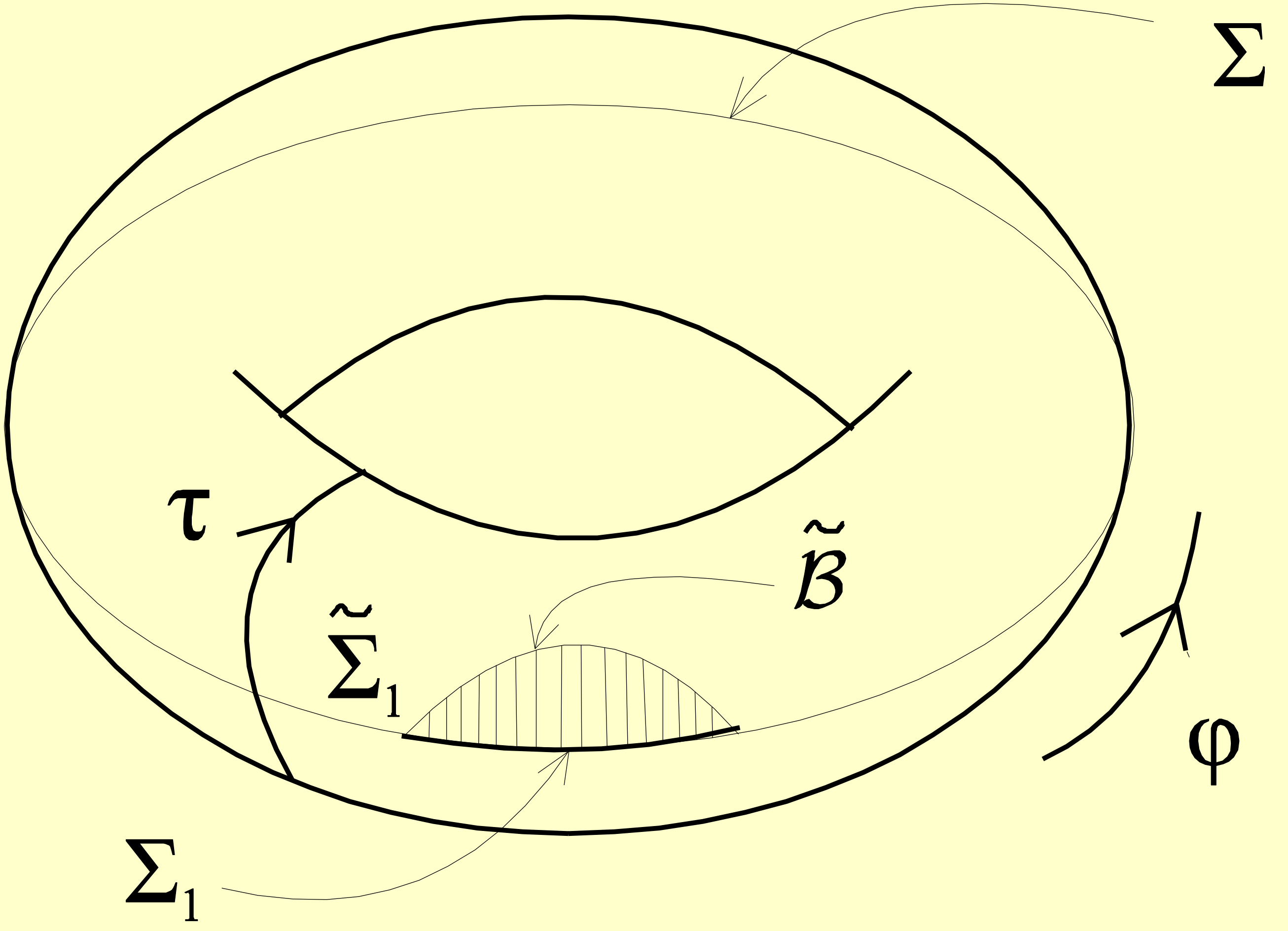
$$S = \frac{A}{4G_3} = \frac{c}{3}\ln\left(e^{\rho_0}\sin\left(\frac{\pi L_1}{L}\right)\right)$$

-holographic formula reproduces the entropy for a ground state entanglement

$$c = \frac{3l}{2G_3}$$ - central charge in d=2 CFT

a CFT on a circle at a finite temperature and BTZ black hole

Euclidean BTZ black hole



The holographic formula at work: higher dimensions

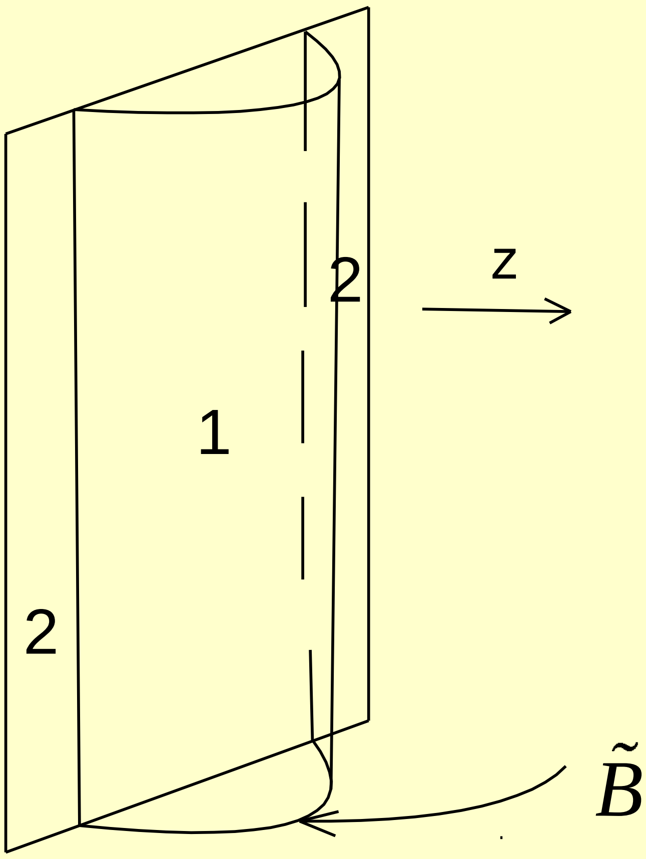
$$S = \frac{A(\tilde{B})}{4G_N^{(D+1)}}$$

$$A(\tilde{B}) - \text{volume of } \tilde{B};$$

Consider the entanglement entropy for N=4 super Yang-Mills in d=4

$$AdS_5$$

a simple example: entanglement of a strip



$$ds_5^2 = \frac{l^2}{z^2}(dz^2 + ds_4^2)$$

$$\tilde{A} \sim \frac{l^3}{a^2} A$$

a – is IR cutoff

$$S = \frac{\tilde{A}}{4G_5} \sim \frac{l^3}{a^2 G_5} A \sim \frac{N^2}{a^2} A$$

$$\frac{l^3}{G_5} \sim N^2, \qquad (SU(N))$$

Tests of the holographic formula by using anomalies (subleading terms!)

Logarithmic term in EE in d=4 (lecture 1)

$$s_{\log} = aF_a + cF_c + bF_b$$ (no boundaries)

- Ryu,Takayanagi, JHEP 0608, 045 (2006),
- Solodukhin, PLB 665, 305 (2008)
- Fursaev, Patrushev, Solodukhin, PRD 88, 044054 (2013)

$c = b$ for CFT's

conformal charges in the trace anomaly of a CFT uniquely fix the logarithmic term in EE

trace anomaly in d=4 (lecture 1)

local conformal anomaly

$$\langle T^{\mu}_{\mu} \rangle = -2aE - cI - \frac{c'}{24\pi^2}\nabla^2 R$$

$$E = \frac{1}{16\pi^2}\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \quad -\text{"density" of the Euler n.}$$

$$I = -\frac{1}{16\pi^2}C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}, \quad C_{\mu\nu\lambda\rho} \quad -\text{the Weyl tensor}$$

"bulk charges" a, c

a- monotonically decreases under RG flow from UV to IR

suggested by J. Crardy, PLB 215, 749-752 (1988),

proved by Z.Komargodski and A.Schwimmer, JHEP 12 (2011)099

Derivation of this result from the Plateau problem in AdS

$\tilde{B} -$ is a holographic surface in the bulk;

$\partial\tilde{B} -$ belongs to conformal class of B;

$\tilde{M} -$ asymptotically AdS solution to:

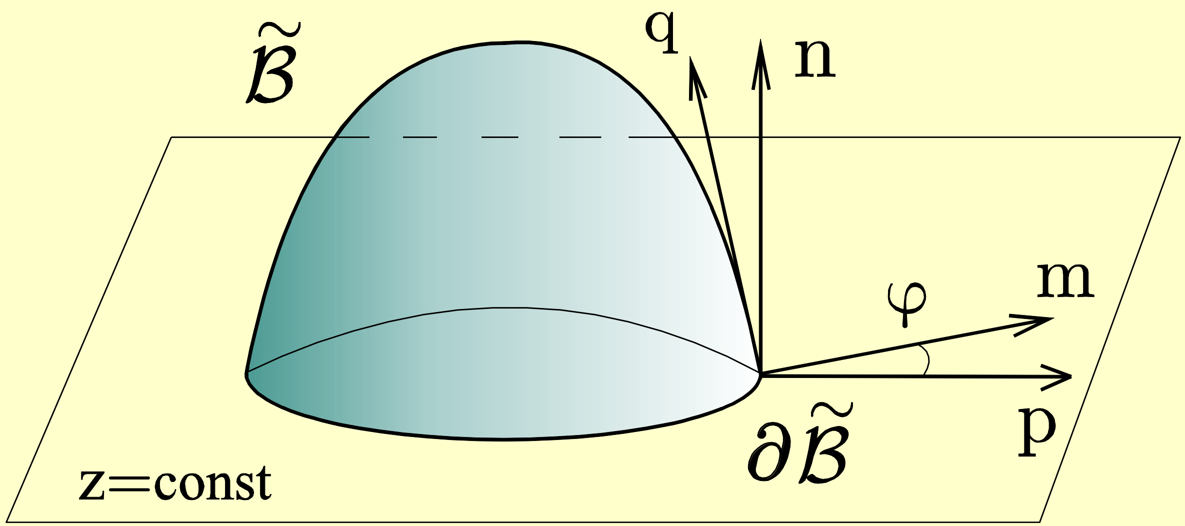
$$\tilde{R}_{MN} - \frac{1}{2}\tilde{R}\,\tilde{g}_{MN} - \frac{3}{l^2}\tilde{g}_{MN} = 0$$

$$ds^2 = \tilde{g}_{MN}dx^M dx^N = z^{-2}(dz^2 + g_{\mu\nu}(z,x)dx^\mu dx^\nu),$$

Fefferman-Graham (FG) expansion in $d = 4$

$$g_{\mu\nu}(z,x) = g_{\mu\nu}(x) - \frac{z^2}{2}\left(R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}R\right) + \ldots$$

FG expansion for the area of a minimal surface



Let $\tilde{B}$ – be a minimal codimension 2 hypersurface in $\tilde{M}$, ($\partial\tilde{B}$ conformal to B)

one needs to find an analog of FG expansion for the metric on $\tilde{B}$

$$ds^2(\tilde{B}) = z^{-2}\left(\frac{dz^2}{\cos^2\varphi} + \sigma_{ab}(z,y)dy^a dy^b\right)$$

FG expansion (continued)

tilt angle: $\varphi = \frac{z}{2}k + \ldots, \quad k \quad -$ extrinsic curvature of B

$x^\mu = x^\mu(z, y), \quad \mu \neq \tau, \quad \tau = const \quad -$ embedding equations of $\tilde{B}$

$x^\mu = x^\mu(y), \quad z = const -$ embedding equations of B

$$x^\mu(z,y) = x^\mu(y) + x^\mu{}_{(1)}(y)z + x^\mu{}_{(2)}(y)z^2 + \ldots$$

$$x^\mu(z,y) = x^\mu(y) - \frac{k}{2(d-2)}\bar{p}^\mu z^2 + \ldots, \quad \mu \neq \tau,$$

$$\bar{p}^\mu = zp^\mu, \quad k_{ab} = x^\mu_{,a} x^\nu_{,b} \bar{p}_{\mu;\nu} \; ;$$

$$\sigma_{ab}(z,y) = \sigma_{ab}(y) - \frac{z^2}{2}\left(kk_{ab} + R_{ab} - \frac{1}{6}\sigma_{ab}R\right) + \ldots, \quad R_{ab} = R_{\mu\nu} x^\mu_{,a} x^\nu_{,b},$$

Holographic anomaly

volume of $\tilde{B}$: $\quad A(\tilde{B}) = \frac{1}{2\varepsilon^2} A(B) + \frac{\pi}{2}(F_a + F_c + F_b) \ln\frac{\mu}{\varepsilon} + \ldots$

$z = \varepsilon -$ position of the boundary (a UV cutoff in CFT)

$$S(B) = \frac{A(\tilde{B})}{4G_5} \sim \frac{N^2\Lambda^2}{4\pi} A(B) + \frac{1}{4} N^2 (F_a + F_c + F_b) \ln \mu\Lambda + \ldots$$

use AdS/CFT dictionary: $\frac{1}{G_5} = \frac{2N^2}{\pi}, \quad \varepsilon = 1/\Lambda$

one reproduces correctly the structutre of the leading divergences and exact value

of the logarithmic part of the entropy

$$a = b = c = \frac{N^2}{4}$$

‘derivation’ of holographic entanglement entropy:

D.F. JHEP 0609 (2006) 018, e-Print: hep-th/0606184

an attempt to find more arguments

different arguments have been provided in 2013 by a number of other authors

Why ‘derivation’ of Ryu-Takayanagi formula is important?

- practical issues like its modifications by quantum corrections and etc;
- fundamental issues like understanding entanglement entropy
in quantum gravity

the idea of origin of the holographic formula

$$Z_{CFT} = Z_{AdS}$$

$$Z_{AdS}(M_n) = \int_{\tilde{M}_n:\, \partial\tilde{M}_n = M_n} [Dg] e^{-W[g]},$$ – partition function for Renyi entropy of ordrer n

$\ln Z_{AdS}(M_n) \simeq -W(n)$ – action at a stationary point, the holographic entropy is

$$S_{AdS}(n) \equiv \frac{1}{1-n}\left(\ln Z_{AdS}(M_n) - n \ln Z_{AdS}(M_{n=1})\right) \simeq \frac{1}{n-1}\left(W(n) - nW(1)\right)$$

taking a naive limit ($n \to 1$) one has (assuming bulk spaces have conical singularities)

$$W[M_n] \to I[M_n] = -\frac{1}{16\pi G_{d+1}} \int_{\tilde{M}_n/\tilde{B}} R\sqrt{g}\, d^{d+1}x + \frac{1}{4G_{d+1}}(n-1)A[\tilde{B}],$$

$$S_{AdS}(n \to 1) \simeq \frac{1}{4G_{d+1}} A[\tilde{B}] = S_{CFT}$$

saddle point approximation requires $A[\tilde{B}]$ to be a minimal hypersurface!

problems with these arguments:

1. The derivation does not reproduce the entanglement Renyi entropy (M. Headrick, Phys. Rev. D82 (2010) 126010, arXiv: 1006.0047[hep-th]);

2. Bulk manifolds with conical singularities cannot be stationary configurations of the AdS partition function (M. Headrick,…);

3. The behavior of string theory on singular manifolds is not known;

4. Path integral integral representation of the Renyi entropy for non-integer index (n -> 1) requires certain conditions which are not always satisfied (H.Cassini, M.Huerta, arXiv: 1203.4007[hep-th]);

5. …..

Maldacena-Lewkowicz approach (JHEP (2013) 1308):

'replicated' boundary manifolds but regular solutions

$$Z_{AdS}(M_n) = \int_{\substack{\tilde{M}_n:\, \partial\tilde{M}_n = M_n \\ \text{regular}}} [Dg] e^{-I[g]} \simeq e^{-I[\tilde{M}_n]}$$

1. $\mathbb{Z}_n$ (replica permutation) symmetry is extened to the bulk;
2. there is a fixed point surface B_n with respect to the action of $\mathbb{Z}_n$
3. $\tilde{M}_n$ is a solution with 'harmless' singularities near B_n;

$$ds^2 \simeq r^2 d\tau^2 + n^2 dr^2 + (\gamma_{ij} + 2 r^n c^{1-n} (\cos\tau k_{ij}^{(1)} + \sin\tau k_{ij}^{(2)})) dv^i dv^j$$

$$0 \le \tau \le 2\pi n$$

4. singularities do not present in the components of $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

when B_n is a minimal surface: $k_{ij}^{(p)} \gamma^{ij} = 0$

5. then $S = A(B)/(4G)$

problems with ML arguments:

1. No proof that such solutions with harmless singularities exist for given boundary conditions.

The boundary manifolds with conical singularities may not have regular solutions;

2. Quantum effects may make singularities worse (usually happens)

3. There are evidences that singularities are not eliminated in higher curvature gravities for entangling surfaces which are minima of the entropy functional

Approach by Fursaev and ML approach may compliment each other

boundary conditions imply two relevant sets of manifolds in the bulk

$$Z_{AdS}(M_n) = \int_{\substack{\tilde{M}_n :\, \partial\tilde{M}_n = M_n \\ \text{singular}}} [Dg]\, e^{-W[g,n]} + \int_{\substack{\tilde{M}_n :\, \partial\tilde{M}_n = M_n \\ \text{regular}}} [Dg]\, e^{-I[g]}$$

the dominant configuration is either a manifold with conical singularities

or a manifold with 'harmless' singularities depending on the fact whose action

has the least value;

Fursaev and ML disagree on the Renyi entropies

Some developments: generalization of RT formua

Fursaev, Patrushev, Solodukhin (2013) - quadratic curvature gravities

Xi Dong (2014), J. Camps (2014) - higher curvature gravities

Xi Dong (2016) - holographic prescription for Renyi entropies

Holographic formula enables one to compute entanglement entropy in strongly correlated systems with the help of classical methods (the Plateau problem)

What about entanglement in quantum gravity?

Can one define an entanglement entropy, S(B),
of fundamental degrees of freedom spatially separated by a surface B?

How can the fluctuations of the geometry be taken into account?

the hypothesis

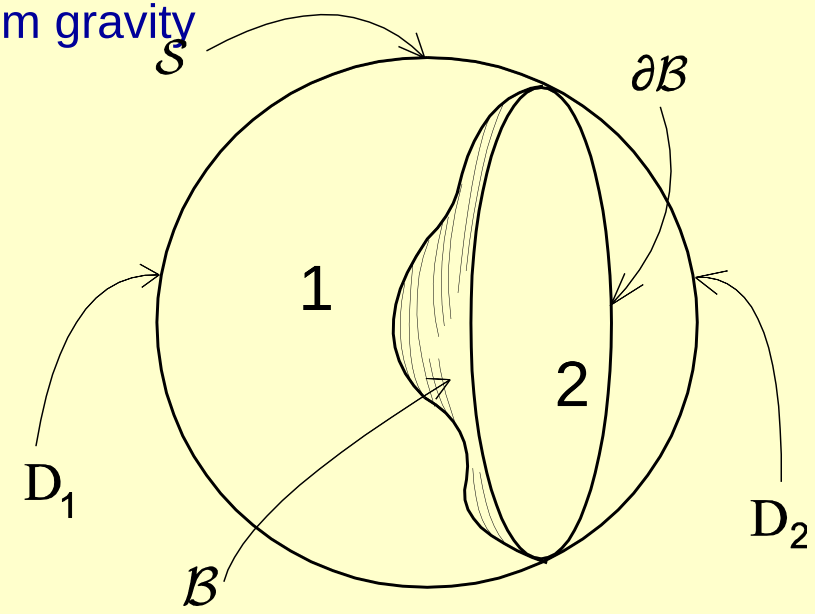
- S(B) is a macroscopical quantity (like thermodynamical entropy);

- S(B) can be computed without knowledge of a microscopical content of the theory (for an ordinary quantum system it can't)

- the definition of the entropy is possible at least for a certain type of boundary conditions

Suggestion (Fursaev, 06,07): EE in quantum gravity between degrees of freedom separated by a surface B is

$$S(B) = \frac{A(B)}{4G}$$



B is a least area minimal hypersurface in a constant-time slice

conditions:

- static space-times

the system is determined by a set of boundary conditions; subsets, “1” and “2” , in the bulk are specified by the division of the boundary

The shape of the separating surface is
formed under fluctuations of the geometry;

As a result the surface is minimal, i.e. has a least area

Details: D.V. Fursaev, Phys. Rev. D77 (2008) 124002,
e-Print: arXiv:0711.1221 [hep-th]

Generalized gravitational entropy (associated with minimal surfaces):

"We consider classical Euclidean gravity solutions with a boundary. The boundary contains a non-contractible circle. These solutions can be interpreted as computing the trace of a density matrix in the full quantum gravity theory, in the classical approximation. When the circle is contractible in the bulk, we argue that the entropy of this density matrix is given by the area of a minimal surface. This is a generalization of the usual black hole entropy formula to Euclidean solutions without a Killing vector."

J. Maldacena and A, Lewkowycz, arXiv:1304.4926 [hep-th]

thank you for attention