The spectrum and separability of 2-qubit mixed X-states

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Qubit

A generic mixed state [1] of an $n-$level quantum system is described by an $n \times n$ complex matrix - the density matrix ρ, satisfying the following conditions:

1. *Hermicity*: $\rho = \rho^{\dagger}$,
2. *finite trace*: $Tr(\rho) = 1$,
3. *positive semidefiniteness*: $\rho \geq 0$.

The state of a qubit is given by a density matrix:

$$\rho = \frac{1}{2}(1 + \boldsymbol{\alpha} \cdot \boldsymbol{\sigma}), \qquad \boldsymbol{\alpha}^2 \leq 1. \tag{1}$$

where $\boldsymbol{\alpha} = Tr(\boldsymbol{\sigma}\rho)$ is the expectation and $\boldsymbol{\sigma}$ is the set of Pauli matrices.

[1] The special class of idempotent matrices, satisfying $\rho^2 = \rho$, corresponds to the so-called *pure states*. A mixed state is a mixture of pure states.

Composite states

The space of states of the system, obtained by joining two systems 1 and 2, is a subspace of the tensor product of their individual Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{2}$$

The density matrix ρ, describing mixed states of system $\mathcal{H}$, is separable, if it allows the convex decomposition:

$$\rho = \sum_k \omega_k \rho_1^k \otimes \rho_2^k, \qquad \sum_k \omega_k = 1, \quad \omega_k \geq 0, \tag{3}$$

where ρ_1^k and ρ_2^k represent the density matrices, acting on the corresponding multiplier of $\mathcal{H}$. Otherwise it is entangled.

Two qubits

Consider the density matrix of two qubits, parametrized in the Fano form:

$$\rho = \frac{1}{4}[\mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbf{a}\cdot \boldsymbol{\sigma} \otimes \mathbb{I}_2 + \mathbf{b}\cdot \mathbb{I}_2 \otimes \boldsymbol{\sigma} + c_{ij}\sigma_i \otimes \sigma_j], \tag{4}$$

where

- $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are the Bloch vectors of the constituent qubits,
- $C = \|c_{ij}\|$ is the so-called "correlation matrix",
- $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

Are there mixed states, which are separable for an arbitrary spectrum of ρ?

2−qubit $X-$states

The density matrices of the form:

$$\rho_X := \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \qquad \begin{cases} \rho_{11}, \rho_{22}, \rho_{33}, \rho_{44} \in \mathbb{R}, \\ \rho_{14} = \overline{\rho}_{14}, \ \rho_{23} = \overline{\rho}_{32}, \\ \sum_{i=1}^{4} \rho_{ii} = 1, \end{cases} \tag{5}$$

are called the $X-$states.
The matrix (5) is unitary equivalent to the diagonal matrix

$$\rho_X = KWP \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\ PW^\dagger K^\dagger, \tag{6}$$

where $K = \exp\left(i\frac{u}{2}\sigma_3\right) \otimes \exp\left(i\frac{v}{2}\sigma_3\right) \in SU(2) \otimes SU(2)$ and

$$W = \left(\begin{array}{c|c} e^{i\frac{\phi_1}{2}\sigma_2} & 0 \\ \hline 0 & e^{i\frac{\phi_2}{2}\sigma_2} \end{array}\right), \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

Spectrum of $2-$qubit $X-$states

The spectrum $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of the diagonal density matrix ρ_X [2] forms the partially ordered simplex [3] $\underline{\Delta}_3$ (Fig. 1):

$$\begin{cases} \sum_{i=1}^{4} \lambda_i = 1, \\ 0 \leq \lambda_2 \leq \lambda_1 \leq 1, \\ 0 \leq \lambda_4 \leq \lambda_3 \leq 1. \end{cases} \tag{8}$$

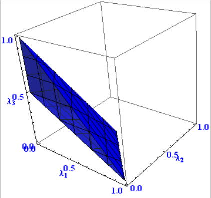

Figure 1: The partially ordered simplex $\underline{\Delta}_3$.

[2] $\rho_X = KWP\,\mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\; PW^\dagger K^\dagger$.

[3] Partially ordered simplex is the quotient of a standard simplex by action of transposition subgroup $P_2 \times P_2 \subset P_4$.

The separability as a function of density matrices eigenvalues $\{\lambda\}$

According to the Peres-Horodecki criterion, which is a necessary and sufficient condition of separability for $2 \otimes 2$ and $2 \otimes 3$ dimensional systems, a state ρ is separable iff its partial transposition is semi-positive as well.

The partial transposition ρ^{T_2} of a $2-$qubit density matrix with respect to the ordinary transposition operation T in the second subsystem is defined as:

$$\rho^{T_2} = I \otimes T\rho, \qquad T(\sigma_1, \sigma_2, \sigma_3) \to (\sigma_1, -\sigma_2, \sigma_3). \tag{9}$$

Similarly, one can use the alternative action: $\rho^{T_1} = T \otimes I\rho$.

The separability conditions

Applying the Peres-Horodecki separability criterion to the $X-$state density matrix ρ_X [4], we conclude, that it is separable iff:

$$(\lambda_1-\lambda_2)^2\cos^2\phi_1+(\lambda_3-\lambda_4)^2\sin^2\phi_2\le(\lambda_1+\lambda_2)^2, \qquad (10)$$
$$(\lambda_3-\lambda_4)^2\cos^2\phi_2+(\lambda_1-\lambda_2)^2\sin^2\phi_1\le(\lambda_3+\lambda_4)^2. \qquad (11)$$

New variables (x, y) and parameters (a, b, c, d) as functions of the density matrix eigenvalues and angles ϕ_1 and ϕ_2:

$$\begin{cases} x=(\lambda_1-\lambda_2)^2\cos^2\phi_1, \\ y=(\lambda_3-\lambda_4)^2\cos^2\phi_2, \end{cases} \qquad (12) \qquad \begin{cases} a=(\lambda_1+\lambda_2)^2-(\lambda_3-\lambda_4)^2, \\ b=-(\lambda_1-\lambda_2)^2+(\lambda_3+\lambda_4)^2, \\ c=(\lambda_1-\lambda_2)^2,\ d=(\lambda_3-\lambda_4)^2. \end{cases} \qquad (13)$$

[4] $\rho_X = KWP\,\mathrm{diag}\,(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\ PW^\dagger K^\dagger$.

The parameters (a, b, c, d) obey the inequalities:

$$a + b \geq 0, \quad a + d \geq 0, \quad b + c \geq 0, \tag{14}$$

Thus, the separability conditions in the form of two inequalities (10) [5] and (11) [6] linearize:

$$\begin{cases} x - y \leq a, & 0 \leq x \leq c, \\ y - x \leq b, & 0 \leq y \leq d. \end{cases} \tag{15}$$

Hence, the inequalities (15) have solutions for all possible values of parameters from the restrictions (14).

[5] $(\lambda_1 - \lambda_2)^2 \cos^2 \phi_1 + (\lambda_3 - \lambda_4)^2 \sin^2 \phi_2 \leq (\lambda_1 + \lambda_2)^2$.

[6] $(\lambda_3 - \lambda_4)^2 \cos^2 \phi_2 + (\lambda_1 - \lambda_2)^2 \sin^2 \phi_1 \leq (\lambda_3 + \lambda_4)^2$.

For eigenvalues from the partially ordered simplex Δ_3 the separability conditions [7] of the $X-$state density matrix ρ_X [8] determine non empty domain (Fig. 2) for angles ϕ_1 and ϕ_2.

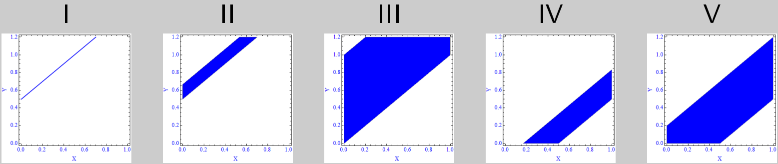

Figure 2: Plots (I-V) - families of solutions: Domain (I) : $a < 0,\ b = -a,\ c \geq 0,\ d \geq b$; Domain (II): $a < 0,\ b > -a,\ c \geq 0,\ d \geq -a$; Domain (III): $a = 0,\ b \geq 0,\ c \geq 0,\ d \geq 0$; Domain (IV): $a > 0,\ -a \leq b \leq 0,\ c \geq -b,\ d \geq 0$; Domain (V): $a > 0,\ b > 0,\ c \geq 0,\ d \geq 0$.

There exists $4-$parametric family of separable mixed $X-$states of $2-$qubits with an arbitrary spectrum of the density matrix.

[7] $(\lambda_1 - \lambda_2)^2 \cos^2 \phi_1 + (\lambda_3 - \lambda_4)^2 \sin^2 \phi_2 \leq (\lambda_1 + \lambda_2)^2$, $(\lambda_3 - \lambda_4)^2 \cos^2 \phi_2 + (\lambda_1 - \lambda_2)^2 \sin^2 \phi_1 \leq (\lambda_3 + \lambda_4)^2$.

[8] $\rho_X = KWP\,\mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\ PW^\dagger K^\dagger$.

The absolute separability conditions

The states of *n*-dimensional quantum system, remaining separable under the adjoint action of the $SU(n)$-transformations of $n \times n$ density matrices, are called absolute separable.

Are there X-states, which are separable for arbitrary angles ϕ_1 and ϕ_2?

The inequalities in the eigenvalues of $X-$matrices, defining the absolutely separable $X-$states, read:

$$\begin{cases} \lambda_1 - \lambda_2 \leq 2\sqrt{\lambda_3\lambda_4}, \\ \lambda_3 - \lambda_4 \leq 2\sqrt{\lambda_1\lambda_2}. \end{cases} \tag{16}$$

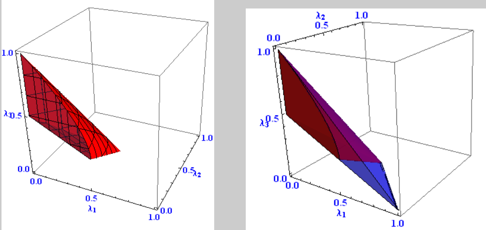

Figure 3: The absolute separability region.

On the generalization to an arbitrary $2-$qubit states

The Peres-Horodecki separability criterion can be written in the form of polynomial inequalities in the $SU(4)$ Casimir invariants $\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$ and two $SU(2) \times SU(2)-$invariant polynomials [9].

In general case,

- the determinants $det(C)$, $det(M)$ are analogues of angles ϕ_1, ϕ_2 of $X-$states,
- the Casimir invariants $\mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$ are analogues of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $X-$states.

[9] Here the determinants of correlation $C = \|c_{ij}\|$ and Schlienz-Mahler matrix $M = \|c_{ij} - a_i b_j\|$ are the $SU(2) \times SU(2)-$ polynomial invariants.

Conjecture

CONJECTURE: The inequalities

$$\begin{cases} 0 \leq 3\mathfrak{C}_2 - 2\mathfrak{C}_3 - 4det(C) \leq 1, \\ 0 \leq (1 - 3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4 + 16det(M) \leq 1, \end{cases} \quad (17)$$

have solutions for unknown $det(C)$ and $det(M)$ for all values of Casimir invariants $\mathfrak{C}_2$, $\mathfrak{C}_3$ and $\mathfrak{C}_4$, which are constrained by the inequalities:

$$\begin{cases} 0 \leq \mathfrak{C}_2 \leq 1, \\ 0 \leq 3\mathfrak{C}_2 - 2\mathfrak{C}_3 \leq 1, \\ 0 \leq (1 - 3\mathfrak{C}_2)^2 + 8\mathfrak{C}_3 - 12\mathfrak{C}_4 \leq 1. \end{cases} \quad (18)$$

Thank you for attention