Problems with solutions for:

Linear optics

Case study, Module 3

- Problem 1. Estimate the dipole's field of RHIC collider, operating with fully stripped gold ions at 100 GeV/u energy. Ring perimeter 3.834km.

Solution. Fully stripped Au^{79+} with atomic weight 197 is highly relativistic, β~1.

$$B\rho = \frac{10}{2.998}\frac{1}{79}100\cdot 197 = 831.8\ \mathrm{T\cdot m}$$

Let's assume dipoles occupy 30% of the ring.

$$\rho = \frac{0.3\cdot \Pi}{2\pi} = 183\ \mathrm{m} \qquad B = \frac{B\rho}{\rho} = 4.5\ \mathrm{T}$$

(In reality RHIC dipoles field is 3.458 T. Thus, dipole occupies ~40%.)

- Problem 2. Derive stability condition for FODO lattice. Assume strong focusing thin lens approximation.

Solution. One-period matrix, calculated explicitly:

$$M_{x,y}=\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \pm P_2 & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \mp P_1 & 1\end{pmatrix}=\begin{pmatrix}1\pm LP_2 & L\\ \pm P_2 & 1\end{pmatrix}\begin{pmatrix}1\mp LP_1 & L\\ \mp P_1 & 1\end{pmatrix}=\dots$$

$$\cos\mu_{x,y}=\frac{1}{2}\mathrm{Tr}M_{x,y}=1\mp LP_1\pm LP_2-\frac{1}{2}LP_1LP_2=\frac{1}{2}(2\mp LP_1)(2\pm LP_2)-1$$

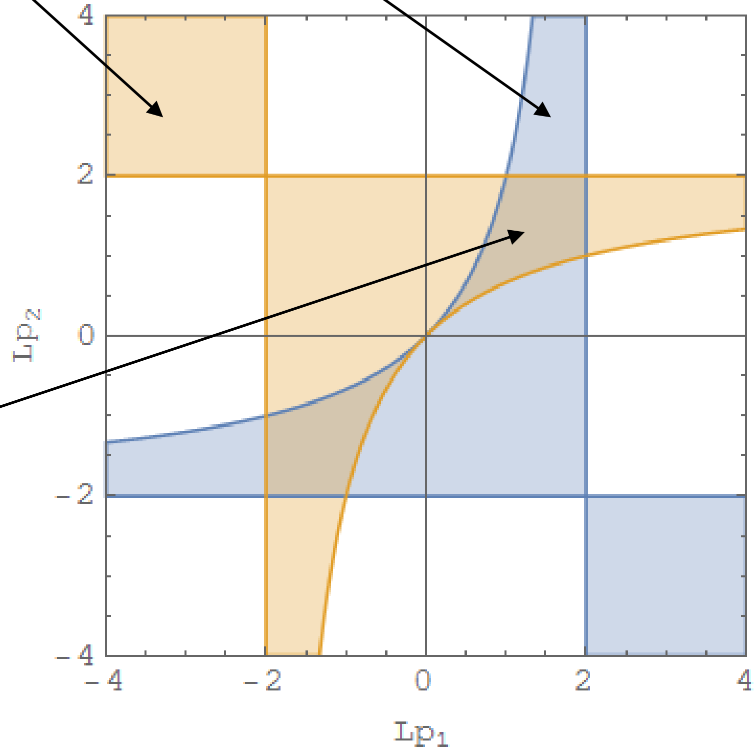
$$-1<\cos\mu_{x,y}<1$$ Stability condition

$$\begin{cases} (2 - LP_1)(2 + LP_2) > 0 \\ LP_2(2 - LP_1) < 2LP_1 \\ (2 + LP_1)(2 - LP_2) > 0 \\ -LP_2(2 + LP_1) < -2LP_1 \end{cases}$$

X degree of freedom

Y degree of freedom

The “tie” of stability



- <u>Problem 3</u>. Assume FODO lattice with lenses of equal absolute strength. Find min and max values of β-function.

<u>Solution</u>.

$$\cos\mu_x = 1 - \frac{1}{2}(LP)^2 \longrightarrow LP = 2\left|\sin\frac{\mu}{2}\right|$$

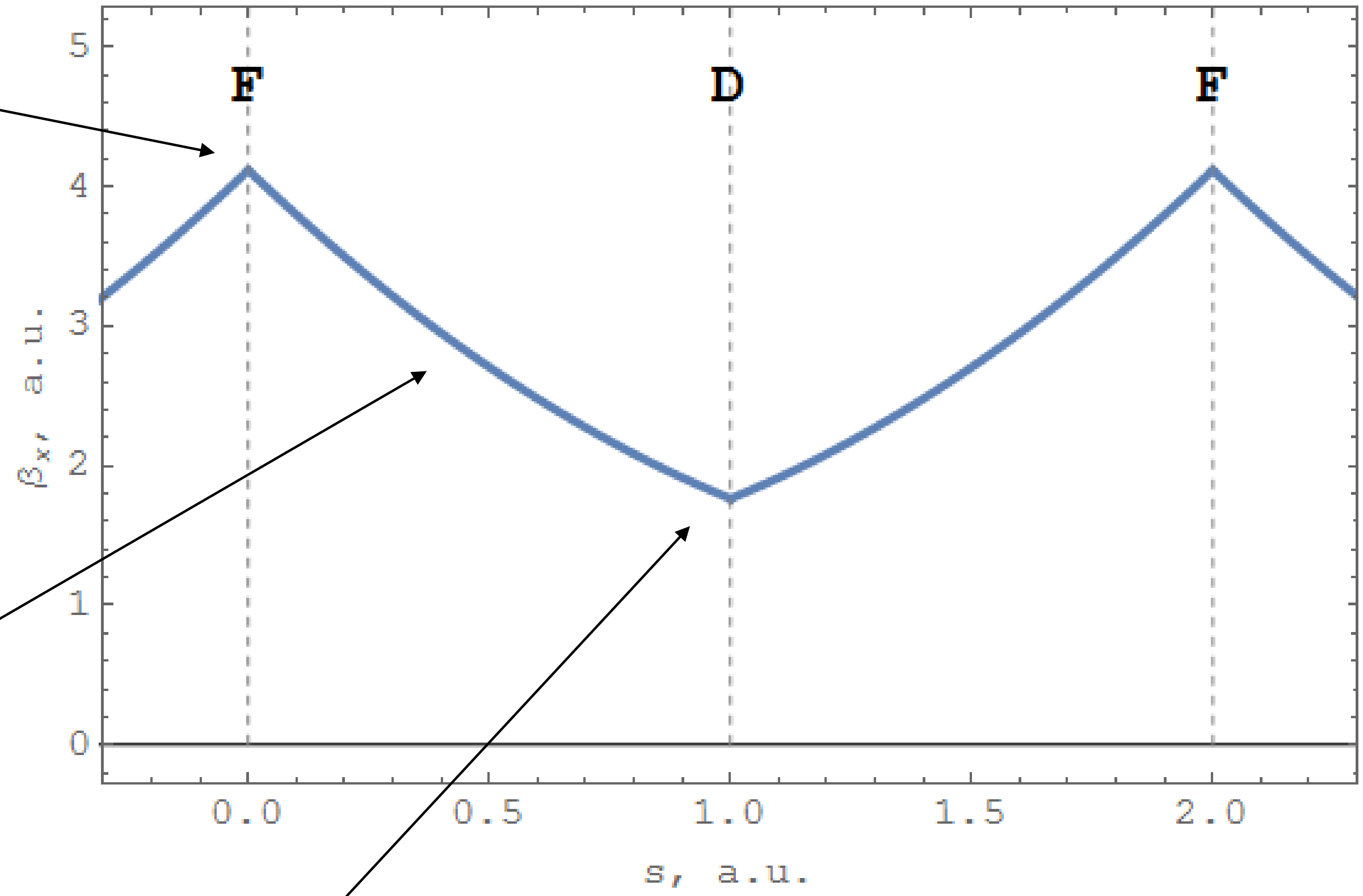
$$\sin\mu = \sqrt{1-(\cos\mu)^2} = \frac{LP}{2}\sqrt{(2-LP)(2+LP)}$$

$$M_{F-0} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ P & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} = \begin{pmatrix} \dots & L(2+LP) \\ \dots & \dots \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$\beta_{max} = \frac{2L}{|LP|}\sqrt{\frac{2+LP}{2-LP}} = L\frac{1+\sin\mu/2}{\sin\mu/2\cos\mu/2} \qquad \beta_{\min} = \frac{2L}{|LP|}\sqrt{\frac{2-LP}{2+LP}} = L\frac{1-\sin\mu/2}{\sin\mu/2\cos\mu/2}$$

Beta-function in FODO lattice

β_{max}



Beta inside the drift:

$$\beta(s) = \beta_0 + \frac{(s - s_0)^2}{\beta_0}$$

$\beta_{\min}$

- Problem 4. Find the dispersion function in the arc with regular FODO lattice. Assume small bending angle per one cell.

Solution. Let's start from the middle of F-quad

$$M_{3\times3}=\begin{pmatrix}1 & 0 & 0\\ -P/2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & L & \frac{L\theta}{2}\\ 0 & 1 & \theta\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ P & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & L & \frac{L\theta}{2}\\ 0 & 1 & \theta\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -P/2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1-\frac{(LP)^2}{2} & 2L\left(1-\frac{LP}{2}\right) & 2L\theta\left(1+\frac{LP}{4}\right)\\ P\left(\frac{(LP)^2}{4}-\frac{LP}{2}\right) & 1-\frac{(LP)^2}{2} & 2\theta\left(1-\frac{LP}{4}-\frac{(LP)^2}{8}\right)\\ 0 & 0 & 1\end{pmatrix}$$

Closure condition

$$\begin{pmatrix}D\\ D'\\ 1\end{pmatrix}_f=M_{3\times3}\begin{pmatrix}D\\ D'\\ 1\end{pmatrix}_f$$

$D'_f=0$ - symmetry condition

$$D_f = \frac{L\theta}{\left(LP/2\right)^2}\left(1+\frac{LP}{4}\right) = L\theta\frac{\left(1+\frac{1}{2}\sin\mu/2\right)}{\sin^2\mu/2}$$

$$D_d = \frac{L\theta}{\left(LP/2\right)^2}\left(1-\frac{LP}{4}\right) = L\theta\frac{\left(1-\frac{1}{2}\sin\mu/2\right)}{\sin^2\mu/2}$$

Transport D-vector into drift:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_s = \begin{pmatrix} 1 & s & \frac{s^2}{2r_0} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{f+0}$$

$$D(s) = D_f - D_f\frac{P}{2} + \frac{s^2}{2r_0}$$

Momentum compaction:

$$\alpha_p = \frac{1}{\Pi}\int\frac{D(s)}{r_0(s)}ds = \frac{1}{L}\int_0^L\left(D_f - D_f\frac{P}{2} + \frac{s^2}{2r_0}\right)\frac{ds}{r_0} = \theta^2\left(\frac{1-\left(LP\right)^2/48}{\left(LP\right)^2/4}\right) = \theta^2\left(\frac{1-\frac{1}{12}\sin^2\mu/2}{\sin^2\mu/2}\right)$$

- Problem 5. Calculate natural tune chromaticity for FODO lattice with quads of equal absolute strength P. What sextupole strength is needed to cure chromaticity?

Solution.

See problem 3

$$\Delta \nu_x \approx \frac{\beta_{xf} P \cdot (-\delta)}{4\pi} + \frac{\beta_{xd} (-P) \cdot (-\delta)}{4\pi} = -\delta \frac{P}{4\pi} \left(\beta_{xf} - \beta_{xd} \right) = -\delta \frac{1}{\pi} \frac{LP}{\sqrt{4 - (LP)^2}} = -\delta \frac{1}{\pi} tg\, \mu/2$$

If we'll put sextupole component directly into quad (note: non-zero dispersion needed):

$$S = \frac{G}{D}$$

$$IS_{f,d} = \frac{\pm P}{D_{f,d}} = \pm \frac{1}{L^2 \theta} \frac{\sin^3 \mu/2}{2 \pm \sin \mu/2}$$

See problem 4

Sextupole's integrated strength grows rapidly with betatron phase advance and with decrease of bending angle per cell