Recap of Transverse Particle Dynamics

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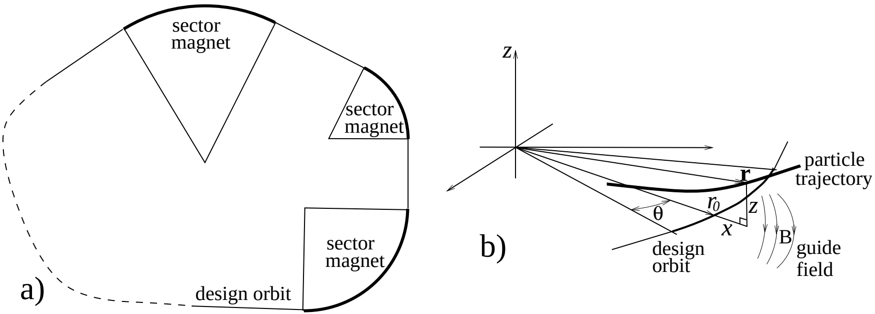
Outline

Design Orbit

Desired shape of the Closed Orbit (CO) is determined by appropriate placement of the bending magnets. In the present lecture we assume CO to be a planar closed curve combined from circular arcs.



The equilibrium particle with charge e, rest mass m , and momentum $\mathbf{p}$ (with appropriate initial conditions!)
will perform periodical rotation along the design orbit.

Sector Bending Magnet

In the cylindrical coordinates,
the particle's position is

$$\mathbf{r} = (z, r, \theta)$$

and its velocity is

$$\mathbf{v} = (\dot{z}, \dot{r}, r\dot{\theta})$$

In a sector magnet the magnetic field configuration $\mathbf{B}$ is identical in each plane $\theta = \text{const}$

$$\mathbf{B} = (B_z(z, r), B_r(z, r), 0)$$

Vector potential $\mathbf{A}$ of the magnetic field, $\mathbf{B} = \text{rot}\mathbf{A}$, can be written as a single component:

$$\mathbf{A} = (0, 0, A_\theta(z, r))$$

Derivation of Equations of Motion in Cylindrical Coordinates

From the Lagrangian: $\mathcal{L} = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - e\varphi + \frac{e}{c}\mathbf{vA}$
we find the canonical momenta by differentiation:

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \gamma m\mathbf{v} + \frac{e}{c}\mathbf{A}$$

here $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the particle's Lorentz-factor.

Lagrangian equations of motion $\frac{d\mathcal{P}}{dt} = \frac{\partial \mathcal{L}}{\partial \mathbf{r}}$ can be expressed in the conventional form with the Lorentz force in their R.H.S.:

$$\frac{d}{dt}(\gamma m\mathbf{v}) = e\mathbf{E} + \frac{e}{c}[\mathbf{v} \times \mathbf{B}]$$

when the fields are expressed via potentials

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \nabla\varphi\,, \qquad \mathbf{B} = \mathrm{rot}\mathbf{A}$$

Equations of Motion in a Sector Magnet

From the Lagrangian written in the cylindrical coordinates,

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{1}{c^2}(\dot{z}^2 + \dot{r}^2 + r^2\dot{\theta}^2)} - e\varphi + \frac{e}{c}r\dot{\theta}A_\theta$$

we obtain a set of equations of motion:

$$\text{z)} \quad \frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{z}}\Big) = \frac{d}{dt}(\gamma m\dot{z}) = -e\frac{\partial\varphi}{\partial z} + \frac{e}{c}r\dot{\theta}\frac{\partial A_\theta}{\partial z} = eE_z - \frac{e}{c}r\dot{\theta}B_r$$

$$\text{r)} \; \frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{r}}\Big) = \frac{d}{dt}(\gamma m\dot{r}) = \gamma mr\dot{\theta}^2 - e\frac{\partial\varphi}{\partial r} + \frac{e}{c}r\dot{\theta}\cdot\frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} = \gamma mr\dot{\theta}^2 + eE_r + \frac{e}{c}r\dot{\theta}B_z$$

$$\theta) \quad \frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\Big) = \frac{d}{dt}\Big(\gamma mr^2\dot{\theta} + \frac{e}{c}rA_\theta\Big) = -er\cdot\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \frac{e}{c}r\frac{\partial A_\theta}{\partial\theta}(=0)$$

An extra term $\gamma mr\dot{\theta}^2$ in the radial equation is due to the curvilinear coordinates, its physical meaning is "inertial centrifugal force".
In what follows we leave out the potential electric fields, putting $\varphi = 0$; meanwhile, the induction electric field E_θ is still here in the θ)-equation, when $\dot{\mathbf{A}} \neq 0$.

Positions of Eqiulibrium in z and r

The equilibrium conditions follow from vanishing forces on the R.H.S.:

$$\text{z)} \quad B_r\big|_{z=0} = 0$$

This condition holds for the guide fields symmetric w.r.t. $z = 0$ plane, where the design orbit belongs to. In the case of such a symmetry the plane $z = 0$ is called *the median plane*.

$$\text{r)} \quad \gamma m v_\theta + \frac{e}{c} r B_z\big|_{z=0} = 0$$

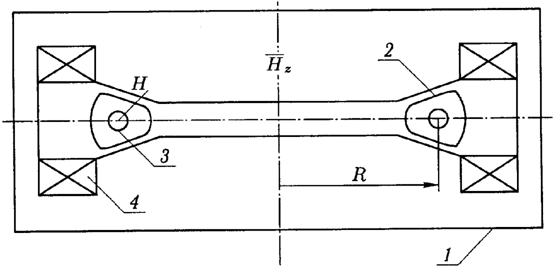
With a given law $B_z(0, r)$ in the median plane,
we can find the orbit radius r_0 that corresponds to the equilibrium particle with momentum p_0.
Using notation $B_0 = B_z(0, r_0)$ and relation $\gamma m v_\theta = p_0$ which assumes no transverse velocities of the equilibrium particle, we rewrite:

$$\text{r)} \quad p_0 c = -eB_0 r_0 \quad \text{or: } (p_0 c/e)\text{eV} = 300(B_0 r_0)\text{Gauss} \cdot \text{cm}$$

Longitudinal Motion

Considering the longitudinal motion of equilibrium particle,
we put: $z = 0$, $r = r_0$, $p = p_0$, then in θ)-equation we have: $\gamma m r^2 \dot{\theta} = p_0 r_0$.
In an azimuthally-symmetric example, magnetic flux Φ enclosed by the equilibrium orbit is readily expressed via the vector potential using the Stokes theorem:

$$\Phi = \int \mathbf{B} d\mathbf{S} = \int \mathrm{rot}\mathbf{A} d\mathbf{S} = \oint \mathbf{A} d\mathbf{l} = 2\pi r_0 A_\theta \Big|_{r=r_0, z=0}$$



Schematic cross-sectional view of a "classical" betatron:
1 — magnet yoke, 2 — vacuum pipe, 3 — beam, 4 — windings.

2 : 1 Condition for Induction Acceleration

Then:

$$\theta) \quad \frac{d}{dt}(p_0 r_0 + \frac{e}{c}\frac{\Phi}{2\pi}) = \frac{d}{dt}(-\frac{e}{c} r_0^2 B_0 + \frac{e}{c}\frac{\Phi}{2\pi}) = 0$$

Finally, putting $r_0 = const(t)$, we obtain a connection between the guide field growth rate on the orbit and the growth rate of average field enclosed by the orbit. The latter generates the electromotive force that accelerates the equilibrium particle.

$$\dot{B}_z\Big|_{\text{orbit}} = \frac{\dot{\Phi}}{2\pi r_0^2} = \frac{1}{2}\dot{\overline{B}}_z\Big|_{\text{average enclosed}}$$

This relation should be kept in the induction accelerator (betatron) for the constant position and curvature radius of the orbit. It was discovered by Wideröe and is also known as the 2 : 1 condition.

In what follows we will solely focus on the **transverse** motion.

From Equations of Motion to Equations of Trajectories

Change of the independent variable in equations, $t \to s$,
where $ds = r_0 d\theta = r_0 \dot{\theta} dt$ is the arc element along the equilibrium orbit, yields:

$$\frac{d}{dt} = r_0 \dot{\theta} \frac{d}{ds} = \frac{r_0}{r}(r\dot{\theta})\frac{d}{ds}, \quad \text{let's denote } \tfrac{d}{ds}(...) = (...)'$$

We have to know $\frac{ds}{dt}$, and we use the invariance of the particle's velocity in a static magnetic field, $|\mathbf{v}| = v = \text{const}$.
In the cylindrical coordinates the exact $\frac{ds}{dt}$ is available, however *a paraxial approximation*, $|z'|, |r'| \ll 1$ gives a sufficient accuracy, in view of linearization as our next step,

$$v = \sqrt{r^2\dot{\theta}^2 + \dot{z}^2 + \dot{x}^2} = \frac{ds}{dt}\sqrt{\frac{r^2}{r_0^2} + z'^2 + r'^2} \approx \frac{r}{r_0}\frac{ds}{dt} + O(p_\perp^2)$$

Thus, in the paraxial approximation we get: $\dfrac{ds}{dt} \approx \dfrac{r_0}{r} v$

Paraxial Approximation

Use this change of variable in the horizontal equation of motion:

$$\frac{ds}{dt}\frac{d}{ds}(\gamma m\frac{ds}{dt}\frac{dr}{ds}) = \gamma m\frac{mr}{r_0^2}\frac{ds}{dt}^2 + \frac{e}{c}\frac{r}{r_0}\frac{ds}{dt}B_z$$

$$(\gamma mv\frac{r_0}{r}r')' = \frac{\gamma mv}{r_0} + \frac{e}{c}\frac{r}{r_0}B_z$$

$$\text{Finally:} \quad \text{r)} \quad (\frac{r'}{r})' = \frac{1}{r_0^2} + \frac{eB_z}{pc}\frac{r}{r_0^2}$$

Similar transformations in the vertical equation of motion give:

$$\text{z)} \quad (\frac{z'}{r})' = -\frac{eB_r}{pc}\frac{r}{r_0^2}$$

Linear Approximation of the Guide Field

Trajectories in the neighborhood of the equilibrium orbit are called **stable** if their deviations from CO remain **limited** in course of motion. Above found positions of equilibrium correspond to motion along the equilibrium orbit.

Stability analysis requires that our equations of trajectories be replaced by their linear approximations. We have to expand the equations w.r.t. small displacements from the equilibrium position, to 1st order in $z, x, \Delta p$, defined by:

$$r = r_0 + x; p = p_0 + \Delta p \,,$$

and approximate the guide field with its power series in small displacements:

$$B_z = B_z(0, r_0) + \frac{\partial B_z}{\partial r}\bigg|_{CO} x + \cdots + \Delta B_z \approx B_0 + Gx + \Delta B_z$$

Here ΔB_z denotes small corrections to the linear terms
(e.g. those due to small azimuthal variations in a realistic sector magnet).
Factor G is commonly called "guide field gradient".

Linearization in Equation of Horizontal Displacements

The term linear in the vertical displacement z was discarded because the symmetry w.r.t. the median plane requires:

$$\left.\frac{\partial B_z}{\partial z}\right|_{CO} = 0$$

Limiting ourselves with the linear approximation and using the relation $p_0 c = -eB_0 r_0$, we obtain:

$$\text{x)} \quad \frac{1}{r_0}\frac{d^2x}{ds^2} \approx \frac{1}{r_0^2} + \frac{e}{p_0 c}(1-\frac{\Delta p}{p_0})(B_0 + Gx + \Delta B_z)\frac{r_0 + x}{r_0^2} \approx \frac{1}{r_0^2} + \frac{eB_0}{p_0 c r_0} +$$

$$+\frac{eB_0 x}{p_0 c r_0^2} + \frac{eGx}{p_0 c r_0} - \frac{eB_0}{p_0 c r_0}\frac{\Delta p}{p_0} + \frac{eB_0}{p_0 c r_0}\frac{\Delta B_z}{B_0} = -(\frac{1}{r_0^2} - \frac{eG}{p_0 c})x + \frac{1}{r_0}(\frac{\Delta p}{p_0} - \frac{\Delta B_z}{B_0})$$

Thus we arrive at the linearized equation of horizontal motion:

$$\text{x)} \quad \frac{d^2x}{ds^2} + (\frac{1}{r_0^2} + \frac{G}{B_0 r_0})x = \frac{1}{r_0}\frac{\Delta p}{p_0} - \frac{\Delta B_z}{B_0 r_0}$$

Linearization in Equation of Horizontal Displacements: Results

The homogeneous equation above results in the stability condition: horizontal displacements are stable (and the solution is oscillatory) if the rigidity K_x of this oscillator is positive.

$$\text{x)} \quad K_x = \frac{1}{r_0^2} + \frac{G}{B_0 r_0} > 0$$

We remind that $r_0, B_0, G = const$. Besides, very slow longitudinal motion allows a rather accurate assumption $\Delta p \approx const$.
Amplitude and phase being available from the initial condition, we can write the solution as oscillation around the equilibrium orbit:

$$\text{x)}\, x(s) = A_x \cos(\sqrt{K_x} s + \phi_x) + \text{partial solution of inhomogeneous equation}$$

Under an unstable condition, $K_x < 0$, displacements grow exponentially.

Linearization in Equation of Vertical Displacements

Now we implement similar transformations in the equation of vertical displacements:

$$B_r = B_r(0, r_0) + \frac{\partial B_r}{\partial z}\Big|_{CO} z + \cdots + \Delta B_r \approx 0 + Gz + \Delta B_r$$

Use has been made of $(\mathrm{rot}\mathbf{B})_\theta = 0$, in a current-free domain, that results in the following relation:

$$\frac{\partial B_r}{\partial z}\Big|_{CO} = \frac{\partial B_z}{\partial r}\Big|_{CO} = G$$

Field symmetry w.r.t. the median plane yields $B_r \equiv 0$ in the orbit plane, $z = 0$, and allows to discard the $x - z$ coupling term in the field expansion,

$$\frac{\partial B_r}{\partial r}\Big|_{CO} x = 0 \; .$$

Thus we obtain for vertical displacements from the equilibrium orbit:

$$\text{z)} \quad \frac{d^2 z}{ds^2} \approx -\frac{e}{p_0 c}\frac{\partial B_r}{\partial z}\Big|_{CO} z - \frac{e\Delta B_r}{p_0 c} = -\frac{eG}{p_0 c} z - \frac{e\Delta B_r}{p_0 c}$$

Linearization in Equation of Vertical Displacements: Results

or:

$$\text{z)} \quad \frac{d^2 z}{ds^2} - \frac{eG}{B_0 r_0} z = \frac{\Delta B_r}{B_0 r_0}$$

Stability of vertical displacements (and oscillatory solution) requires a positive rigidity K_z of the vertical oscillator:

$$\text{z)} \quad K_z = -\frac{G}{B_0 r_0} > 0$$

Similarly to horizontal motion, we can find the vertical oscillation amplitude and phase from the initial condition, then write the stable solution as oscillation around the equilibrium orbit:

z) $z(s) = A_z \cos(\sqrt{K_z} s + \phi_z) +$ partial solution of inhomogeneous equation

Betatron Oscillations. Weak Focusing.

Transverse oscillations around CO are called *betatron oscillations*.
Guide fields that provide simultaneous stability of both transverse degrees of freedom are of special practical interest,

$$(K_x > 0 \;\&\; K_z > 0) \quad \Longrightarrow \quad -\frac{1}{r_0^2} < \frac{G}{B_0 r_0} < 0$$

These conditions for transverse displacements are conventionally called *weak focusing*.

Traditionally, for weak focusing the field gradient is characterized by a dimensionless "field index" n:

$$n = -\frac{r}{B_z}\frac{\partial B_z}{\partial r}\Big|_{CO}$$

Weak Focusing

Being a logarithmic derivative of the bending field, the field index is independent of r when the bending field in the median plane obeys the power law:

$$B_z(0, r) = B_0(\frac{r_0}{r})^n$$

We can express the gradient in terms of n:

$$G = -\frac{n}{r_0^2} B_0 r_0$$

and write the equations for displacements in an azimuthally-symmetric magnet structure (consisting from a single full-turn sector magnet, i.e. with a 360^o bending angle):

$$\text{x)} \quad x'' + \frac{1-n}{r_0^2} x = \frac{1}{r_0} \frac{\Delta p}{p_0} \quad \text{stability for } n < 1$$

$$\text{z)} \quad z'' + \frac{n}{r_0^2} z = 0 \quad \text{stability for } n > 0$$

Limitations of Weak Focusing

Then, simultaneous stability of both transverse degrees of freedom is reached for $0 < n < 1$.
Solutions in an azimuthally-symmetric magnet structure take a simple form:

$$\text{x)} \quad x(s) = A_x \cos(\sqrt{1-n}\frac{s}{r_0} + \phi_x) \qquad \text{z)} \quad z(s) = A_z \cos(\sqrt{n}\frac{s}{r_0} + \phi_z)$$

We define a dimensionless frequency of betatron oscillations ν, conventionally called *the betatron tune*, as a number of trajectory oscillations over the full turn around CO, $s \to s + C$, where C is the CO circumference. In an azimuthally-symmetric magnet structure, $C = 2\pi r_0$.
Evaluating the oscillation phase advance over the full turn around CO and dividing it by 2π, we find the horizontal and vertical betatron tunes:

$$\nu_x = \sqrt{1-n}, \qquad \nu_z = \sqrt{n}, \qquad 0 < \nu_{x,z} < 1$$

Be aware that expression of $K_{x,z}$ in terms of n is meaningless in an important special case of sector magnet with $B_0 = 0$, i.e. in a magnetic quadrupole lens where the equilibrium orbit is straight, $r_0 = \infty$.

Limitations of Weak Focusing and the Idea of Strong Focusing

From the above solutions we can evaluate the trajectory slope:

$$z' = -\frac{A_z \nu_z}{r_0} \sin(\dots)$$

Hence, the spread of transverse momenta accepted in a vacuum tube of limited size $A_{x,z}$ is proportional to $\nu_{x,z}$.
A machine with $\nu_{x,z} \gg 1$, will thus have an advantage of greater accepted transverse momenta and therefore potentially higher beam intensities.
However, in weak-focusing magnet structures betatron tunes are limited,

$$\nu_x^2 + \nu_z^2 \simeq 1\,.$$

Magnet structures with an alternating-sign gradient of a high absolute value (that corresponds to $|n| \gg 1$) are needed for *strong focusing* regime where the betatron tunes are much greater than unity.

General Theory of Linear Betatron Oscillations in Periodic Focusing Lattices

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Outline

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Hill's Equation and Transport Matrix

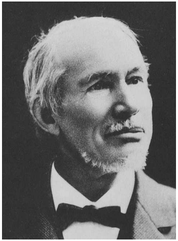

Figure: G. W. Hill (1838–1914)

Consider a linear oscillator with a *periodic* variable rigidity (G.W.Hill, 1886):

$$x'' + K(s)x = 0,$$

here $K(s)$ is an arbitrary periodic function with a period C: $K(s+C) = K(s)$.

Select two linearly independent solutions of this linear equation $\mathcal{C}(s), \mathcal{S}(s)$,

$$\text{with a cosine-like,} \quad \mathcal{C}(s_0) = 1, \mathcal{C}'(s_0) = 0;$$

$$\text{and a sine-like,} \quad \mathcal{S}(s_0) = 0, \mathcal{S}'(s_0) = 1,$$

initial conditions, respectively.

Transfer Matrix

2×2-matrix of transfer $T(s|s_0)$, combined from these functions (and their derivatives)

$$T(s|s_0) = \begin{pmatrix} \mathcal{C}(s) & \mathcal{S}(s) \\ \mathcal{C}'(s) & \mathcal{S}'(s) \end{pmatrix}$$

gives a convenient expression of general solutions of a linear differential equation via the initial conditions: the matrix transforms a column-vector of initial values specified at the initial azimuth $s = s_0$, into the current solution $x(s)$:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = T(s|s_0) \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = \begin{pmatrix} \mathcal{C}(s) & \mathcal{S}(s) \\ \mathcal{C}'(s) & \mathcal{S}'(s) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

The resulting vector may serve as an initial condition for transport through the next element: multiplying by its matrix yields continuation of the current solution through the following element providing the continuity of x and x' at the boundary of the elements.

Properties of Transport Matrices

Therefore, a sequence of element-by-element transformations is represented by successive multiplication of elements' transfer matrices. Notice that the matrices are ordered from right to left:

$$T(s_n|s_0) = T(s_n|s_{n-1}) \cdots T(s_2|s_1)T(s_1|s_0)$$

The transfer matrix $T(s|s_0)$ is also called *transport matrix* of solutions of our equation where the force is independent of x'. Indeed, we ignore any dissipation, moreover the charged particle motion in a static magnetic field is Hamiltonian. This results in invariance of the Wronskian, specifically, $\det T$:

$$\frac{d}{ds}(\det T) = \begin{vmatrix} \mathcal{C}' & \mathcal{S}' \\ \mathcal{C}' & \mathcal{S}' \end{vmatrix} + \begin{vmatrix} \mathcal{C} & \mathcal{S} \\ \mathcal{C}'' & \mathcal{S}'' \end{vmatrix} = 0 + \begin{vmatrix} \mathcal{C} & \mathcal{S} \\ -K\mathcal{C} & -K\mathcal{S} \end{vmatrix} = 0$$

Consequently, $\det T(s|s_0) = const(s)$.
Obviously, $\det T(s_0|s_0) \equiv 1$, hence: $\forall s : \det T(s|s_0) \equiv 1$.

Properties of Transport Matrices

Thus follows a physically important consequence:
transformation of the phase plane x, x' performed by a transport matrix T **is area-preserving**.

Areas are 2-dimensional phase-space volumes,
here we have a particular case of the Liouville theorem.

A proof is easy for area $(x_1 x_2' - x_2 x_1')$ of a parallelogram element spanning vectors $(x_1, x_1'), (x_2, x_2')$ of any two solutions,
then we can generalize for an area of arbitrary figure being divided in such elements.

Examples of Transport Matrices: Constant Focusing

Consider first the simple special case of $K = \mathrm{c}onst$. For $K > 0$, we can take the cosine trajectory, and the sine trajectory,

$$\mathcal{C}(s) = \cos\sqrt{K}s\,,\ \begin{cases} \mathcal{C}(0) = 1 \\ \mathcal{C}'(0) = 0 \end{cases} \qquad \mathcal{S}(s) = \frac{1}{\sqrt{K}}\sin\sqrt{K}s\,,\ \begin{cases} \mathcal{S}(0) = 0 \\ \mathcal{S}'(0) = 1 \end{cases}$$

as a complete set of two linearly independent particular solutions of Hill's equation, which yields simple harmonic oscillations in this special case.

Two found solutions thus form a "focusing" transport matrix:

$$T(s|0) = \begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s \end{pmatrix}$$

Examples of Transport Matrices: Constant Focusing

For $K < 0$, we should replace the above solutions by the respective hyperbolic functions,

$$\mathcal{C}(s) = \cosh\sqrt{-K}s, \begin{cases} \mathcal{C}(0) = 1 \\ \mathcal{C}'(0) = 0 \end{cases} \quad \mathcal{S}(s) = \frac{1}{\sqrt{-K}}\sinh\sqrt{-K}s, \begin{cases} \mathcal{S}(0) = 0 \\ \mathcal{S}'(0) = 1 \end{cases};$$

and this motion is locally unstable, the deviations from the design orbit grow exponentially.

These two found solutions form a "defocusing" transport matrix:

$$T(s|0) = \begin{pmatrix} \cosh\sqrt{-K}s & \frac{1}{\sqrt{-K}}\sinh\sqrt{-K}s \\ \sqrt{-K}\sinh\sqrt{-K}s & \cosh\sqrt{-K}s \end{pmatrix}$$

For a field-free element, $K = 0$, we obtain a transport matrix of a drift space:

$$T(s|0) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Alternating-Gradient Focusing

Aiming at simultaneous stability of horizontal and vertical displacements, we must envisage the focusing function $K(s)$ of both signs.

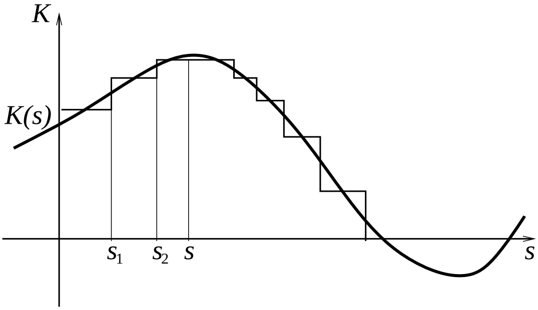

Figure: Approximation of $K(s)$ by step functions.

With a stepwise-constant approximation, the transport matrix at each step is available from previous examples. The total transport is found by their multiplication.

One-Period Matrix and Stability

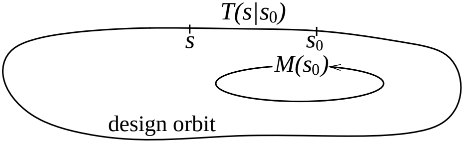

Figure: Transformation by a one-period matrix $M(s_0)$.

Introduce a one-period transfer matrix $M(s)$,

$$M(s) = T(s + C|s),$$

For stable motion, we should have limited values of x and x' when applying M repeatedly to any initial condition.
Transport over N periods is given by M^N, therefore stability requires that eigenvalues λ of M must be limited, $|\lambda| \leq 1$.
Otherwise, with $|\lambda| > 1$, λ^N means a possibility of unlimited growth of displacements.
Condition $\det M = 1$ means $\lambda_1 \lambda_2 = 1$
hence the stability condition is reduced to $|\lambda_{1,2}| = 1$.

One-Period Matrix and Stability

Rewriting M via its matrix elements,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

we find the eigenvalues of M from the characteristic equation

$$\det(M - \lambda I) = 0,$$

or, explicitly,

$$\begin{vmatrix} m_{11} - \lambda & m_{12} \\ m_{21} & m_{22} - \lambda \end{vmatrix} = \lambda^2 - (m_{11} + m_{22})\lambda + \det M = 0.$$

Using $\det M = 1$ and denoting the trace of M, $m_{11} + m_{22} = \mathrm{t}r M$, we solve this equation for λ,

$$\lambda_{1,2} = \frac{1}{2}\mathrm{t}r M \pm i\sqrt{1 - \left(\frac{1}{2}\mathrm{t}r M\right)^2} \equiv \cos\mu \pm i \sin\mu = e^{\pm i\mu},$$

where $\cos\mu = \frac{1}{2}\mathrm{t}r M$ determines the phase advance μ.

Stability Condition of Hill's Equation

Finally, the stability condition $|\cos\mu| \leq 1$ requires that complex eigenvalues of M belong to the circle of unity radius.
The stability condition can be expressed in terms of matrix M,

$$-2 \leq \mathrm{t}rM \leq 2\,.$$

Stable solutions of Hill's equation are called *betatron oscillations*, and the meaning of parameter μ is the phase advance of these oscillations over one period of the focusing structure.

Twiss Parameters

The one-period matrix M may be rewritten via Twiss parameters,

$$M = I\cos\mu + J\sin\mu = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}.$$

Among the matrix elements of J,

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix},$$

there are only two independent parameters, since the relation $\det M = 1$ bounds these matrix elements, $\gamma\beta - \alpha^2 = 1$, or $\det J = 1$. Hence, $J^2 = -I$, and the matrix exponent form of M follows, $M = \exp(\mu J)$. The matrix elements of $M(s)$ are apparently periodic functions of s, $M(s + C) = M(s)$, and so are the Twiss functions $\beta(s)$, $\alpha(s)$ and $\gamma(s)$. Provided $M(s_0)$ is known, transformation to another point s is given by the transfer matrix $T(s|s_0)$,

$$M(s) = T\,M(s_0)\,T^{-1}.$$

Twiss Parameters

To derive differential equations for the Twiss parameters, we rewrite Hill's equation as a set of 1st-order equations

$$X' = \frac{d}{ds}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = DX \quad \text{where } D = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}.$$

The differential equation for the transfer matrix T has the same form.

$$\frac{d}{ds}T = \begin{pmatrix} \mathcal{C}' & \mathcal{S}' \\ \mathcal{C}'' & \mathcal{S}'' \end{pmatrix} = \begin{pmatrix} \mathcal{C}' & \mathcal{S}' \\ -K\mathcal{C} & -K\mathcal{S} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}\begin{pmatrix} \mathcal{C} & \mathcal{S} \\ \mathcal{C}' & \mathcal{S}' \end{pmatrix},$$

$$\text{or} \qquad T' = D\,T\,.$$

The differential equation for a one-period matrix $M(s) = T(s + C|s)$ is quite different. We start from

$$M = M(s) = T\,M_0\,T^{-1}, \qquad \text{and rewrite it as} \quad M\,T = T\,M_0\,,$$

then differentiate with respect to s,

$$M'\,T + M\,T' = T'\,M_0\,.$$

Differential Equations for Twiss Parameters

Using $T' = D\,T$ we arrive at

$$M' = DM - MD\,.$$

Substituting here the Twiss form of M, we find a set of differential equations for the Twiss functions,

$$\beta' = -2\alpha\,, \qquad \alpha' = K\beta - \gamma\,, \qquad \gamma' = 2K\alpha\,.$$

Elimination of α and substitution of $\gamma = (1+\alpha^2)/\beta$ yields a rather cumbersome equation for the β-function alone,

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K\beta^2 = 1\,.$$

These equations should be solved with periodic boundary conditions, since the Twiss functions are periodic.
For $w(s) = \sqrt{\beta(s)}$, we get a nice equation,

$$w'' + Kw = \frac{1}{w^3}, \qquad \text{again with periodic boundary conditions.}$$

Eigenvectors of $M(s)$

Now we find a pair of eigenvectors $F^T = (f, f')$ of the one-period matrix $M(s)$, using its Twiss form and knowing its eigenvalues $\lambda_{1,2} = e^{\pm i\mu}$. From $MF = e^{\pm i\mu} F$,

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \begin{pmatrix} f_\pm \\ f'_\pm \end{pmatrix} = e^{\pm i\mu} \begin{pmatrix} f_\pm \\ f'_\pm \end{pmatrix},$$

$$\text{we have} \qquad \frac{f'_\pm}{f_\pm} = \frac{\pm i - \alpha}{\beta}.$$

Using $\alpha = -\beta'/2$ we come to a differential equation $\quad \dfrac{f'_\pm}{f_\pm} = \dfrac{\beta'}{2\beta} \pm \dfrac{i}{\beta}.$

Integration yields a fundamental relation of the eigenvector to the β-function,

$$f_\pm(s) = f_0\sqrt{\beta(s)}\exp\left[\pm i\int^s \frac{ds'}{\beta(s')}\right],$$

where f_0 is the integration constant. Using freedom of normalization, we choose $f_0 = 1$.

Eigenvectors of $M(s)$. The Floquet Theorem

We get the complex-conjugate pair of eigenvectors,

$$\begin{pmatrix} \beta \\ \pm i - \alpha \end{pmatrix} \frac{e^{\pm i\psi}}{\sqrt{\beta}}, \quad \psi = \int^s \frac{ds'}{\beta(s')}.$$

We can also write this complex-conjugate pair of normalized eigenvectors F, F^* in terms of w and w',

$$F = \begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} w \\ w' + i/w \end{pmatrix} e^{i\psi}, \quad \psi' = \frac{1}{w^2}.$$

The above derivation provides for the proof of the **Floquet Theorem**: For Hill's equation

$$x'' + K(s)\,x = 0 \qquad K(s+C) = K(s),$$

there exist normal solutions $f(s)$, $f'' + K(s)\,f = 0$, for which advance by one period means multiplication by a phase factor,

$$f(s+C) = e^{i\mu} f(s)\,.$$

Eigenvectors of $M(s)$ and the Floquet Theorem

Indeed, the above constructed eigenvectors of M, with $f(s) = w(s)e^{i\psi(s)}$, whose absolute value is a periodic function, are transformed by M when advanced by one period, and this transformation is reduced to multiplication by the eigenvalue $e^{i\mu}$ of M,

$$\begin{pmatrix} f \\ f' \end{pmatrix}_{s+C} = M \begin{pmatrix} f \\ f' \end{pmatrix}_s = e^{i\mu} \begin{pmatrix} f \\ f' \end{pmatrix}_s .$$

Moreover, the phase advance is related to the amplitude function w,

$$\psi(s+C) - \psi(s) = \mu = \oint d\psi = \oint \frac{ds}{w^2} .$$

The normal solutions $f(s) = w(s)e^{i\psi(s)}$ are often called Floquet functions. Any solution of Hill's equation can be decomposed in this basis,

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \frac{A}{2} \begin{pmatrix} f \\ f' \end{pmatrix} + \frac{A^*}{2} \begin{pmatrix} f^* \\ f^{*\prime} \end{pmatrix} = \mathrm{Re}[AF] .$$

Eigenvectors of $M(s)$ and Courant-Snyder Invariant

"Orthonormality" of this special basis is expressed in Wronskian form, it is suited for the phase-space geometry,

$$\begin{vmatrix} f & f^* \\ f' & f^{*\prime} \end{vmatrix} = e^{i\psi} e^{-i\psi} \begin{vmatrix} w & w \\ w' + i/w & w' - i/w \end{vmatrix} = -2i\,,$$

These relations help to find the decomposition constant A

$$A = -ie^{-i\psi} \begin{vmatrix} w & x \\ w' - i/w & x' \end{vmatrix} .$$

The Courant-Snyder invariant then follows from the fact that A is a constant determined by the initial conditions of the trajectory, and independent of s,

$$|A|^2 = (wx' - w'x)^2 + \frac{x^2}{w^2} = \gamma x^2 + 2\alpha xx' + \beta x'^2 \equiv \epsilon. \tag{1}$$

When the solution $x(s)$ is propagated in an AG focusing lattice, the quadratic form remains constant because of appropriate variation of the Twiss functions. The physical meaning of this invariant is that it is proportional to the action variable in the particle motion.

Pseudo-Harmonic Oscillations

Using expression for the displacement $x(s)$ via the eigenvectors of $M(s)$ we arrive at the pseudo-harmonic form of the solutions to Hill's equation,

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\psi(s)\,, \quad x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\Big(\sin\psi(s) + \alpha(s)\cos\psi(s)\Big)\,.$$

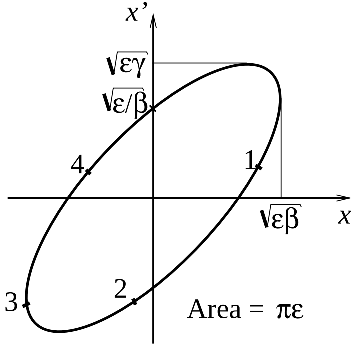

Figure: Elliptic phase-space trajectory of the betatron oscillation.

Pseudo-Harmonic Oscillations

The Courant-Snyder quadratic form defines the ellipse with area $\pi\epsilon$. The phase-space ellipse illustrates the meaning of the Twiss parameters. Being a locus of points 1, 2, ... representing one-period mapping, the ellipse is often called a phase-space trajectory of the betatron oscillation. We can see that $w(s) = \sqrt{\beta(s)}$ is the envelope function enclosing all betatron trajectories with given $|A|$.

The pseudo-harmonic oscillation is related to the simple harmonic oscillation by a linear transformation:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} \sqrt{\epsilon}\cos\psi \\ -\sqrt{\epsilon}\sin\psi \end{pmatrix}.$$

and by the change of independent variable from s to ψ. The new variables are called the **normalized variables**.

Conclusion and References

A versatile formalism is available (in different forms)
to fully support linear lattice analysis and
to simplify the formulation of nonlinear dynamics problems.

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