Linear Imperfections Measurements and Correction in Storage Rings Part II

Contents: Part I

- Closed Orbit Perturbations:
  - Sources
  - Corrections

Part II

- Optics Perturbation:
  - Gradient errors, detection and correction
  - Linear coupling: sources, detection and correction

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Focusing errors and Optics Measurements

We recall the equation for the β function used to "disentangle" the motion of the single particles from the machine optics:

$$\frac{1}{2}\beta_z\beta_z'' - \frac{1}{4}\beta_z'^2 + \beta_z^2 K_z = 1 \qquad (z = x, y)$$

with

$$K_x \equiv \left(\frac{1}{\rho^2} + K\right) \qquad \text{and} \qquad K_y \equiv -K$$

and

$$K(s) \equiv \frac{e}{p}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0}$$

Errors in the machine focusing elements (gradient errors in quadrupoles, but also *feed-down* effects) lead to a β perturbation.

A direct way [a] for finding the equation describing the perturbation in first approximation consists in

- writing $\boldsymbol{K} = \boldsymbol{K_0} + \boldsymbol{\Delta K}$ and $\boldsymbol{\beta} = \boldsymbol{\beta_0} + \boldsymbol{\Delta\beta}$;
- inserting those expressions in the $\boldsymbol{\beta}$ function equation;
- recognizing that $\boldsymbol{\beta_0}$ is the solution to the unperturbed equation;
- keeping only the linear terms in $\boldsymbol{\Delta K}$ and $\boldsymbol{\Delta\beta}$;
- using $\boldsymbol{\beta_0\beta_0'' + 2\beta_0^2 K_0 = 2(1 + \beta_0'^2/4)}$ (from the unperturbed equation);
- writing the derivatives in terms of $\boldsymbol{\phi}$ (see Part I).

[a] In the literature it is in general obtained by introducing a thin lens perturbation in the one turn transport matrix.

The $\Delta\beta$ equation, which looks awful when s is used, takes a simple and enlightening form when $\Delta\beta/\beta_0$ and ϕ are used

$$\frac{d^2}{d\phi^2}\left(\frac{\Delta\beta}{\beta_0}\right) + 4Q^2\left(\frac{\Delta\beta}{\beta_0}\right) = -2Q^2\beta_0^2\Delta\bar{K}(\phi) \qquad \phi \equiv \mu/Q$$

It has the same form as the equation for $\eta \equiv z/\sqrt{\beta}$ (see Part I) with

$$Q \rightarrow 2Q$$

and

$$Q^2\beta^{3/2}f(\phi) \rightarrow -2Q^2\beta_0^2\Delta\bar{K}(\phi)$$

and we can use the results found for η here. For an integrated gradient error, $\Delta K\ell$, at $s = s_k$

$$\frac{\Delta\beta}{\beta_0}(s) = -\frac{1}{2\sin(2\pi Q)}\beta_k \cos[2Q\pi - 2|\mu(s) - \mu_k|]\Delta K\ell$$

$\Delta\beta/\beta_0(s)$ (beta-beat)

- oscillates with *twice* the betatron frequency and is thus sensitive to error harmonics near to $2Q \rightarrow$ true β-beat is reach in harmonics close to $2Q$;
- is large when Q approaches a *half integer*.

The change of Q due the quadrupole error can be found

$$Q = \frac{1}{2\pi}\oint\frac{ds}{\beta} \simeq Q_0 - \frac{1}{2\pi}\oint\frac{ds}{\beta_0}\frac{\Delta\beta}{\beta_0} = Q_0 - \frac{1}{2\pi}\oint d\mu\frac{\Delta\beta}{\beta_0}$$

$$\oint d\mu\frac{\Delta\beta}{\beta_0} = \frac{\beta_k \Delta K \ell}{2\sin(2\pi Q)}\oint d\mu \cos[2Q\pi - 2|\mu(s) - \mu_k|] = \frac{1}{2}\beta_k\Delta K\ell$$

$$\Delta Q = \frac{1}{4\pi}\beta_k\Delta K\ell$$

By the way, we can make use of this result to easily find the equation of the (linear) chromaticity. Therefore let's make an excursus on chromaticity...

Particles with different momentum wrt the nominal one, experience a different force:

$$K = \frac{e}{p_0 + \Delta p} \frac{\partial B_x}{\partial y}\bigg|_{x=y=0} \simeq K_0 - \frac{\Delta p}{p_0} K_0$$

$$\Delta K = -\frac{\Delta p}{p_0} K_0$$

The tune change due to a single quadrupole is

$$(\Delta Q)_k = \frac{1}{4\pi} \beta_k \Delta K \ell = -\frac{1}{4\pi} \beta_k \frac{\Delta p}{p_0} K_0 \ell$$

and by integrating over the machine length we get the total tune change

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} \oint ds \beta K_0$$

Linear chromaticity:

$$\xi \equiv \frac{\Delta Q}{\Delta p / p_0} = -\frac{1}{4\pi} \oint ds \beta K_0$$

- The chromaticity is large in large machines.
- Strong quadrupoles at large β function values are the main contributors! Future Circular Collider $e^{+}e^{-}$ ring

Optics		ξ_x	ξ_y
45 GeV	all sexts off	-361	-1540
	IR setxs off	+3.5	-1230
80 GeV	all sexts off	-359	-1331
	IR setxs off	+3	-1017

First small accelerators could live w/o chromaticity correction!

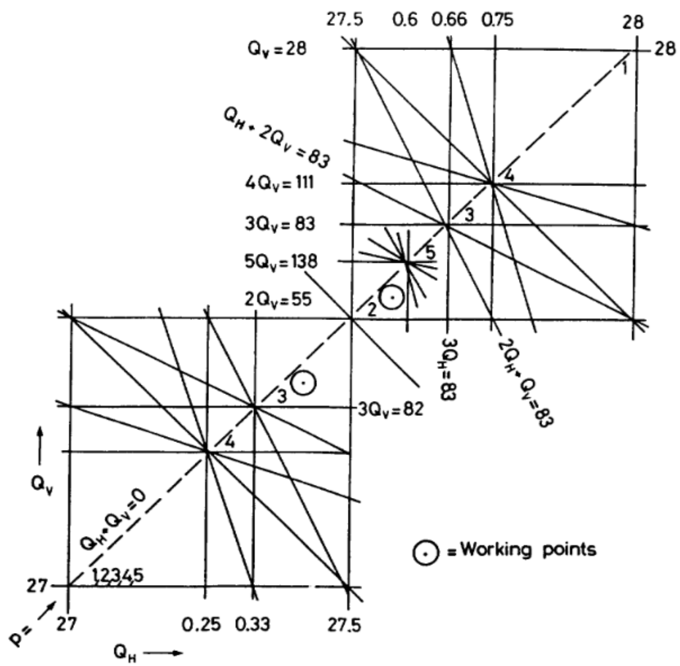
First time it was realized that a correction was needed was during the commissioning of the Fermilab Main Ring in 1971.

Chromaticty leads to a spread of the particle tunes. The particles which tune lies on a resonance may be lost[a]. Resonance condition

$$n_x Q_x + n_y Q_y = p$$

with n_z integer. Resonance order: $|n_x| + |n_y|$.

SPS working point diagram (from E. Wilson, CAS 1984) with resonances up to 4th order



[a] depending whether there are field driving the resonance.

For correcting the chromaticity we need a quadrupole which strength depends linearly on momentum. Sextupole magnets placed in locations where the horizontal dispersion $D_x \neq 0$ are exactly that

$$B_x = Sxy = S(D_x \frac{\Delta p}{p_0} + x_\beta)y_\beta = SD_x \frac{\Delta p}{p_0}\, y_\beta + Sx_\beta y_\beta$$
$$B_y = \frac{1}{2}S(x^2 - y^2) \simeq SD_x \frac{\Delta p}{p_0} x_\beta + \frac{1}{2}S(x_\beta^2 - y_\beta^2)$$

The simplest correction scheme consists in placing sextupoles in the arcs, where $D_x \neq 0$,

$$\Delta\xi_x = \frac{1}{4\pi} \sum_{i=1}^{NS} \beta_{x,i} D_{x,i} S_i \ell_i$$
$$\Delta\xi_y = \frac{1}{4\pi} \sum_{i=1}^{NS} \beta_{y,i} D_{x,i} S_i \ell_i$$

Arranging the sextupoles into two families, the values of their strength, S_F and S_D, for a given $\Delta\xi_x$ and $\Delta\xi_y$ are obtained by inverting a system of two equations

$$\begin{pmatrix} \Delta\xi_x \\ \Delta\xi_y \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} S_F \\ S_D \end{pmatrix}$$

For minimizing the strengths is convenient to place the sextupoles at location where D_x is large and $\beta_z^2 >> \beta_x\beta_y$ so that the corrections are (almost) orthogonal.

Remarks:

- Sextupoles introduce non-linearities and drive 3th order resonances unless attention is paid to the phase advance $\boldsymbol{\Delta\mu_z}$ between them.
- Colliders with very low β^* need a sophisticated *local correction* of the chromaticity generated by the IR quadrupoles.

End excursus on chromaticity

Back to the β-beat. Errors in the focusing structure have bad consequences

- *unpredictable response* to any machine parameter change;
- *uncontrolled beam size* with consequences on aperture, luminosity, beams separation (when two counter-rotating beams share the same vacuum chamber).

The β function value at a quadrupole location may be evaluated by changing its current and measuring the tune change

$$\beta = 4\pi \frac{\Delta Q}{\Delta K \ell}$$

This is a good old method, but

- it requires *independently* powered quadrupoles (or trims): this may be the case for the IR quads;
- results are affected by
  - orbit perturbations if the beam is *off-center* at the quadrupole;
  - magnet *histeresys*;
- the quadrupole *calibration* must be well known.

Today BPM systems allow sophisticated techniques and several methods for measuring the linear optics and fitting measurement to a model have been developed in the last years.

Two main *philosophies*:

- Closed Orbit response to the excitation of correctors;
- Analisys of beam oscillations excited by a single kick or AC dipoles (TBT analysis); data acquisition is fast.

Orbit Response Matrix

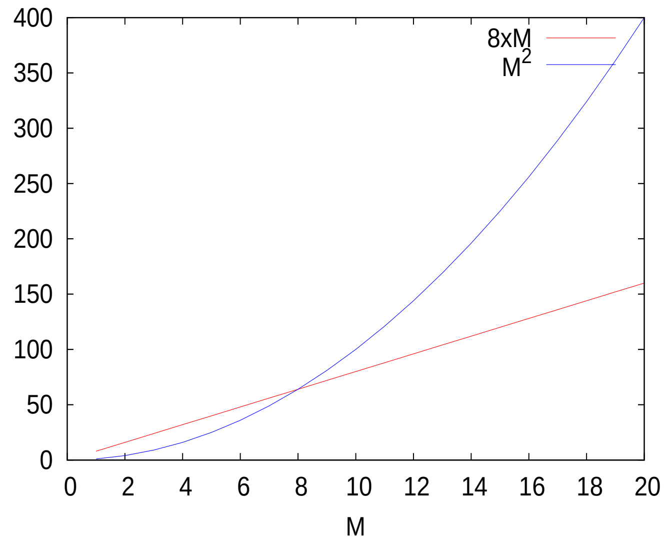
Orbit change due to the jth corrector at the ith BPM (w/o coupling):

$$\delta z_i = T_{ij}\Theta_j = \frac{1}{2\sin(\pi Q)}\sqrt{\beta_i^m \beta_j^c}\cos\left(Q\pi - |\mu_i^m - \mu_j^c|\right)$$

with $z = x$ or y. The response is proportional to β values both at corrector and monitor position as well it depends on the phase advance between them. In presence of errors the *actual* Twiss parameters may be determined by *measuring* the actual orbit response matrix.

By powering one corrector and reading its effect at *all* M BPMs one get M conditions and 2$\times M$ (ie β_i^m and μ_i^m)+ 2 unknowns (ie β_j^c and μ_j^c).
By using *all* N correctors the number of unknown parameters increases to 2$\times M$ + 2$\times N$ but the number of constraints becomes $M \times N$. The number of unknown parameters increases by 2$\times M$+ 2$\times N$ if also BPM and correctors roll angles and calibrations are considered.

To have a constrained system, assuming $M \simeq N$ it must be $M^2 \geq 8 \times M$ ie $M \geq 8$



The (usually) large number of constraints allows to compute accurately the unknown parameters at BPMs and correctors by "simple" computations.

The equations being non-linear in the unknown parameters, some iterations may be needed.

One can do more by attempting to *change* the machine model so to fit the measured orbit changes. In general (eventually coupled machine)

$$\vec{Z} = M^{meas}\vec{\Theta} \qquad \text{with } \vec{Z} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{\Theta} \equiv \begin{pmatrix} \Theta_x \\ \Theta_y \end{pmatrix}$$

M_{ij}^{meas} being the measured beam position at the i^{th} BPM due to a unitary kick at the j^{th} corrector. One can *compute* the response matrix, M^{mod}, for the theoretical optics, by using any (coupled motion handling) optics code. Machine parameters as quadrupole gradients, roll angles etc., as well as gauge factors and roll angles of BPMs and correctors are varied so to minimize the difference between the model matrix and the measured one

$$\chi^2 = \Sigma_{ij} \frac{[M_{ij}^{mod} - M_{ij}^{meas}]^2}{\sigma_i^2} \qquad \sigma_i \equiv \text{BPMs rms noise}$$

smaller weight for noisy BPMs

Two ways for solving in practice the problem

- CALIF orginal algorithm by Corbett, Lee and Ziemann (SLAC) uses a first-order perturbation

$$M_{ij}^{mod} = M_{ij}^{mod,0} + \Sigma_q \frac{\partial M_{ij}^{mod,0}}{\partial K_q} \Delta K_q$$

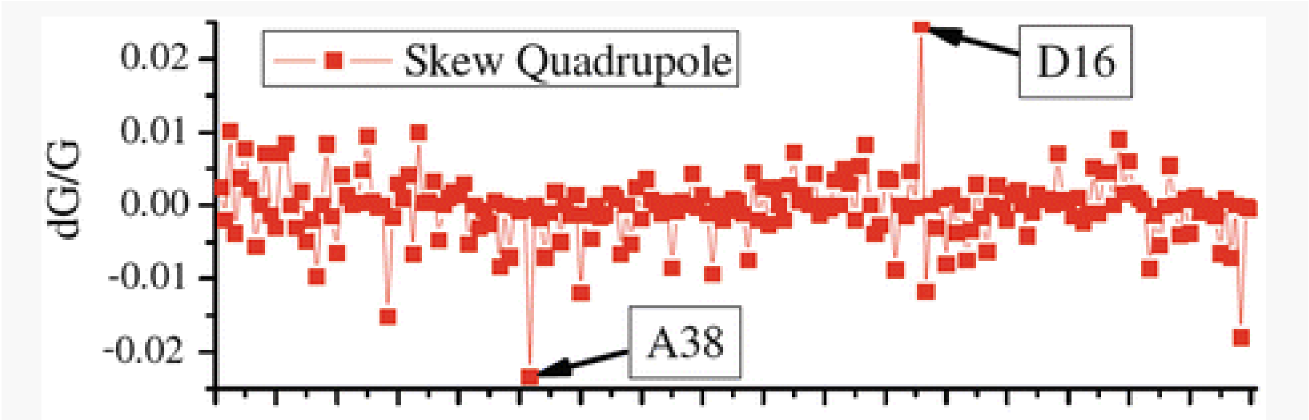
  with $M_{ij}^{mod,0}$ and its derivatives fixed.

- LOCO (Linear Optics from Closed Orbits) by Safranek (BNL) iterates the above procedure recomputing $M_{ij}^{mod,0}$ and derivatives at each step. It is slower but more accurate.

Due to the large number of equations involved the SVD is the best suited method for finding the mathematical solution.

Ideally the lower limit to the difference between model-expected orbit and measured orbit is the BPMs resolution. These techniques were first developed for *small* machines as SPEAR (SLAC) and the NSLS X-Ray Ring (BNL). The largest machine where LOCO has been (routinely) applied is Tevatron, by using a sub-set of correctors.

Relative skew-quadrupole errors fitted by LOCO for Tevatron show large skew quadrupoles at A38 and D16.



It was found that the quadrupoles at A38 and D16 were erroneously assembled with a large roll angle undetectable by alignment measurements outside of the magnet.

- LOCO can only resolve relative BPMs and corrector calibration errors (ie non systematic).
- Large kicks increase signal-to-noise ratio, but introduce large systematic errors (nonlinearities): common problem.
- For large machine one can divide the ring into sections to be analyzed and corrected separately (like open beam lines) with fixed conditions at the boundaries.

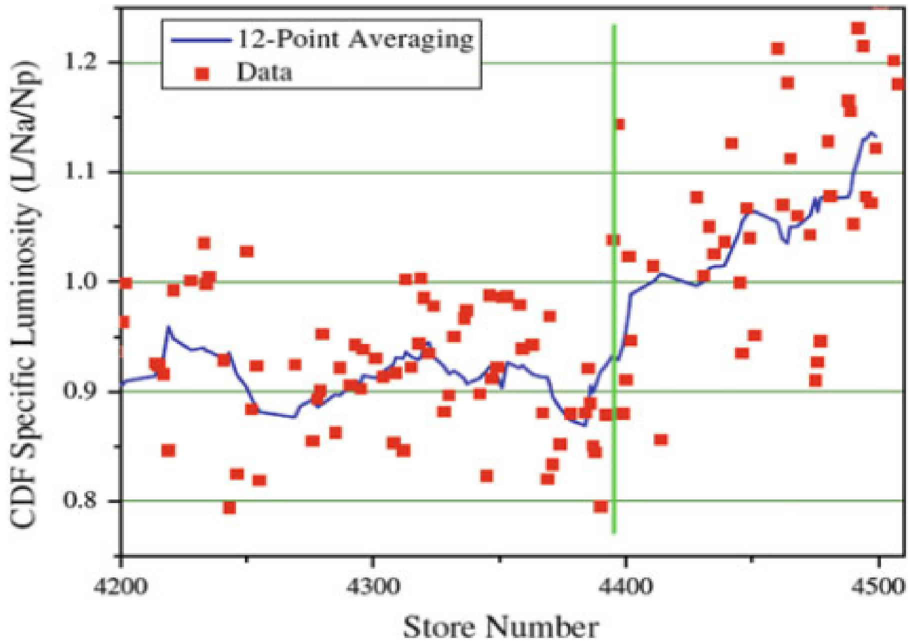
In some circumstances

- large machines where no all quadrupoles may be include in a global fit

rather then the *true* errors, an *equivalent* model may be found which correction may still improve the machine performance.

After correcting the Tevatron optics on the basis of LOCO analysis, it was possible to

- reduce the β-beating;
- correct the discrepancy in the values of β^* between the two experiments, D0 and CDF;
- load a new optics with lower β^* increasing the luminosity

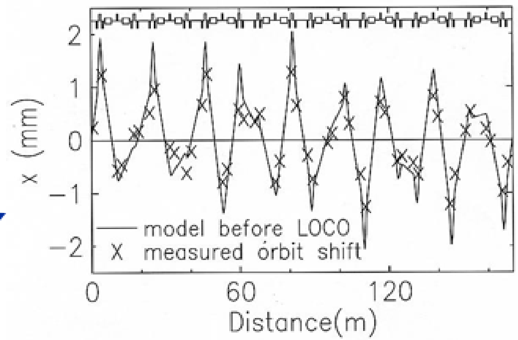


↑ New optics

LOCO allowed to reach a $\Delta\beta/\beta \approx$ 1-2% at the BNL National Synchrotron Light Source (NSLS2) and is currently used in many synchrotron light sources.

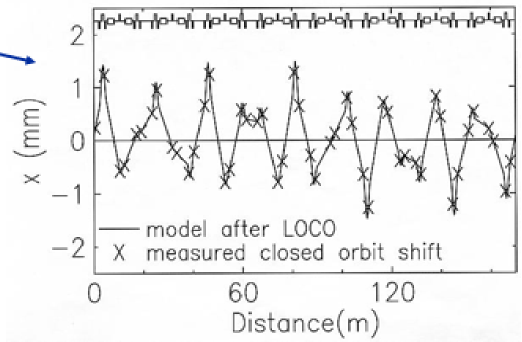
LOCO fit for NSLS X-Ray Ring

Before fit, measured and model response matrices agree to within ~20%.



After fit, response matrices agree to 10^{-3}.

$(M_{meas}-M_{model})_{rms} = 1.17\ \mu m$



Action and phase jump

In very large machine LOCO may be unpractical. For detecting optics errors the Action and phase jump has been developed at the BNL Relativistic Heavy Ion Collider. Idea

- Use a pair of BPMs upstream and a pair downstream the Point of Interest for computing (z_1, z_1') upstream and (z_2, z_2') downstream.
- The trajectory upstreams is given by $z = A_1\sqrt{\beta}\cos(\mu + \delta_1)$, the trajectory downstream is given by $z = A_2\sqrt{\beta}\cos(\mu + \delta_2)$ with constant of motion determined by the measured coordinates:
  - if there are no errors in between s_1 and s_2, it will be $A_1 = A_2$ and $\delta_1 = \delta_2$,
  - the constant of motion will differ if there is an optic error instead.

Assumption:

- no errors between the BPMs of each pair, which allows to use the unperturbed optics (although the Twiss functions *are* perturbed by errors outside).

Evaluation of the error strength: the transport matrix from s_1 to s_2 is

$$M_{(s_1 \to s_2)} = M_{0(s_k \to s_2)} m M_{0(s_1 \to s_k)}$$

with (thin lens quadrupole)

$$m = \begin{pmatrix} 1 & 0 \\ -\Delta K \ell & 1 \end{pmatrix}$$

$$M_{0(s_k \to s_2)} m M_{0(s_1 \to s_k)} \vec{z}_1 = \vec{z}_2 \qquad m M_{0(s_1 \to s_k)} \vec{z}_1 = M^{-1}_{0(s_k \to s_2)} \vec{z}_2$$

$$\Delta K \ell = \frac{1}{(M_{0(s_1 \to s_k)} \vec{z}_1)_1} \left[(M_{0(s_1 \to s_k)} \vec{z}_1)_2 - (M^{-1}_{0(s_k \to s_2)} \vec{z}_2)_2 \right]$$

or by writing explicitily the trajectory at s_2 after the perturbation as the superposition of the unperturbed and perturbed trajectories

$$z_2 = A_2 \sqrt{\beta_2} \cos(\mu_2 + \delta_2) = A_1 \sqrt{\beta_2} \cos(\mu_2 + \delta_1) + \sqrt{\beta_k} \Theta_k \sqrt{\beta_2} \sin(\mu_2 - \mu_k)]$$

with $\Delta K \ell = \Theta_k / z_k = \Theta_k A_1 \sqrt{\beta_k} \cos(\mu_k + \delta_1)$

RHIC simulation of action and phase jump using one turn trajectory:

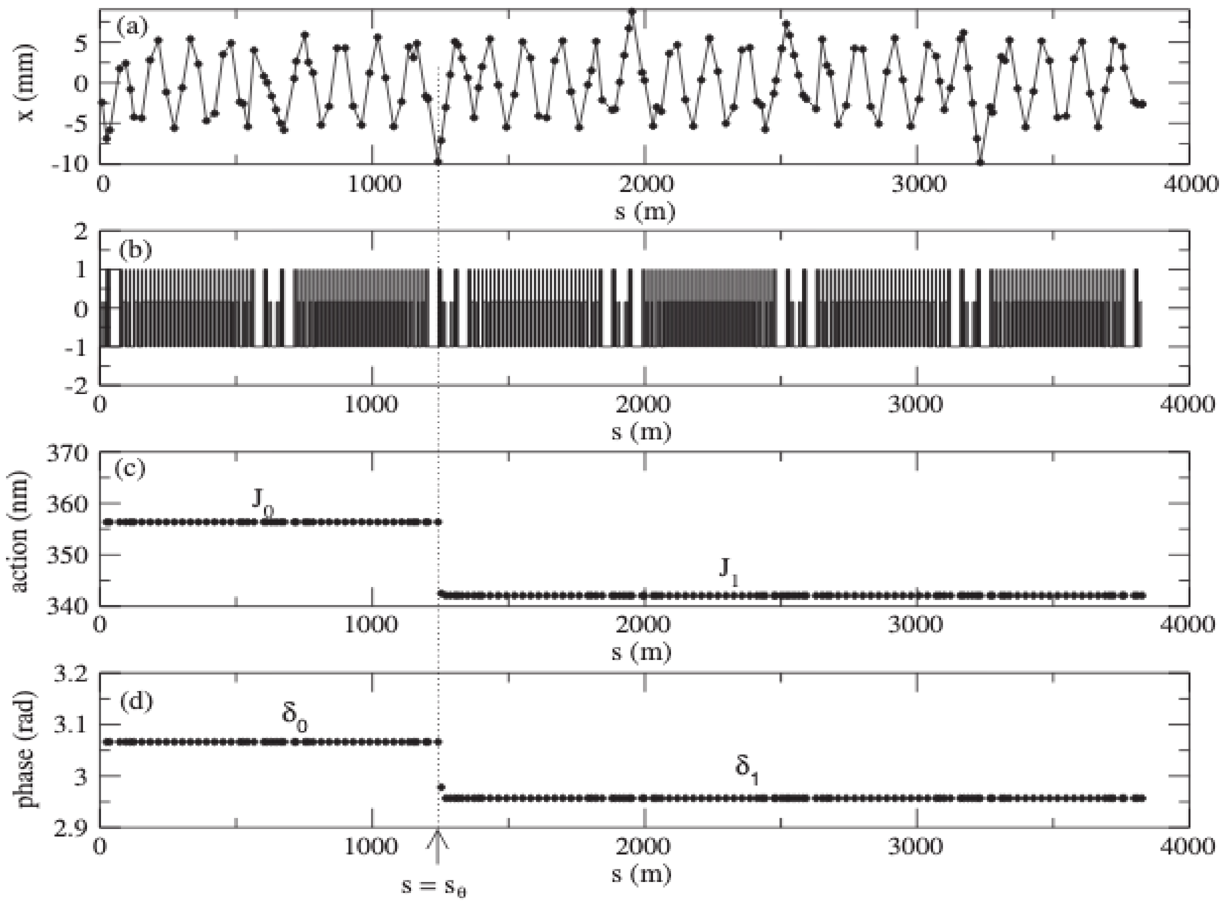


FIG. 1. (a) Simulation of a RHIC first turn trajectory with every dot representing a BPM measurement, in the presence of a magnetic kick with strength θ_z at $s_\theta = 1241$ m. (b) RHIC optics with short bars representing dipoles and long bars representing quadrupoles. (c) Action analysis of the simulated first turn trajectory. (d) Phase analysis of the same trajectory.

(from J. Cardona and S. Peggs, PRST 12, 2009)

RHIC 2002 measurement of action and phase jump using TBT turn data:

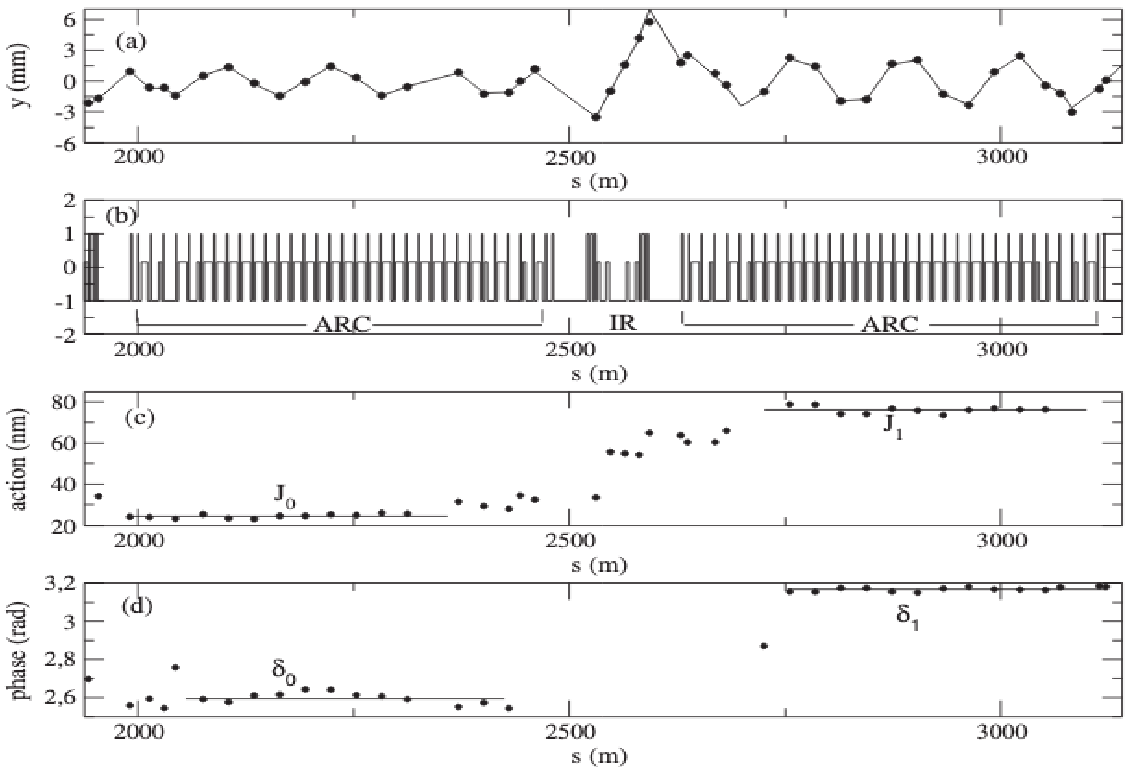


FIG. 8. (a) One of the first difference orbits obtained from beam in RHIC in the year 2000. Each dot represents a BPM measurement. (b) Lattice representation of RHIC. (c) Action plot obtained by applying Eq. (13) to all pairs of adjacent BPM measurements in the ring. (d) Phase plot obtained by applying Eq. (14) to all pairs of adjacent BPM measurements in the ring.

(from J. Cardona and S. Peggs, PRST 12, 2009)

The APJ analysis reveals a large discontinuity at the IR where β is large at the IR quadrupoles.

Beam position at the j[th] BPM after n turns following a single kick in the z plane $(z \equiv x, y)$

$$z_{nj} = A\sqrt{\beta_j}\cos\left(\mu_j + \delta_0 + 2\pi Q n\right) = A\sqrt{\beta_j}\cos\left(\mu_j + \delta_0 + \frac{2\pi}{P}\tilde{Q}n\right)$$

with A and δ_0 constant of motion, P total number of turns and $\tilde{Q} \equiv QP$.

Assuming that P is large so that $\tilde{Q}$ can be approximated by an integer, $z_j(\alpha)$ is a periodic function between 0 and 2π sampled at P equidistant points $\alpha_n = n2\pi/P$ and we can expand it in a Fourier series. qth Fourier coefficient of $z_j(\alpha)$:

$$Z_j(q) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, z_j(\alpha)\mathrm{e}^{-iq\alpha} = \frac{1}{P}\sum_{n=1}^{P} z_{nj}\mathrm{e}^{-iq\alpha_n} \qquad (\alpha_n \equiv n\frac{2\pi}{P})$$

$$= \frac{1}{P}A\sqrt{\beta_j}\Big\{\cos\left(\mu_j + \delta_0\right)\sum_{n=1}^{P}\Big[\cos\tilde{Q}\alpha_n \cos q\alpha_n - i\cos\tilde{Q}\alpha_n \sin q\alpha_n\Big] -$$

$$\sin\left(\mu_j + \delta_0\right)\sum_{n=1}^{P}\Big[\sin\tilde{Q}\alpha_n \cos q\alpha_n - i\sin\tilde{Q}\alpha_n \sin q\alpha_n\Big]\Big\}$$

which is non-vanishing only for $q = \tilde{Q}$. For $q = \tilde{Q} \neq 0$ and $\neq P/2$ it is

$$Z_j(\tilde{Q}) = \frac{A}{2}\sqrt{\beta_j}\Big[\cos(\mu_j + \delta_0) + i\sin(\mu_j + \delta_0)\Big]$$

Twiss functions:

$$\beta_j = 4|Z_j(\tilde{Q})|^2/A^2 \qquad\qquad \mu_j = \arctan\frac{\Im[Z_j(\tilde{Q})]}{\Re[Z_j(\tilde{Q})]} - \delta_0$$

While the phase advance μ is always defined apart from an additive constant, to get the numerical value of β_j we need to know A:

- it depends on the kick amplitude: $A = \sqrt{\beta_k}\Theta_k$
  - β_k is in principle unknown
  - the kicker calibration is only approximately known

Under the reasonable assumption that the tune

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta} = \frac{L}{2\pi} < \frac{1}{\beta} >$$

is adjusted to the design value it is $<1/\beta>=<1/\beta_0>$ and we may compute A as:

$$\beta_j = 4\frac{|Z_j(\tilde{Q})|^2}{A^2} \quad \rightarrow \quad \sum_j \frac{1}{\beta_j} = A^2 \sum_j \frac{1}{4|Z_j(\tilde{Q})|^2}$$

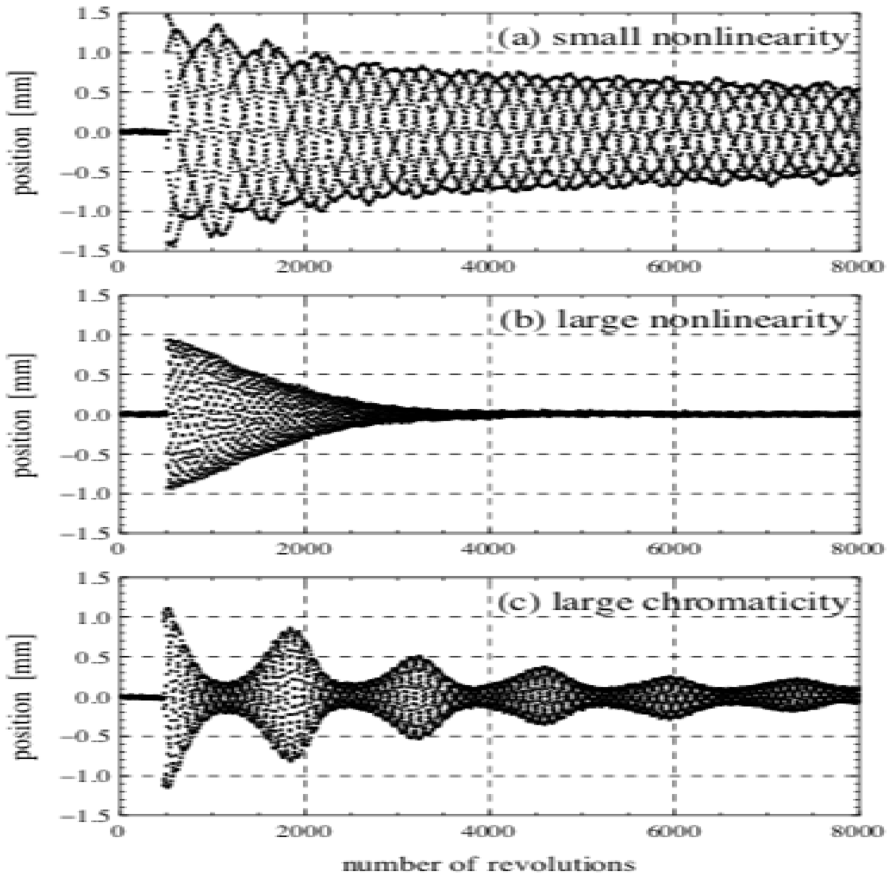
$$A^2 = \sum_j \frac{1}{\beta_j} / \sum_j \frac{1}{4|Z_j(\tilde{Q})|^2} \simeq \sum_j \frac{1}{\beta_{0j}} / \sum_j \frac{1}{4|Z_j(\tilde{Q})|^2}$$

A systematic error for instance in the BPMs or kicker calibration is absorbed by the constant A and does not affect the knowledge of β.

An unbiased value of β can be obtained from the phase advance by using 3 consecutive BPMs under the assumption that there are no optics errors between them. For this purpose one uses the element $M_{12}(s_1 \rightarrow s_2) = \sqrt{\beta_1\beta_2}\sin(\mu_2 - \mu_1)$ of the transport matrix: the values of M_{12} are known from the unperturbed optics while the phase advances are taken from the measurement.

Some Tevatron results

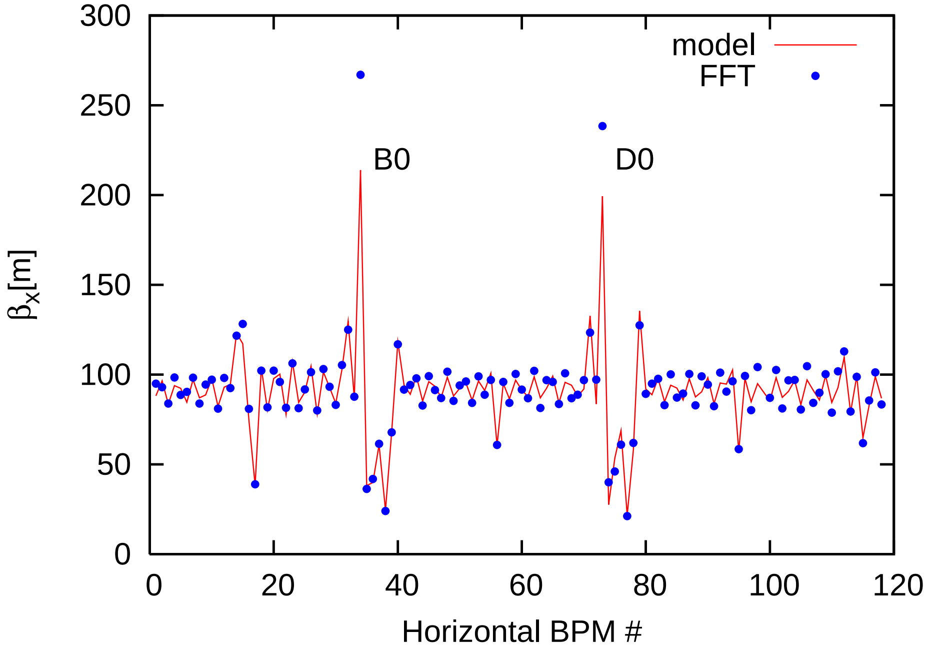
The Tevatron BPMs could store 8192 positions data per BPM with high resolution ($\simeq$ 15- 50 μm).



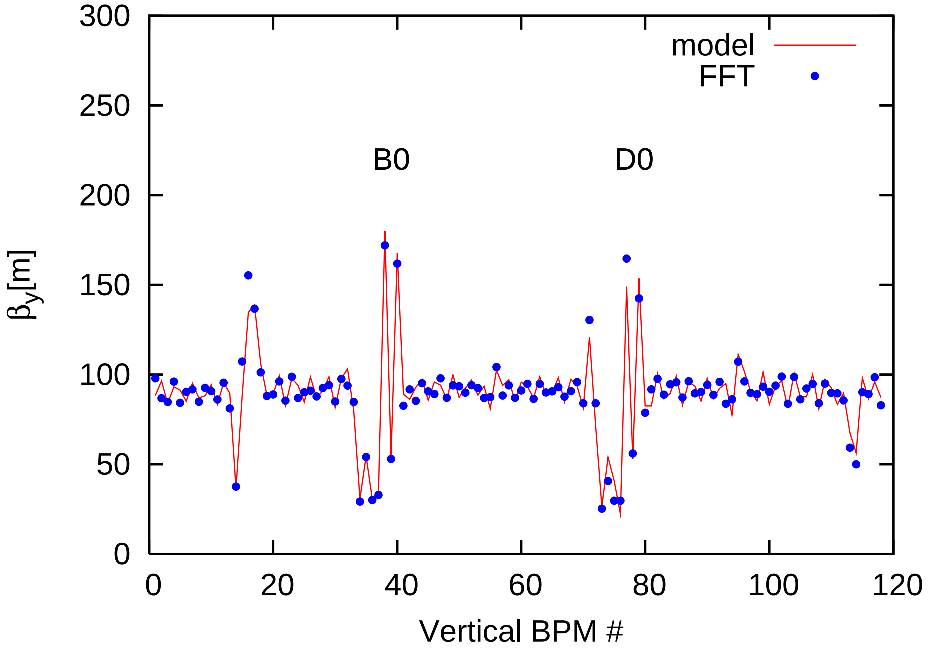
R. Miyamoto, PhD Thesis

- Under ideal condition the coherent oscillation lasts many thousands turns.
- Non-linearities (here an octupole) cause fast decoherence.
- Q_s modulation with large chromaticity.

Reconstructed Injection Optics (November 2005 data)



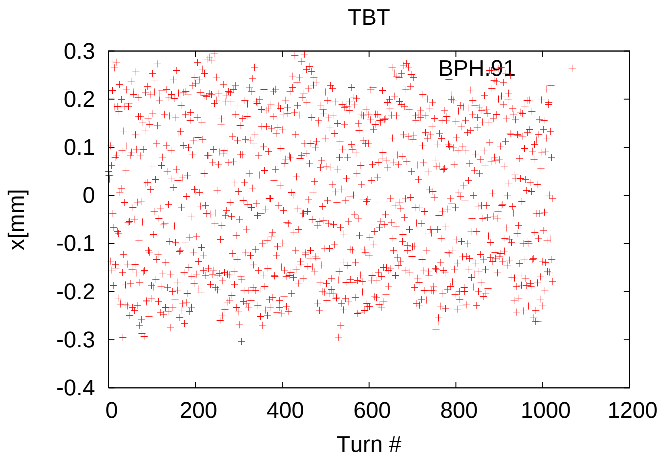
Horizontal



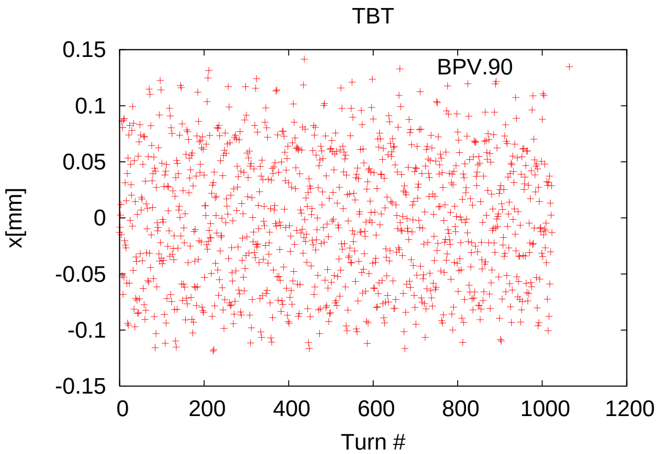
Vertical

Some results for KEK Accelerator Test Facility (June 2010)

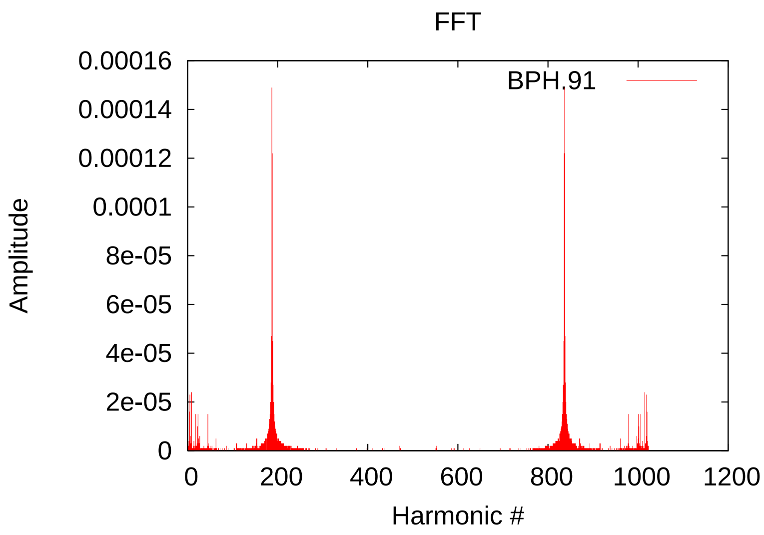
The BPM system may store 1024 TBT data.

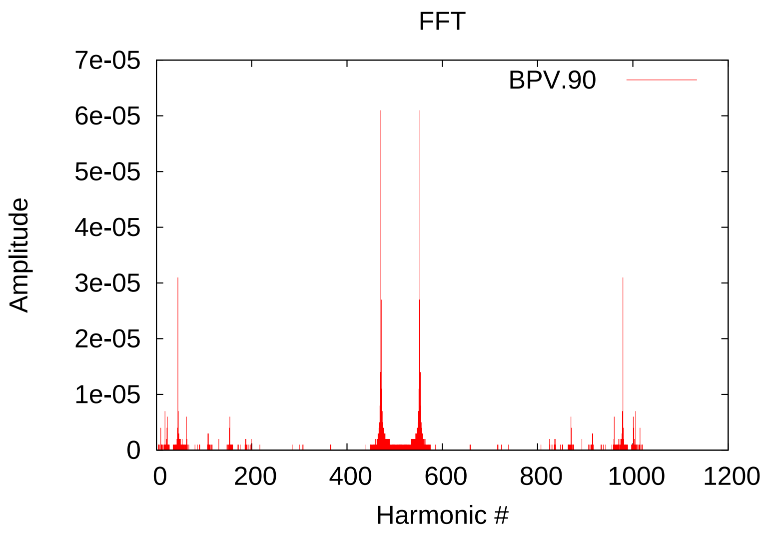


$$\beta_x^0 = 1.6 \text{ m}$$



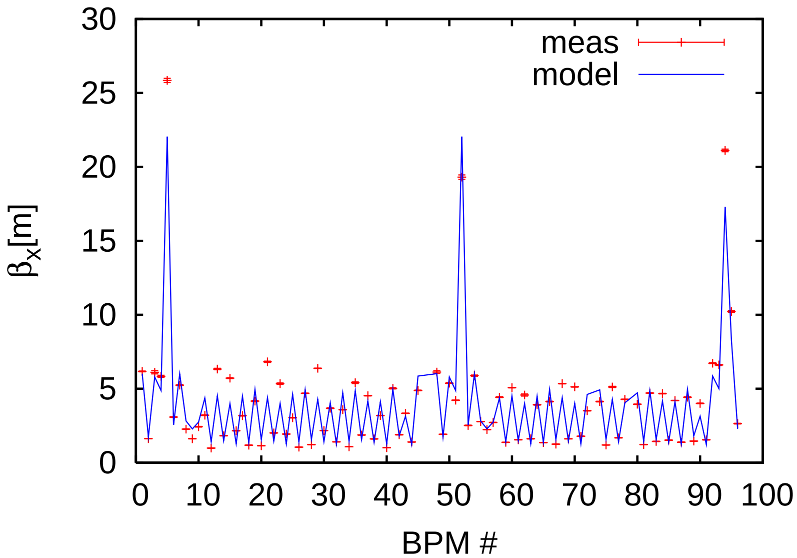
$$\beta_y^0 = 2.0 \text{ m}$$

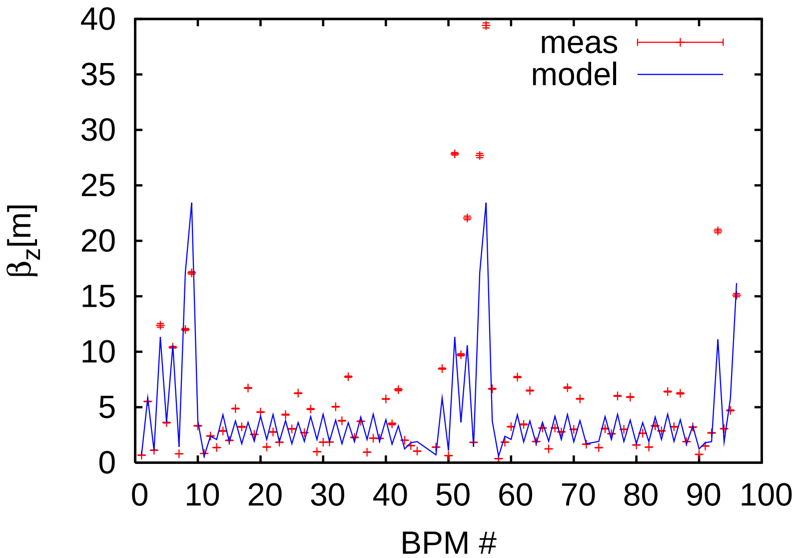




- Tunes: 0.1831 0.5398
- Small coupling between planes, orthogonal plan tune only visible on logarithmic scale.

Optics (average over 9 data set)

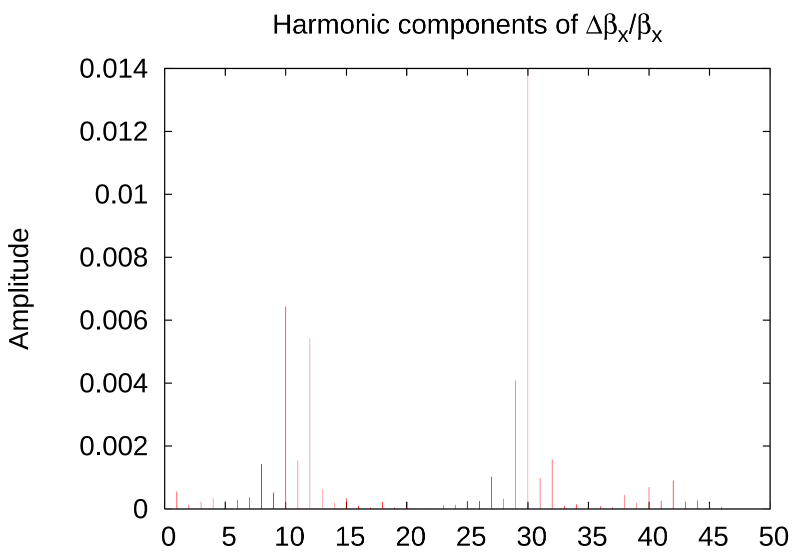


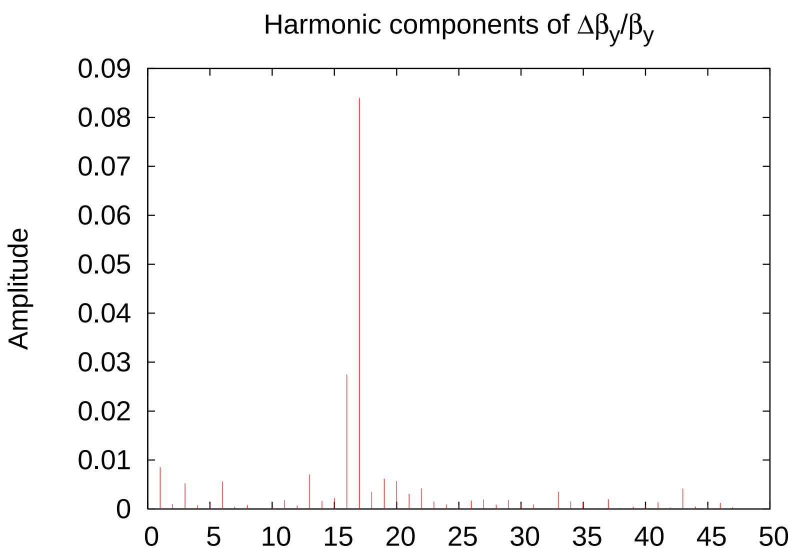


The statistical error is very small also for the the vertical plane where oscillations are smaller.

Harmonic analysis of the average $\boldsymbol{\Delta\beta/\beta}$ shows large components at $\boldsymbol{h}$=30 for the horizontal plane and $\boldsymbol{h}$=17 for the vertical one, which correspond to $2\times\boldsymbol{Q}$ ($\boldsymbol{Q_x}$=15.18, $\boldsymbol{Q_y}$=8.54). Thus the beating is a *true* beating.

The fact that it is larger in the vertical plane is likely due to the fact that the vertical tune is closer to a half integer than the horizontal one is to an integer.





AC dipole

The coherent oscillation following a single kick *decays* more or less quickly and the *emittance growth* makes the beam almost unusable afterwards.

One can use a AC dipole [a] for exciting a driven coherent oscillation. Although the dipole frequency, Q_d, is very close to the natural beam oscillation frequency, Q, if adiabatically ramped up and down and if the field is small enough, it does not blow-up the emittance.

AC dipoles have been employed at BNL (AGS and RHIC), CERN SPS, Fermilab Tevatron and now at the CERN LHC.

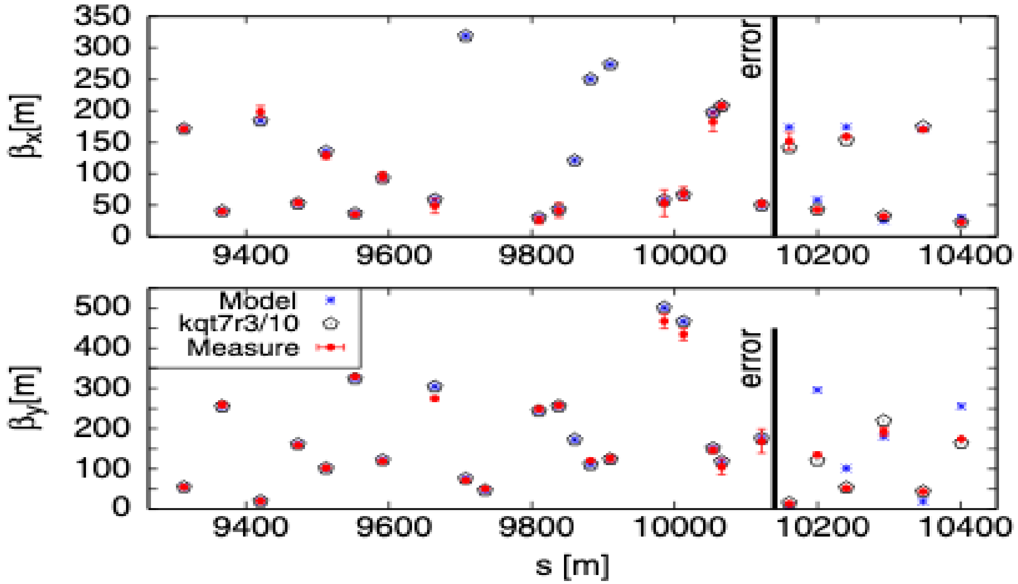
The relationship between TBT analysis results and actual BPMs Twiss parameters is not as straightforward as for free oscillations: the AC dipole is *equivalent* to a *quadrupole perturbation* which vanishes only when $\delta Q = Q - Q_d$ vanishes. For hadron machines this condition cannot be fullfilled, but one can make several measurements for different values of δQ and fit the results to find the *unperturbed* Twiss parameters at the BPMs location.

[a] sinusoidally oscillating magnetic field

At CERN Large Hadron Collider the optics is measured by analyzing the TBT data recorded by exciting the beam with an AC dipole.

Analysis and correction are done *segment-by-segment*

- the ring is chopped into sections;
- the optics functions measured at the section entrance are propagated through the section: when a gradient error is encountered there is a discontinuity between measured and computed values (clear analogy with APJ). Example:

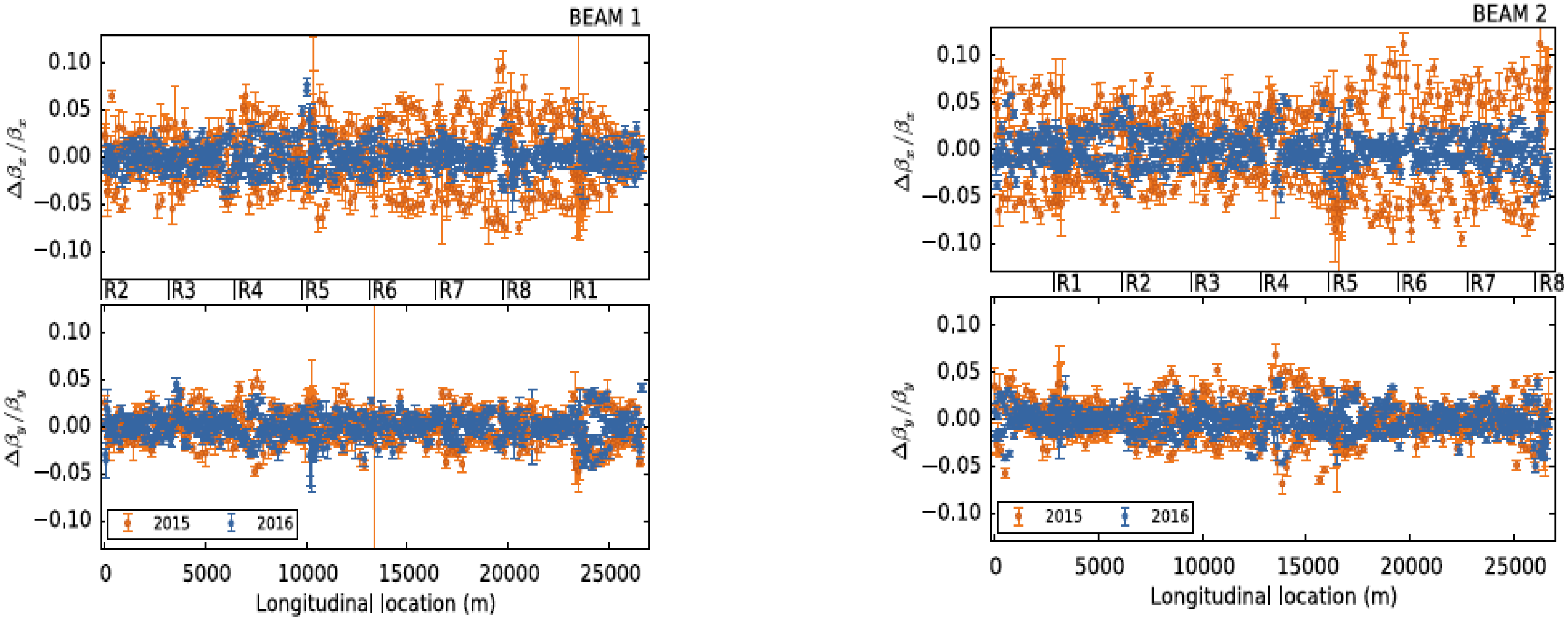


M. Aiba et al.,
PRST 12 (2009).
Quadrupole cabling problem!

- MADX is used for fitting the model to the measurement.

- Combining SBS with $\boldsymbol{K}$-modulation, possible for the IR quadrupoles which are independently powered,
- improving the β computation from the phase advance measurement (using more than 3 consecutive BPMs)
- and some more *tricks*....

allowed to reach a rms $\Delta\beta/\beta$ in the order of 1.3%-1.8% for the first time in a large hadron collider (40 cm β^* optics)!



T. Persson et al., PRAB 20 (2017).

Linear Coupling

Usually a ring is designed so that the motion in the radial and vertical plane are de-coupled. Actually due to

- tilted quadrupoles
- vertical offsets in the sextupoles
- experiment solenoids

the two planes may be coupled. The Hill's equation are then

$$x'' + \left(\frac{1}{\rho^2} + K\right)x + (N - H')y - 2Hy' = -\frac{e}{p}\Delta B_y$$

$$y'' - Ky + (N + H')x + 2Hx' = \frac{e}{p}\Delta B_x$$

with

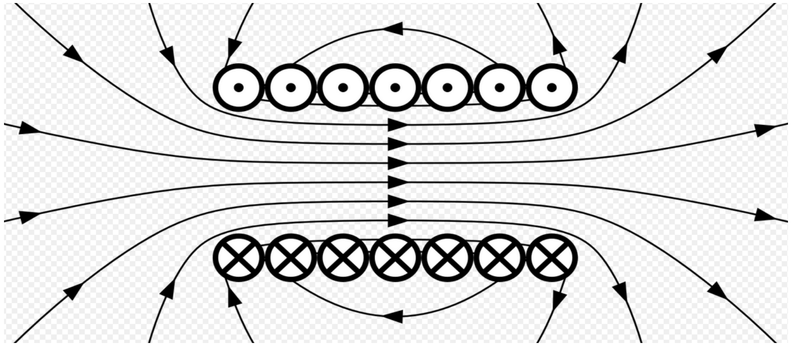
$$K(s) \equiv \frac{e}{p}\Big(\frac{\partial B_y}{\partial x}\Big)_{x=y=0} \quad \text{and} \quad N \equiv \frac{1}{2}\frac{e}{p}\Big(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\Big)_{x=y=0}$$

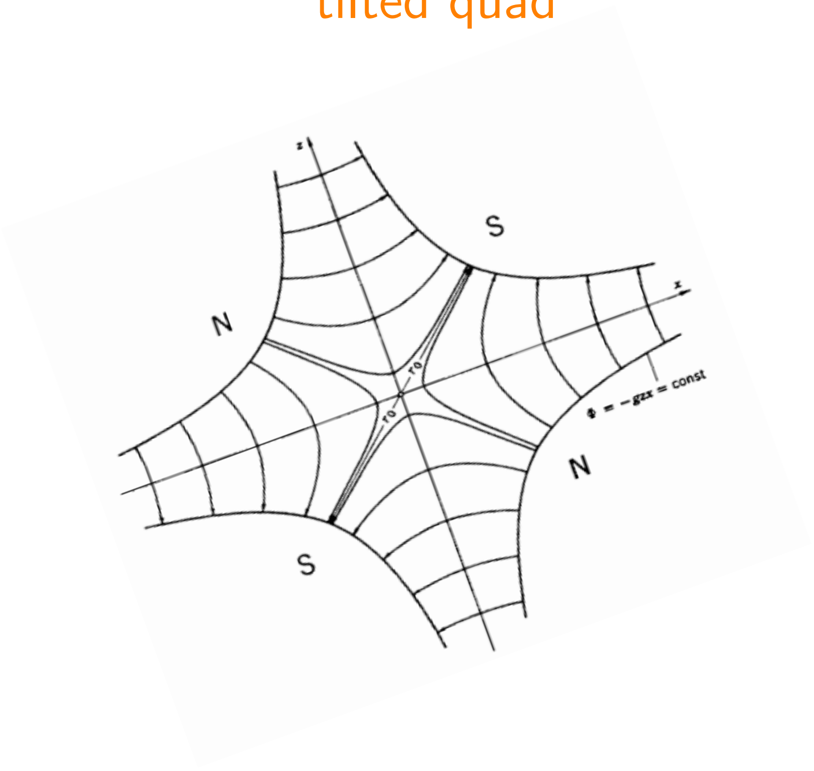
↑ normal quad

↑ tilted quad

$$H \equiv \frac{1}{2}\frac{e}{p}B_s$$

← solenoid





For a perfectly aligned pure skew quadrupole (a normal quadrupole rotated by 45 degrees around the longitudinal axis) the equations of motion are simply

$$x'' + Ny = 0$$

$$y'' + Nx = 0$$

which can be easily solved for the new variables $x + y$ and $x - y$

$$(x + y)'' + N(x + y) = 0$$

$$(x - y)'' + N(x - y) = 0$$

for which the equations are decoupled.

The solenoid case too can be easily studied. We assume for simplicity that the solenoid is perfectly aligned and therefore $\Delta B_x = \Delta B_y = 0$ on the axis. The field is longitudinal inside the solenoid, however it is easy to see that the fields at the entrance and exit have a radial (rotation symmetric) component. For symmetry reasons, it must be

$$\left(\frac{\partial B_x}{\partial x}\right)_{x=y=0} = \left(\frac{\partial B_y}{\partial y}\right)_{x=y=0} \quad \rightarrow \quad N = 0$$

and

$$\left(\frac{\partial B_x}{\partial y}\right)_{x=y=0} = \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} = 0 \quad \rightarrow \quad K = 0$$

and therefore ($\rho^2 = \infty$ at the solenoid)

$$x'' - 2Hy' - H'y = 0$$

$$y'' + 2Hx' + H'x = 0$$

Introducing a frame which rotates around the longitudinal axis by $\alpha(s) = \int_0^s ds H$, the particle coordinates in the rotating frame, $\bar{x}$ and $\bar{y}$, are

$$\bar{x} + i\bar{y} = e^{-i\alpha(s)}(x + iy)$$

and using

$$\frac{d\alpha}{ds} = H(s) \quad \text{and} \quad \frac{d^2\alpha}{ds^2} = H'(s)$$

we get

$$(\bar{x} + i\bar{y})'' + H^2(\bar{x} + i\bar{y}) = 0$$

- For a solenoid it is possible to decouple the equation of motion in the rotating frame.
- Unless $\int_0^{\ell_s} ds H = 0$ the motion in the x and y coordinates is still coupled outside the solenoid region.
- As $H^2 > 0$, the solenoid acts as a focusing lens in both directions.

At DaΦne, the e^+e^- collider in Frascati, the experiment solenoids were compensated by anti-solenoids so that $\int_0^{\ell_s} ds H = 0$ and quadrupoles in-between were rotated around their axis following the solenoid field integral.

More source of coupling are normal quadrupoles rotated around the longitudinal axis, for instance due to alignment errors.

There are many different general approaches to the problem. We will look at

- Edwards-Teng approach
- Mais-Ripken Twiss functions
- Canonical perturbation theory

Remember: in presence of solenoids the canonical variables are $(\boldsymbol{x}, \boldsymbol{p_x}, \boldsymbol{y}, \boldsymbol{p_y})$ with

$$\boldsymbol{p_x} = \boldsymbol{x'} - \boldsymbol{H}\boldsymbol{y} \quad \text{and} \quad \boldsymbol{p_y} = \boldsymbol{y'} + \boldsymbol{H}\boldsymbol{x}$$

The canonical coordinates are related to $(\boldsymbol{x}, \boldsymbol{x'}, \boldsymbol{y}, \boldsymbol{y'})$ by the matrix

$$\boldsymbol{U} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\boldsymbol{H} & 0 \\ 0 & 0 & 1 & 0 \\ \boldsymbol{H} & 0 & 0 & 1 \end{pmatrix}$$

Edwards-Teng formalism

Edwards-Teng approach uses the matrix formalism. The one-turn transport matrix, $\boldsymbol{T}(\boldsymbol{s})$, around $\boldsymbol{s}$ may be written as

$$\boldsymbol{T}(\boldsymbol{s}) = \underset{\substack{\uparrow \\ \boldsymbol{R}}}{\begin{pmatrix} \boldsymbol{I}\cos\phi & \boldsymbol{D}^{-1}\sin\phi \\ -\boldsymbol{D}\sin\phi & \boldsymbol{I}\cos\phi \end{pmatrix}} \underset{\substack{\uparrow \\ \boldsymbol{U}}}{\begin{pmatrix} \boldsymbol{A} & 0 \\ 0 & \boldsymbol{B} \end{pmatrix}} \underset{\substack{\uparrow \\ \boldsymbol{R}^{-1}}}{\begin{pmatrix} \boldsymbol{I}\cos\phi & -\boldsymbol{D}^{-1}\sin\phi \\ \boldsymbol{D}\sin\phi & \boldsymbol{I}\cos\phi \end{pmatrix}}$$

which is a similarity transformation. In general $\boldsymbol{R}$ is not a simple rotation, unless $\boldsymbol{D} = \boldsymbol{I}$. ϕ and the 2×2 matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are *unknown*. $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ have unitary determinant, which reduces the number of unknown from 13 to 10.
Defining $\vec{\boldsymbol{V}} \equiv \boldsymbol{R}^{-1}\vec{\boldsymbol{Z}}$, with $\vec{\boldsymbol{Z}}=(\boldsymbol{x}, \boldsymbol{p_x}, \boldsymbol{y}, \boldsymbol{p_y})^t$ it is

$$\underbrace{\vec{\boldsymbol{Z}}(\boldsymbol{s}+\boldsymbol{C})}_{\boldsymbol{R}\vec{\boldsymbol{V}}(\boldsymbol{s}+\boldsymbol{C})} = \boldsymbol{R}\vec{\boldsymbol{V}}(\boldsymbol{s}+\boldsymbol{C}) = \underbrace{\boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{-1}}_{\boldsymbol{T}}\underbrace{\boldsymbol{R}\vec{\boldsymbol{V}}(\boldsymbol{s})}_{\vec{\boldsymbol{Z}}} \quad \rightarrow \quad \vec{\boldsymbol{V}}(\boldsymbol{s}+\boldsymbol{C}) = \boldsymbol{U}\vec{\boldsymbol{V}}(\boldsymbol{s})$$

that is $\boldsymbol{U}$ is the (block diagonal) transport matrix for $\vec{\boldsymbol{V}}$.

$\boldsymbol{U}$ is parametrized in Courant-Snyder form as for the un-coupled case:

$$\boldsymbol{A}(s) = \begin{pmatrix} \cos\mu_{\mathrm{I}} + \alpha_{\mathrm{I}}\sin\mu_{\mathrm{I}} & \beta_{\mathrm{I}}\sin\mu_{\mathrm{I}} \\ -\gamma_{\mathrm{I}}\sin\mu_{\mathrm{I}} & \cos\mu_{\mathrm{I}} - \alpha_{\mathrm{I}}\sin\mu_{\mathrm{I}} \end{pmatrix}$$

$$\boldsymbol{B}(s) = \begin{pmatrix} \cos\mu_{\mathrm{II}} + \alpha_{\mathrm{II}}\sin\mu_{\mathrm{II}} & \beta_{\mathrm{II}}\sin\mu_{\mathrm{II}} \\ -\gamma_{\mathrm{II}}\sin\mu_{\mathrm{II}} & \cos\mu_{\mathrm{II}} - \alpha_{\mathrm{II}}\sin\mu_{\mathrm{II}} \end{pmatrix}$$

μ_{I} is related to the eigenvalues of $\boldsymbol{A}$ and μ_{II} is related to the eigenvalues of $\boldsymbol{B}$ in the usual way as for the uncoupled case.

The 6 parameters $\beta_{\mathrm{I}}, \alpha_{\mathrm{I}}, \mu_{\mathrm{I}}, \beta_{\mathrm{II}}, \alpha_{\mathrm{II}}$ and μ_{II}, the angle ϕ and the 3 independent matrix elements of $\boldsymbol{D}$ (it is unitary) are found by inverting $\boldsymbol{T}(s) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^{-1}$. The fact that $\boldsymbol{T}$ is symplectic implies that only 10 out of the 16 matrix elements are independent.

In summary, the transformation $\boldsymbol{R}$ has allowed to find a new vector $\vec{\boldsymbol{V}}$ which is transported by an uncoupled matrix. This approach is *exact* and well suited for coding in an optics program. However it has no evident connection to measurable quantities.

Mais-Ripken formalism

In this approach the the lattice functions definition is generalized to the 4D motion. Here we outline the steps.

- Solve eigenvalue problem for the 4×4 one turn transport matrix $M(s_0 + C, s_0)$.
  - For stable motion the 4 eigenvalues must be unimodular complex conjugate pairs: $\lambda_{\pm k} = \mathrm{e}^{\pm i 2\pi Q_k}$ and $\vec{v}_{-k} = \vec{v}_k^*$ (as for the uncoupled case).
- Build the coupled lattice functions.
  - Build the (real) "generating" vectors $\vec{z}_1$, $\vec{z}_2$, $\vec{z}_3$ and $\vec{z}_4$ defined as

$$\vec{z}_1 = \frac{1}{\sqrt{2}}(\vec{v}_{\mathrm{I}} + \vec{v}_{\mathrm{I}}^*) \qquad \vec{z}_2 = -\frac{i}{\sqrt{2}}(\vec{v}_{\mathrm{I}} - \vec{v}_{\mathrm{I}}^*)$$

$$\vec{z}_3 = \frac{1}{\sqrt{2}}(\vec{v}_{\mathrm{II}} + \vec{v}_{\mathrm{II}}^*) \qquad \vec{z}_4 = -\frac{i}{\sqrt{2}}(\vec{v}_{\mathrm{II}} - \vec{v}_{\mathrm{II}}^*)$$

- Write $x_1 = (\vec{z}_1)_1 = \sqrt{\beta_{xI}} \cos \mu_{xI}$, $x_2 = (\vec{z}_2)_1 = \sqrt{\beta_{xI}} \sin \mu_{xI}$ and so on.
- The generalized periodic β functions and the phase advances are given by

$$\beta_{x\mathrm{I}} = x_1^2 + x_2^2 \quad \beta_{x\mathrm{II}} = x_3^2 + x_4^2 \quad \beta_{y\mathrm{I}} = y_1^2 + y_2^2 \quad \beta_{y\mathrm{II}} = y_3^2 + y_4^2$$

$$\mu_{x\mathrm{I}} = \tan^{-1}\left(\frac{x_2}{x_1}\right) \quad \mu_{x\mathrm{II}} = \tan^{-1}\left(\frac{x_4}{x_3}\right)$$

$$\mu_{y\mathrm{I}} = \tan^{-1}\left(\frac{y_2}{y_1}\right) \quad \mu_{y\mathrm{II}} = \tan^{-1}\left(\frac{y_4}{y_3}\right)$$

In this way we have constructed the generalized lattice functions at $s = s_0$ from the knowledge of the one turn transport matrix eigenvectors.

The lattice functions in the rest of the machine can be obtained for instance by transporting the generating vectors $\vec{z}_i$.

Any solution of the equation of motion can be written in the eigenvector basis and therefore the general solution has the form

$$\vec{z}(s) = \sum_{k=\mathrm{I,II}} a_k \vec{v}_k(s) + a_{-k}\vec{v}_{-k}(s)$$

and therefore

$$\begin{aligned} x = {} & A_I e^{i\delta_I}\sqrt{\beta_{xI}}e^{i\mu_{xI}} + A_I e^{-i\delta_I}\sqrt{\beta_{xI}}e^{-i\mu_{xI}} \\ & + A_{II} e^{i\delta_{II}}\sqrt{\beta_{xII}}e^{i\mu_{xII}} + A_{II} e^{-i\delta_{II}}\sqrt{\beta_{xII}}e^{-i\mu_{xII}} \\ y = {} & A_I e^{i\delta_I}\sqrt{\beta_{yI}}e^{i\mu_{yI}} + A_I e^{-i\delta_I}\sqrt{\beta_{yI}}e^{-i\mu_{yI}} \\ & + A_{II} e^{i\delta_{II}}\sqrt{\beta_{yII}}e^{i\mu_{yII}} + A_{II} e^{-i\delta_{II}}\sqrt{\beta_{yII}}e^{-i\mu_{yII}} \end{aligned}$$

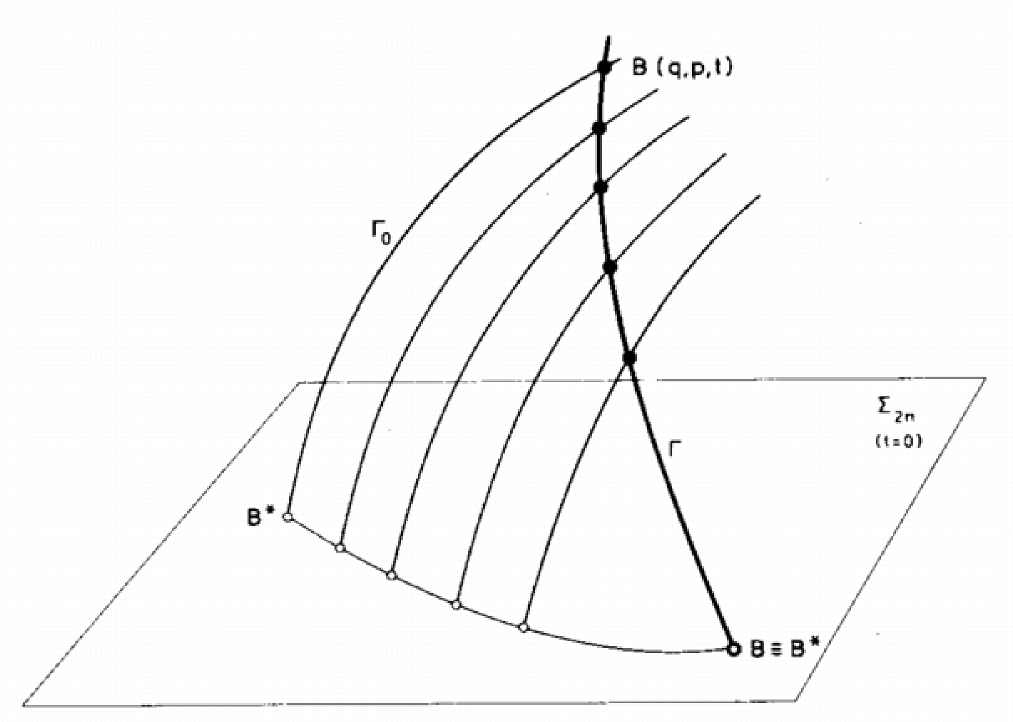
that is

$$\begin{aligned} x &= A_I\sqrt{\beta_{xI}}\cos(\mu_{xI} + \delta_I) + A_{II}\sqrt{\beta_{xII}}\cos(\mu_{xII} + \delta_{II}) \\ y &= A_I\sqrt{\beta_{yI}}\cos(\mu_{yI} + \delta_I) + A_{II}\sqrt{\beta_{yII}}\cos(\mu_{yII} + \delta_{II}) \end{aligned}$$

In the absence of coupling: $\beta_{xII}{=}\beta_{yI}{=}0$.

Method of the variation of constants:

The general solution of the perturbed motion keeps the form of the unperturbed one with constants, a_i, depending on time[a]



(Guignard, CERN 78-11)

Hamiltonian in presence of a perturbation $\mathcal{H}_1$

$$\mathcal{H} = [\mathcal{H}_0 + \mathcal{H}_1](q_1, ...q_n, p_1, ...p_n)$$

$$= [U_0 + U_1](a_1, ...a_{2n})$$

Equations of motion

$$\frac{da_j}{d\theta} = \Sigma_m \underbrace{[a_j, a_m]}_{\uparrow} \frac{\partial U_1}{\partial a_m}$$

Poisson bracket

[a] θ or s in our case

When the unperturbed Hamiltonian describes the betatron motion, the variables may be written as

$$x = a_1\sqrt{\beta_x}\mathrm{e}^{i(\mu_x - Q_x\theta)}\mathrm{e}^{iQ_x\theta} + c.c. \equiv a_1 u(\theta)\mathrm{e}^{iQ_x\theta} + c.c.$$

$$y = a_2\sqrt{\beta_y}\mathrm{e}^{i(\mu_y - Q_y\theta)}\mathrm{e}^{iQ_y\theta} + c.c. \equiv a_2 v(\theta)\mathrm{e}^{iQ_y\theta} + c.c.$$

etc., with a_k complex constant (starting conditons). The perturbed motion is described by the same expression but with $a_k = a_k(\theta)$.

The perturbation hamiltonian for linear coupling is

$$\mathcal{H}_1(\theta) = R^2 N xy + RH(xp_y - yp_x) + \frac{R^2}{2}H^2(x^2 + y^2)$$

where N and H have been defined earlier and $R = C/2\pi$. We must express the perturbation hamiltonian in terms of $\vec{a}$:

$$U_1 = \sum_{j,k,l,m=0}^{2} h_{jklm} a_1^j (a_1^*)^k a_2^l (a_2^*)^m \mathrm{e}^{i[(j-k)Q_x + (l-m)Q_y]\theta}$$

with $j + k + l + m = 2$

The functions $h_{jklm}(\theta)$ are

$$h_{2000} \equiv \frac{R^2}{2} H^2 u^2 \quad h_{0200} \equiv \frac{R^2}{2} H^2 (u^*)^2 \quad h_{1100} \equiv R^2 H^2 u u^*$$

$$h_{0020} \equiv \frac{R^2}{2} H^2 v^2 \quad h_{0002} \equiv \frac{R^2}{2} H^2 (v^*)^2 \quad h_{0011} \equiv R^2 H^2 v v^*$$

$$h_{1010} \equiv R^2 N u v + R H [u(v' + i Q_y v) - v(u' + i Q_x u)]$$

$$h_{1001} \equiv R^2 N u v^* + R H [u(v'^* - i Q_y v^*) - v^*(u' + i Q_x u)]$$

$$h_{0101} = h_{1010}^* \quad h_{0110} = h_{1001}^*$$

As the fields and the functions u and v are periodic in θ it is possible to develop h_{jklm} in Fourier series

$$h_{jklm}(\theta) = \sum_{q=-\infty}^{+\infty} h_{jklmq} \mathrm{e}^{iq\theta}$$

which adds a second sum on q in the expression for U_1 and modify the argument of the exponential to $[(j-k)Q_x + (l-m)Q_y + q]\theta$.

Under the assumption that *slowly* varying part gives the largest contribution, the indices are kept for which

$$(j-k)Q_x + (l-m)Q_y + q = 0$$

In addition neglecting the terms containing H^2 we are left with

$$n_x Q_x + n_y Q_y + q = 0 \quad \rightarrow \quad Q_x \pm Q_y = p$$

Writing u and v explicitly, the important Fourier components are

$$h_{\pm} = \frac{R}{4\pi} \oint d\theta \sqrt{\beta_x \beta_y} \Big\{ \frac{N}{2\pi} + \frac{H}{C} \Big[\Big(\frac{\alpha_x}{\beta_x} - \frac{\alpha_y}{\beta_y} \Big) \\ - i \Big(\frac{1}{\beta_x} \mp \frac{1}{\beta_y} \Big) \Big] \Big\} \mathrm{e}^{i[\mu_x \pm \mu_y + (Q_x \pm Q_y - p)\theta]}$$

which becomes a sum for localized perturbations.

Coupling correction

- Decoupling of the transport matrix
  - If the perturbed transport matrix is known, one can set the off-diagonal block elements to zero by introducing skew quadrupoles in the lattice. The symplecticity of the transport matrix reduces the number of free parameters from 16 to 10. By setting $M_{13}=M_{14}=M_{23}=M_{24}=0$ also the other off-diagonal block will vanish. This means we need at least 4 skew quadrupoles to decouple the transport matrix.
- Compensation of linear coupling driving terms (Guignard approach)
- Local coupling correction

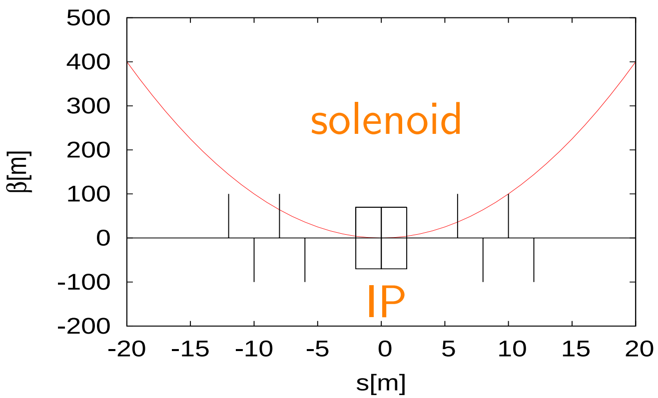
Guignard approach suggests that the driving terms $h_\pm$ must be compensated. For a localized source of coupling as an experiment solenoid the driving terms are

$$h_\pm^{sol} = \frac{1}{4\pi}\oint d\theta\sqrt{\beta_x\beta_y}H\left[\left(\frac{\alpha_x}{\beta_x}-\frac{\alpha_y}{\beta_y}\right)-i\left(\frac{1}{\beta_x}\mp\frac{1}{\beta_y}\right)\right]\mathrm{e}^{i[\mu_x\pm\mu_y+(Q_x\pm Q_y-p)\theta]}$$

$\rightarrow$ In general 4 quantities to be corrected by 4 skew quads.

Experiment solenoids are located around the IP were usually the β functions are symmetric which means that, choosing the origin θ=0 at the IP, $h_\pm^{sol}$ are purely *imaginary*. These 2 quantities may be compensated by using 2 pairs of skew quadrupoles *symmetrically* placed wrt the IP and powered *anti-symmetrically* ($\rightarrow$ 2 knobs) so that the real part of their contribution vanishes.

However introducing two more pairs of such skew quadrupoles (8 skews, 4 free knobs) also the integrals from the first skew to IP and from IP to last skew can be made vanishing (LEP scheme).



An example from HERAe. After the luminosity upgrade the H1 and Zeus solenoids were no more locally compensated by anti-solenoids. 4 skew quadrupoles per IP were used. In particular the H1 solenoid was not symmetrically positioned wrt the IP and was overlapping with machine quadrupoles. The actual solenoid fields had been measured, 3 slightly different transport maps through the solenoids were computed. We had also some simple "sandwich" model to compare with. 3 approaches were used for evaluating the needed strengths for the 4 skew quadrupoles. Here there are the results for two of them and one of the 3 maps.

	matrix method $[m^{-1}]$	Guignard method $[m^{-1}]$
QSKN1	-0.00257	-0.00231
QSKN2	0.00024	0.00028
QSKN3	-0.00747	-0.00649
QSKN4	0.00178	0.00146

We were prepared to empirical fine adjustments of the quadrupole strengths. For this we needed some observable quantities and "orthogonal" optimization knobs.

Orthogonal knobs:

- 4 knobs each changing simultaneously the values of the 4 skew quadrupoles for exciting one components at a time were programmed.

Observables:

- the beam ellipse orientation.
  - In a uncoupled machine the beam (x, y) cross section is an ellipse with axes aligned with x and y. Its tilt is a sign of coupling and depends upon machine azimuth. At HERA it was measured at 4 different positions by the synchrotron light monitor (HERAe was a 27 GeV lepton ring), the transverse polarimeter and the experimental luminosity monitors.
- Minimum tune distance.
  - In an uncoupled machine, it is possible to set the horizontal and vertical tunes to "any" value separately by using the quadrupole circuits. This is not true in presence of coupling in particular when the tunes are close.

This strategy proved to work quite well.

Coupling functions

This approach is based on canonical perturbation theory for solving the coupled equation of motion.

We recall that the perturbation hamiltonian, keeping only the linear terms in the perturbing fields, is

$$
\begin{aligned}
U_1 = {} & h_{1010} a_1 a_2 \mathrm{e}^{i(Q_x+Q_y)\theta} + h_{0101} a_1^* a_2^* \mathrm{e}^{-i(Q_x+Q_y)\theta} \\
& + h_{1001} a_1 a_2^* \mathrm{e}^{i(Q_x-Q_y)\theta} + h_{0110} a_1^* a_2 \mathrm{e}^{-i(Q_x-Q_y)\theta}
\end{aligned}
$$

or in more compact form

$$
U_1 = C_+(\theta) a_x a_y + C_+^*(\theta) a_x^* a_y^* + C_-(\theta) a_x a_y^* + C_-^*(\theta) a_x^* a_y
$$

with

$$
C_+ \equiv h_{1010} = h_{0101}^* \quad C_- \equiv h_{0101} = h_{1010}^*
$$

and

$$
a_x \equiv a_1 \mathrm{e}^{iQ_x\theta} \qquad a_y \equiv a_2 \mathrm{e}^{iQ_y\theta}
$$

"Ansatz"

$$a_x(\theta) = a_{x0}(\theta) + w_-^*(\theta) a_{y0}(\theta) + w_+^*(\theta) a_{y0}^*(\theta)$$

$$a_y(\theta) = a_{y0}(\theta) - w_-(\theta) a_{x0}(\theta) + w_+^*(\theta) a_{x0}^*(\theta)$$

Inserting into the equation of motion

$$\frac{da_j}{d\theta} = \Sigma_m [a_j, a_m] \frac{\partial U_1}{\partial a_m}$$

and keeping 1^{th} order terms one finds the equations for $w_\pm$

$$2i \mathrm{e}^{-iQ_\pm \theta} \frac{d}{d\theta} \mathrm{e}^{iQ_\pm \theta} w_\pm(\theta) = C_\pm(\theta)$$

The periodic solutions are

$$w_{\pm}(\theta) = -\int_0^{2\pi} d\theta' \frac{C_{\pm}(\theta')}{4\sin \pi Q_{\pm}} \mathrm{e}^{-iQ_{\pm}[\theta-\theta'-\pi \mathrm{sign}(\theta-\theta')]}$$

with

$$Q_{\pm} \equiv Q_x \pm Q_y$$

The functions $\tilde{w}_{\pm} \equiv w_{\pm}\mathrm{e}^{iQ_{\pm}\theta}$ are

- constant in coupler free regions
- experience a discontinuity $-iC_{\pm}\ell/2R$ at coupler locations $\Rightarrow$ diagnostics tool !
- are constant on the resonances $Q_x \pm Q_y = int$.

Coupling functions measurement through TBT analysis

TBT beam position at the j[th] vertical BPM following a horizontal kick

$$\boldsymbol{y_n^j} = \left[\sqrt{\beta_y^j}\left(\mathrm{e}^{-i\Phi_y^j} w_+^j - \mathrm{e}^{i\Phi_y^j} w_-^j\right)\right] A_x \mathrm{e}^{iQ_x(\theta_j + 2\pi n)} + c.c.$$

TBT beam position at the j-th horizontal BPM following a vertical kick

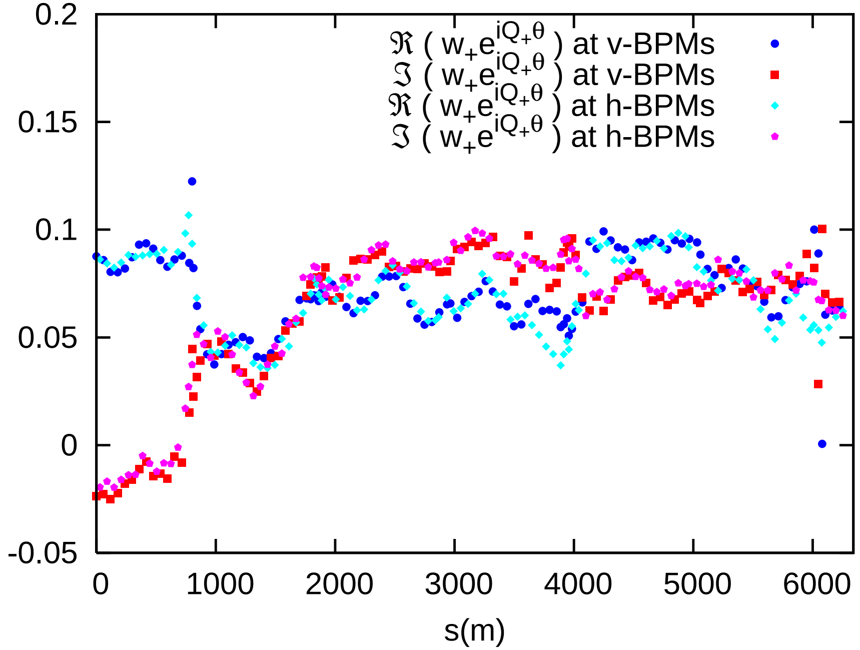
$$\boldsymbol{x_n^j} = \left[\sqrt{\beta_x^j}\left(\mathrm{e}^{-i\Phi_x^j} w_+^j + \mathrm{e}^{i\Phi_x^j} w_-^{*j}\right)\right] A_y \mathrm{e}^{iQ_y(\theta_j + 2\pi n)} + c.c.$$

Here it is $\Phi_z \equiv \mu_z - Q_z \theta$.
The FFT of y^j at Q_x, $Y^j(Q_x)$, for a horizontal kick ($X^j(Q_y)$ for a vertical one) is proportional to the *coupling functions* $w_\pm(\theta_j)$.

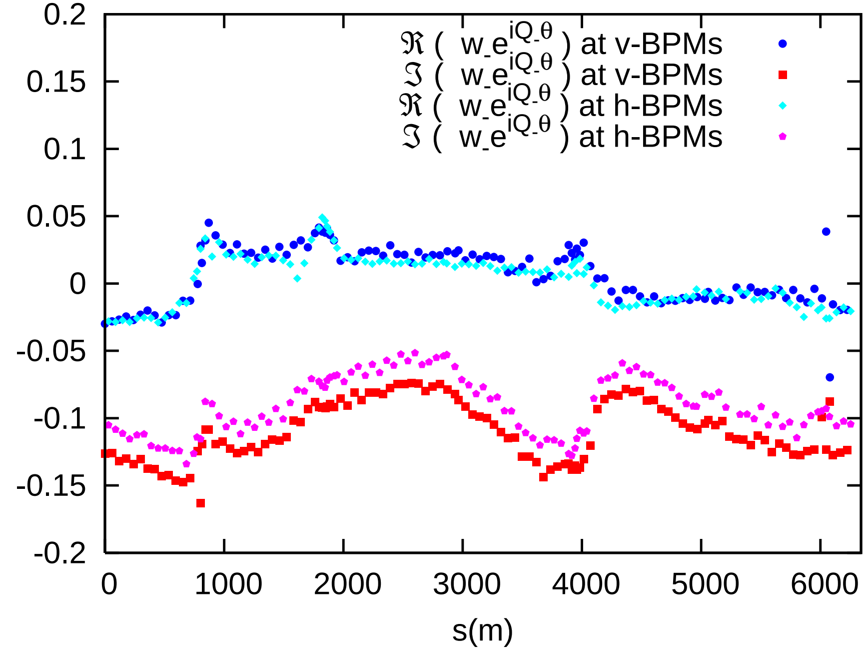
We get per each BPM 2 real equations (amplitude and phase of the Fourier component) in 4 unknowns ($w_\pm$ are complex). When between two consecutive monitors there are no strong source of coupling, the four equations can be solved in favor of $w_\pm(\theta_j) = w_\pm(\theta_{j+1})$.

Examples of Tevatron Measurements

Coupling functions (November 2005 data)



$\tilde{w}^{+}$

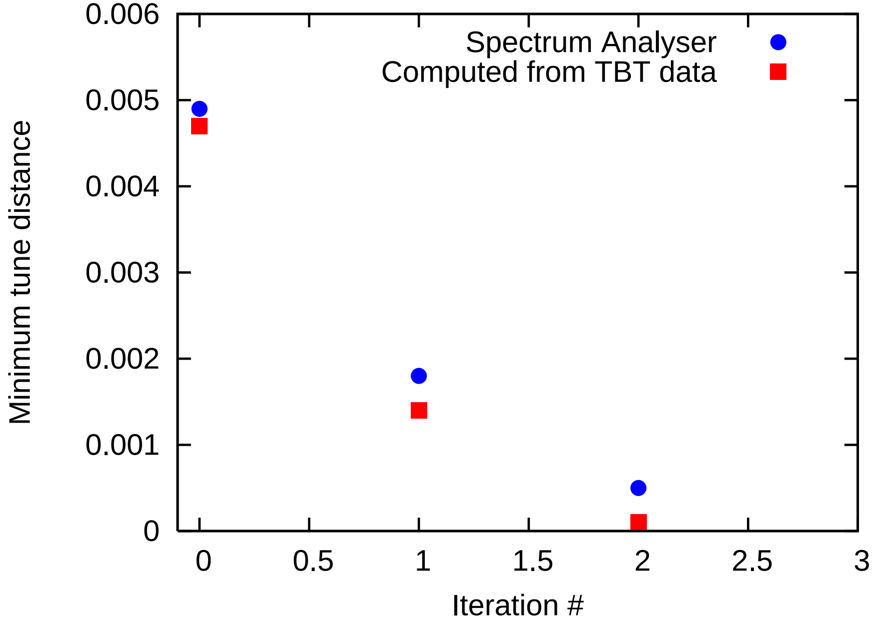


$\tilde{w}^{-}$

Discontinuities visible around 1000 (SQA0), 1500 (A38) and 4000 (D16) meters, confirming LOCO results.

The measured $\boldsymbol{w}^{\pm}$ were used for correcting the *global coupling* with 2 skew quadrupoles (the working point was $\boldsymbol{Q_x}$=20.584 and $\boldsymbol{Q_y}$=20.574 ie close to the difference resonance).

Minimum tune split measured with spectrum analyzer and computed from TBT data

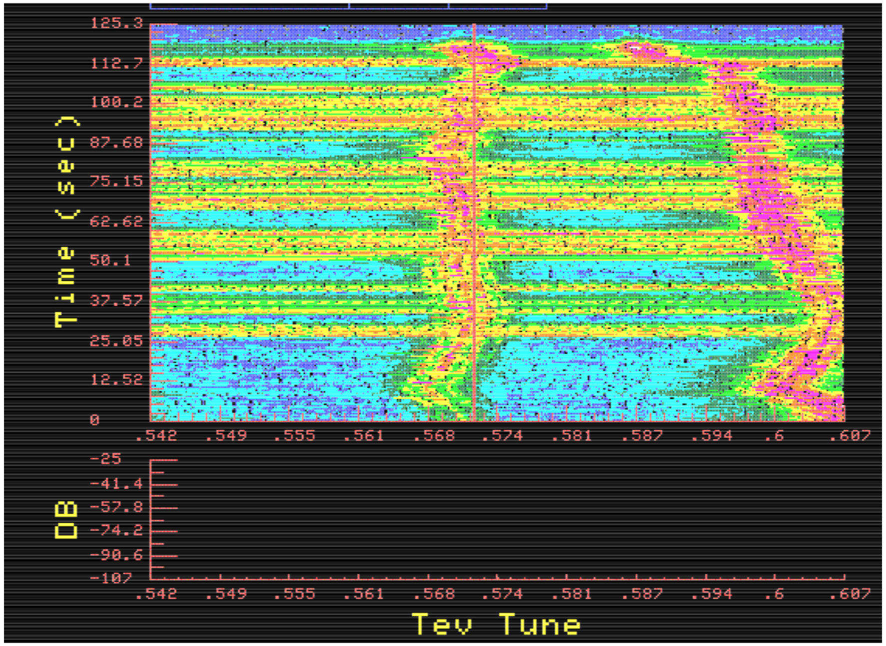


● Spectrum Analyser

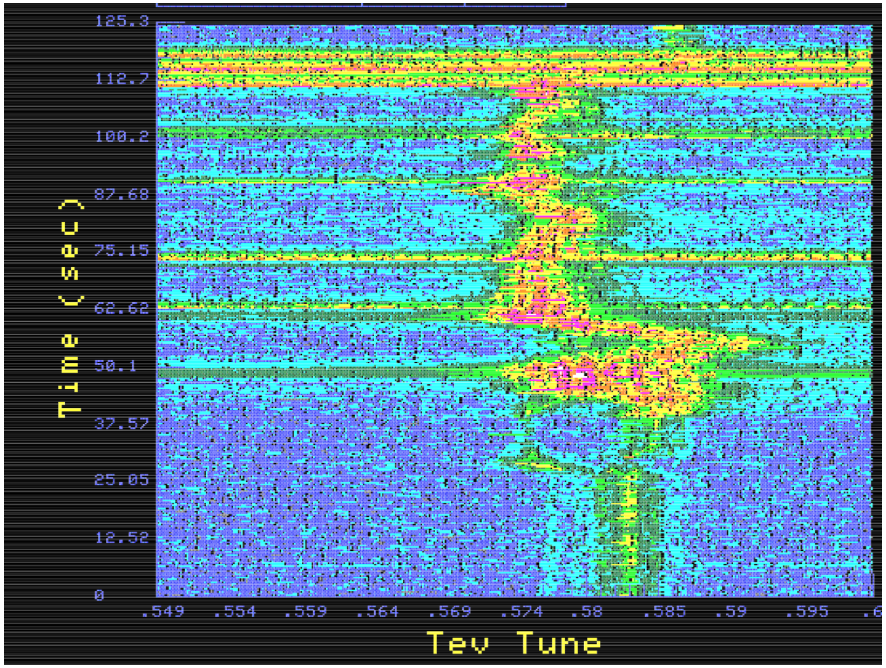
■ TBT Analysis

TEVATRON was a fast ramping machine (83 seconds from 150 to 980 GeV), the TBT analysis was a very practical method for measuring optics and coupling also during acceleration.

First ramp after 2006 shut down (3th June 2006)



After correcting with W118 (6th June 2006)



A local correction is particularly important for $e^{\pm}$ machines when a very small vertical beam size is required.
A part of the vertical beam size comes from the *spurious* vertical dispersion which is generated by

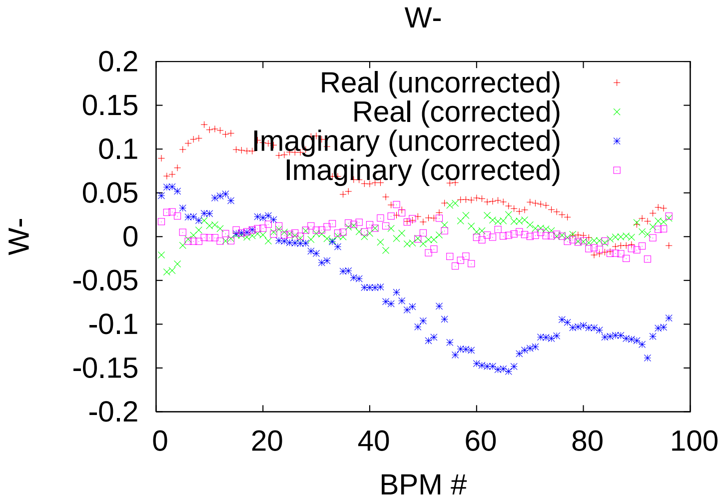
- vertical misalignment of quadrupoles
- rotation of normal quadrupoles at $D_x \neq 0$ locations

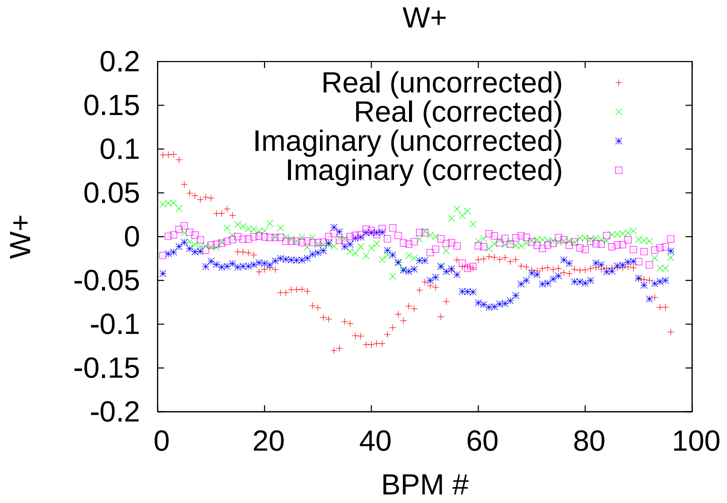
Thus one must correct betatron coupling *and* spurious vertical dispersion.

Simulations for KEK Accelerator Test Facility (ATF), a 139 m test damping ring with a goal of $\epsilon_y \approx 1$ or 2 pm.

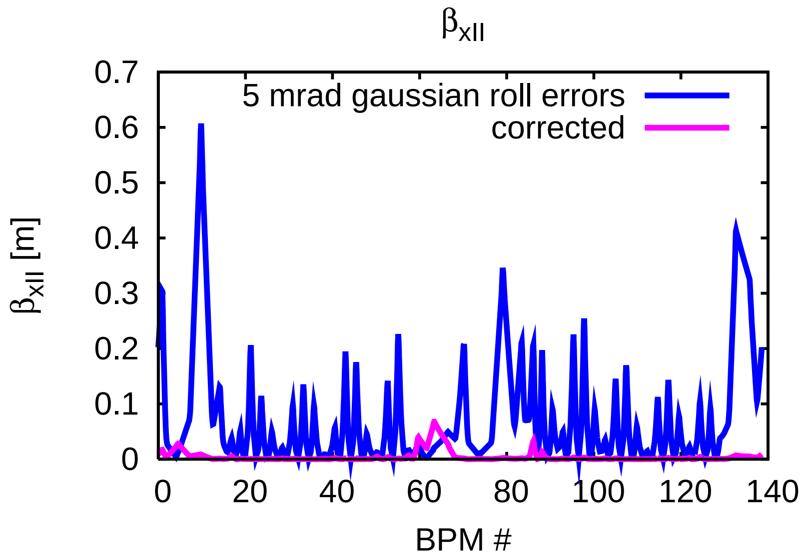
- Gaussian random *roll* errors (5 mrad rms) applied to all normal quadrupoles
- coupling functions $w^{\pm}$ computed
- All 68 skew quadrupoles used for minimizing $w^{\pm}$ *and* spurious vertical dispersion along the ring

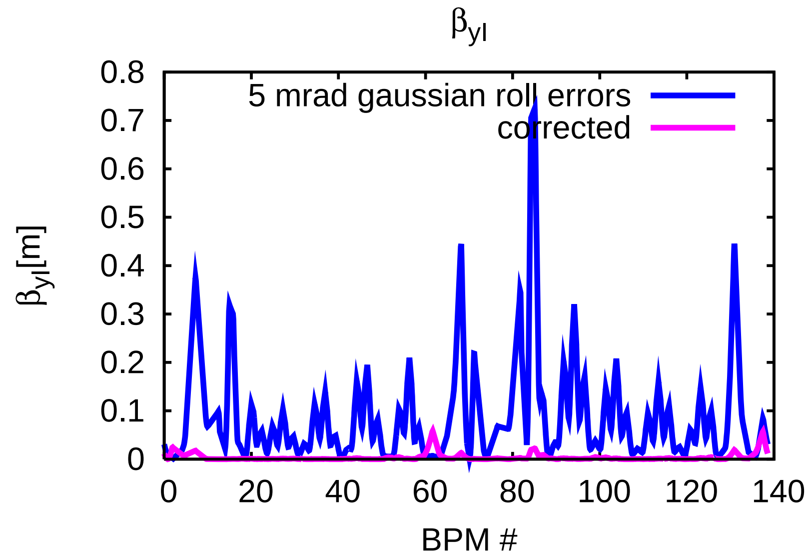
ATF simulations: coupling functions

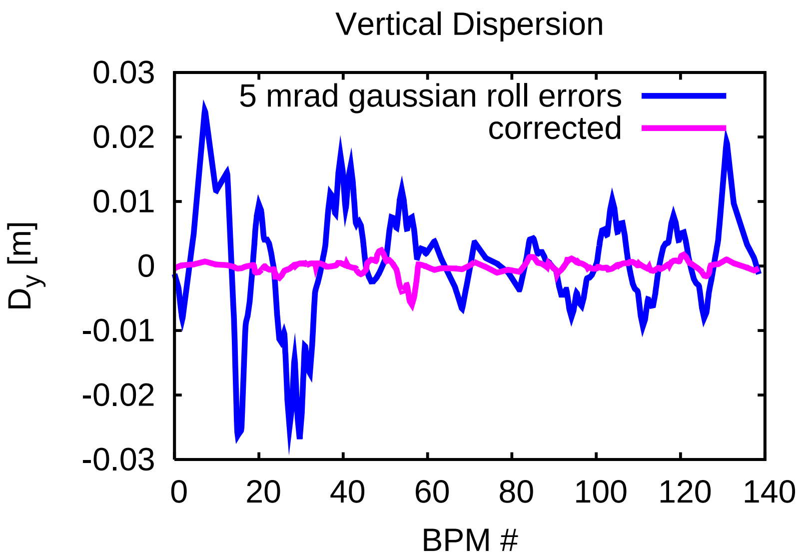




ATF simulations: Mais-Ripken cross functions







Transverse Emittance

	ε_x (nm)	ε_y (nm)
Nominal	0.973	0.000
with errors	0.971	0.042
with β-tron coupling correction	0.973	0.012
with D_y correction	0.970	0.013
correcting both	0.973	0.001

Summary

We have studied two kind of linear perturbations:

- Gradient errors:
  - We have found the equation for the β-beating in lowest order.
    - The same formalism has allowed us to find the equation for the linear chromaticity.
  - We have seen methods for measuring and correcting gradient errors.
- Linear coupling (generated by skew quadrupoles, solenoids and roll angle of normal quadrupoles:
  - Overview of different formalism:
    - Edwards-Teng and Mais-Ripken generalized optics functions.
    - Canonical perturbation theory and linear coupling resonances.
  - Corrections method by using skew quadrupoles have been described.

"The search for truth is more precious than its possession"

(Albert Einstein)

...therefore don't be too disappointed if there are mistakes in my slides!