Southwest State University
Department of computer sciences

ORTHOGONALITY-BASED CLASSIFICATION OF DIAGONAL LATIN SQUARES OF ORDER 10

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Dubna, 2018

What is Latin squares?

$A = \left\| a_{ij} \right\|$

$i, j = \overline{1, N}$

$N = |S|$

$S = \{0, 1, 2, \ldots, N-1\}$

$\forall\, i, j, k = \overline{1, N},\ j \neq k : (a_{ij} \neq a_{ik}) \wedge (a_{ji} \neq a_{ki})$

$\forall\, i, j = \overline{1, N},\ i \neq j : (a_{ii} \neq a_{jj}) \wedge (a_{N-i+1, N-i+1} \neq a_{N-j+1, N-j+1})$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\ 2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\ 3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\ 4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\ 5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\ 6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\ 7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\ 8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\ 9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Normalized LS of order 10

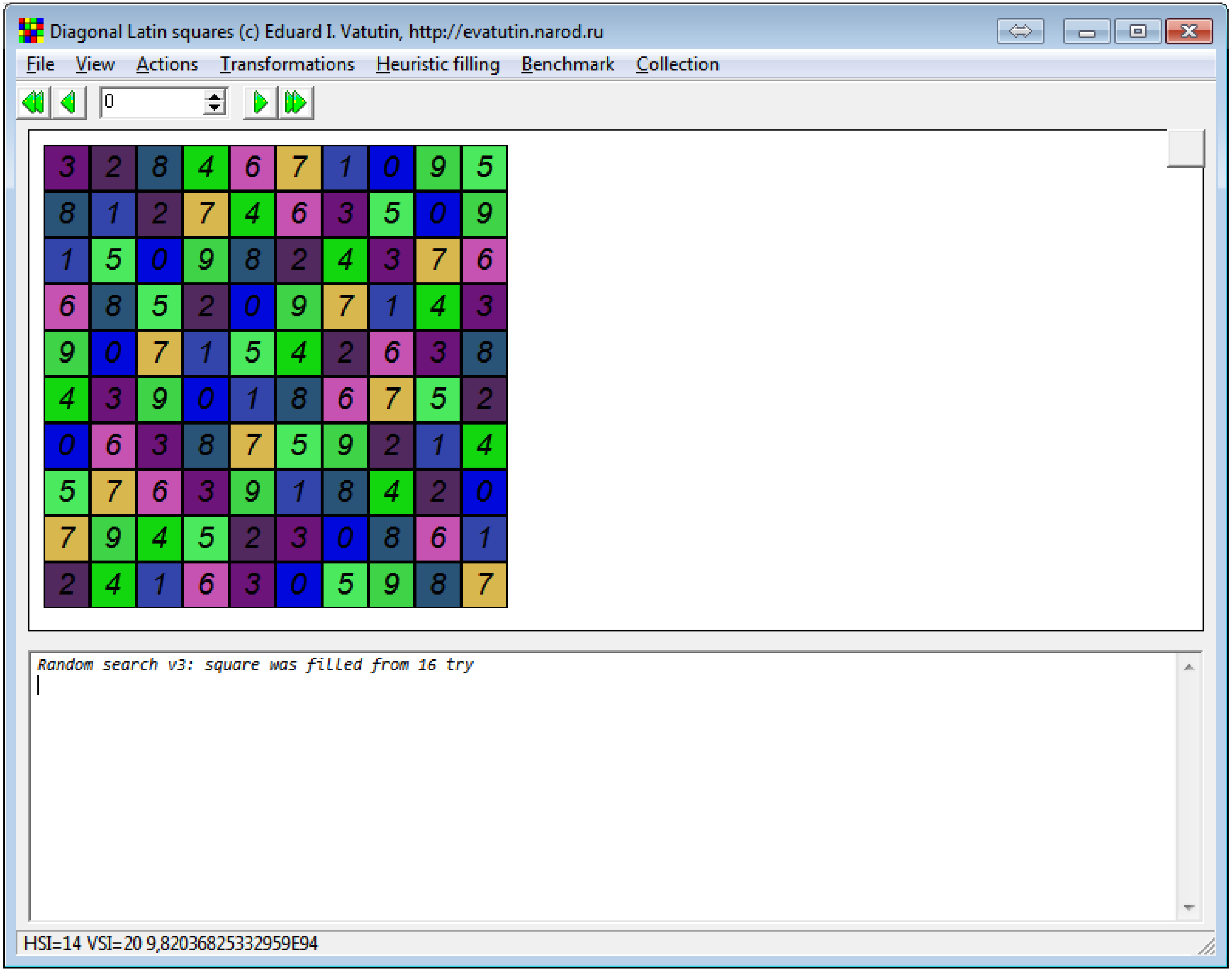
$N! \times (N-1)!$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\ 8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\ 2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\ 5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\ 9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\ 4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\ 3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\ 6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\ 1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4 \end{pmatrix}$$

Normalized DLS of order 10

$(N-1)!$

Lets try to get diagonal Latin square!



- http://evatutin.narod.ru/evatutin_LsEdit.7z

Why is this interesting?

Applied problems:

- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, ...

Mathematical problems:

- existence of a triple of MOLS/MODLS
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

Searching for pairs of ODLS of order 10

L. Euler expected that for N=10 ODLS doesn’t exist
First pair — Parker et al., 1960

0	1	2	3	4	5	6	7	8	9
1	2	0	4	3	7	9	8	5	6
7	3	5	9	0	4	8	6	2	1
3	5	6	8	9	0	4	1	7	2
4	9	7	2	6	8	1	5	0	3
5	8	4	6	7	1	3	2	9	0
8	4	9	1	2	3	7	0	6	5
6	7	3	0	1	2	5	9	4	8
9	0	1	5	8	6	2	4	3	7
2	6	8	7	5	9	0	3	1	4

0	1	2	3	4	5	6	7	8	9
7	5	1	9	2	8	0	4	6	3
1	0	3	4	6	7	5	2	9	8
9	8	4	7	5	2	1	0	3	6
6	7	9	0	8	3	2	1	5	4
4	6	5	1	0	9	8	3	2	7
2	3	8	5	1	6	4	9	7	0
5	2	7	8	3	4	9	6	0	1
3	4	6	2	9	0	7	8	1	5
8	9	0	6	7	1	3	5	4	2

SAT@Home, 04.2015

0	1	2	3	4	5	6	7	8	9
4	9	0	8	5	6	3	1	2	7
2	5	7	9	6	4	0	8	1	3
9	0	4	6	8	7	1	5	3	2
6	7	5	2	1	3	8	0	9	4
1	8	3	5	7	2	9	6	4	0
7	3	1	0	9	8	4	2	6	5
8	2	6	4	0	9	5	3	7	1
3	4	8	1	2	0	7	9	5	6
5	6	9	7	3	1	2	4	0	8

0	1	2	3	4	5	6	7	8	9
6	5	9	7	0	8	2	3	1	4
4	7	1	2	3	9	8	0	6	5
1	2	0	4	5	3	7	6	9	8
2	6	8	0	9	4	1	5	3	7
8	4	6	9	2	7	0	1	5	3
5	0	4	6	8	2	3	9	7	1
9	3	5	1	7	6	4	8	0	2
7	8	3	5	6	1	9	4	2	0
3	9	7	8	1	0	5	2	4	6

Gerasim@Home, 04.2017

Very rare combinatorial objects:
~30 millions DLS of order 10
has only **1 pair of ODLS**!

Isomorphic decisions and canonical forms (CFs)

0	1	2	3	4	5	6	7	8	9
1	2	3	4	9	0	5	6	7	8
4	0	8	7	6	3	2	1	9	5
9	8	7	6	5	4	3	2	1	0
5	9	1	2	3	6	7	8	0	4
3	5	9	8	2	7	1	0	4	6
2	3	4	0	8	1	9	5	6	7
7	6	5	9	1	8	0	4	3	2
6	4	0	1	7	2	8	9	5	3
8	7	6	5	0	9	4	3	2	1

Orthogonality characteristic
74,
citerra
(**world record, 2016**)

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0
5	0	6	8	7	2	1	3	9	4
1	6	4	7	9	0	2	5	3	8
4	9	3	1	2	7	8	6	0	5
8	3	5	2	0	9	7	4	6	1
3	7	0	4	8	1	5	9	2	6
7	4	8	9	6	3	0	1	5	2
2	5	1	0	3	6	9	8	4	7
6	2	9	5	1	8	4	0	7	3

Orthogonality characteristic
74,
evatutin (2017)

- Can characteristic value be increased? It is open question, we are trying...
- Are decisions differ?
- Are decisions have special properties?

Special types of squares and its properties

0	1	2	3	4	5
4	2	0	5	3	1
5	4	3	2	1	0
2	5	4	1	0	3
3	0	1	4	5	2
1	3	5	0	2	4

0	1	2	3	4	5
4	2	5	0	3	1
3	5	1	2	0	4
5	3	0	4	1	2
2	4	3	1	5	0
1	0	4	5	2	3

Symmetric DLS examples

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0
8	0	6	7	9	3	4	5	2	1
1	2	5	4	3	9	7	6	0	8
7	9	3	1	2	6	8	0	4	5
6	5	1	9	0	7	2	8	3	4
5	4	0	8	6	2	1	3	9	7
3	6	4	5	1	8	0	9	7	2
4	3	8	2	7	0	9	1	5	6
2	7	9	0	8	1	5	4	6	3

String-inverse DLS

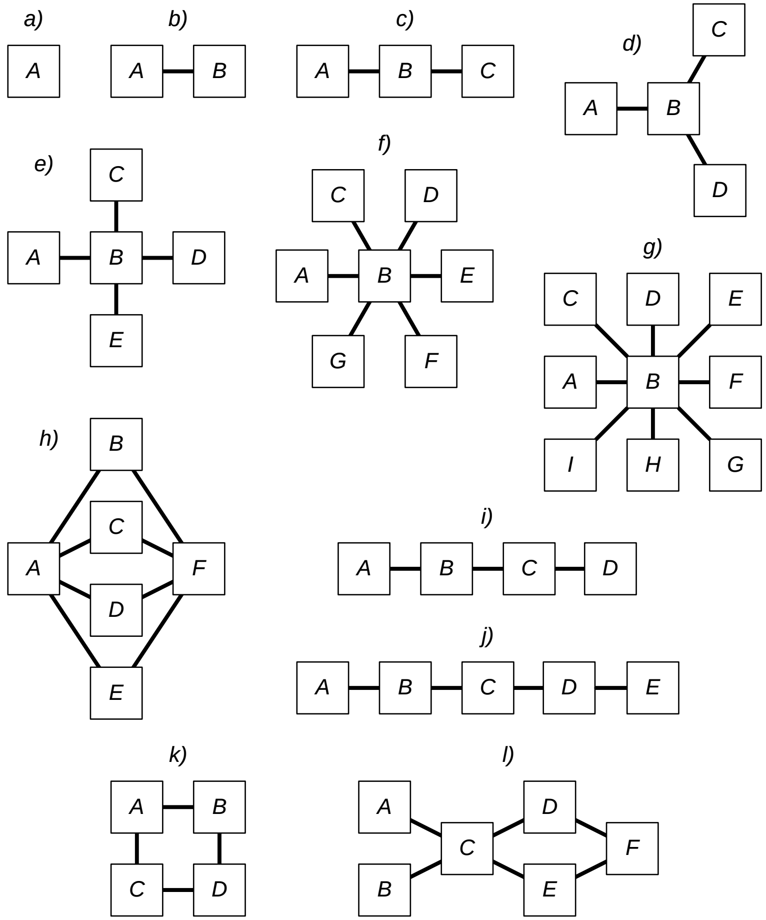
0	1	2	3	4	5	6	7	8	9
5	9	6	4	8	1	3	0	2	7
9	0	1	8	6	2	7	4	5	3
4	6	5	2	0	7	8	3	9	1
2	4	9	7	3	6	1	8	0	5
3	7	8	9	5	4	0	2	1	6
7	8	3	0	2	9	5	1	6	4
8	5	7	1	9	0	4	6	3	2
6	3	4	5	1	8	2	9	7	0
1	2	0	6	7	3	9	5	4	8

SODLS

0	1	2	3	4	5	6	7
6	2	3	7	0	4	5	1
4	5	1	0	7	6	2	3
5	6	7	4	3	0	1	2
7	3	6	2	5	1	4	0
2	7	4	1	6	3	0	5
3	0	5	6	1	2	7	4
1	4	0	5	2	7	3	6

Double symmetric DLS

Combinatorial structures for plane symmetry



Vatutin E.I., Titov V.S., Zaikin O.S., Kochemazov S.E., Manzuk M.O. An analysis of the combinatorial structures from the diagonal Latin squares of order 10 on the binary relation of orthogonality (in Russian) // Information technologies and mathematical modeling of a systems 2017. Moscow: Center of Information Technologies in Mathematical Modeling of RAS, 2017. pp. 167–170.

Some else symmetries?

0	1	2	3	4	5	6	7	8
6	3	0	2	7	8	1	4	5
3	2	1	8	6	7	0	5	4
7	8	6	5	1	3	4	0	2
8	6	4	7	2	0	5	3	1
2	7	5	6	8	4	3	1	0
5	4	7	0	3	1	8	2	6
4	5	8	1	0	2	7	6	3
1	0	3	4	5	6	2	8	7

Centrally symmetric DLS examples

Number of centrally symmetric diagonal Latin squares of order n with constant first row:

1, 0, 0, 2, 8, 0, 2816, 135 168, 327 254 016 (N < 10), evatutin, 2017

Exist only for N != 4n+2, doesn't exist for N=10 :(

Formulas for symmetries

[i,j] <=> [i',j']=f([i,j])

[i,j] <=> [i, N – 1 – j] — horizontal symmetry
[i,j]<=>[N – 1 – i, j] — vertical symmetry
[i,j]<=>[N – 1 – i, N – 1 – j] — central symmetry

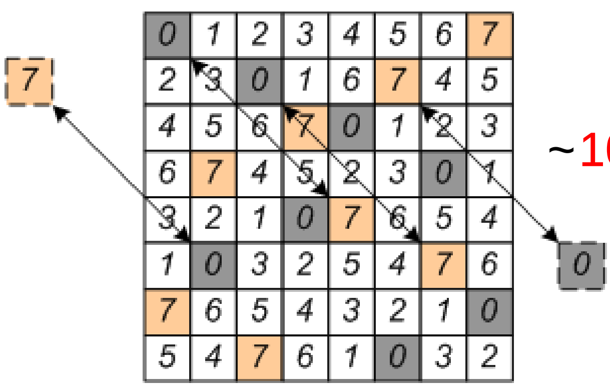
Simple way: using formulas like f(i, j) = Ai + Bj + C and g(i, j) = Di + Ej + F

Tuple **(A, B, C, D, E, F)** identifies the symmetry!

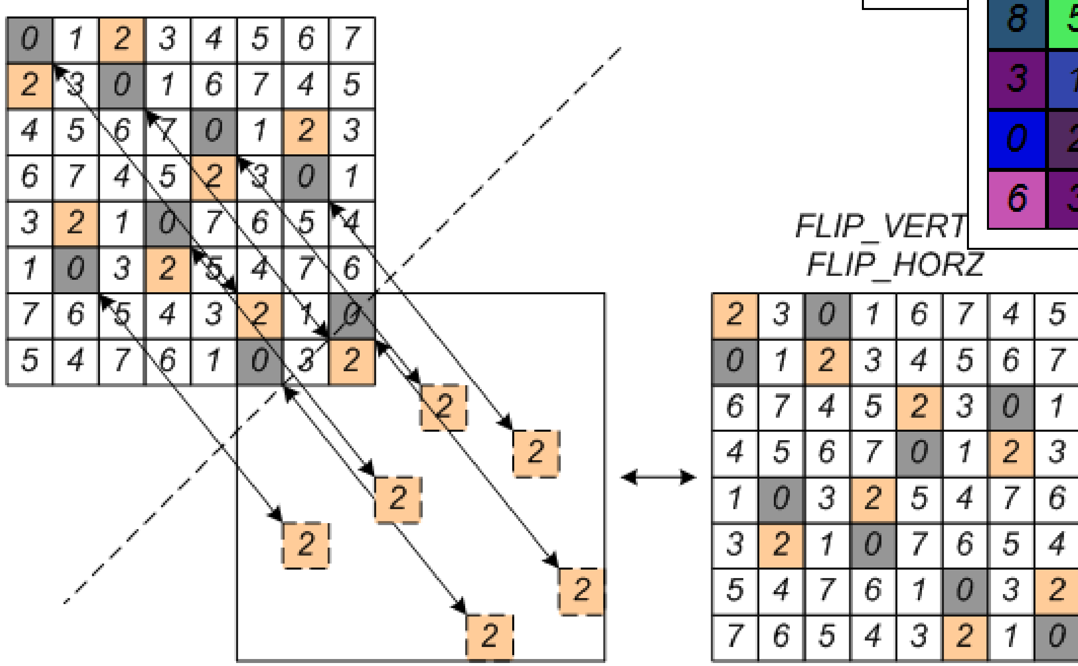
(1, 0, 0, 0, –1, N – 1) — horizontal symmetry
(–1 , 0, N – 1, 0, 1, 0) — vertical symmetry
(–1, 0, N – 1, 0, –1, N–1) — central symmetry

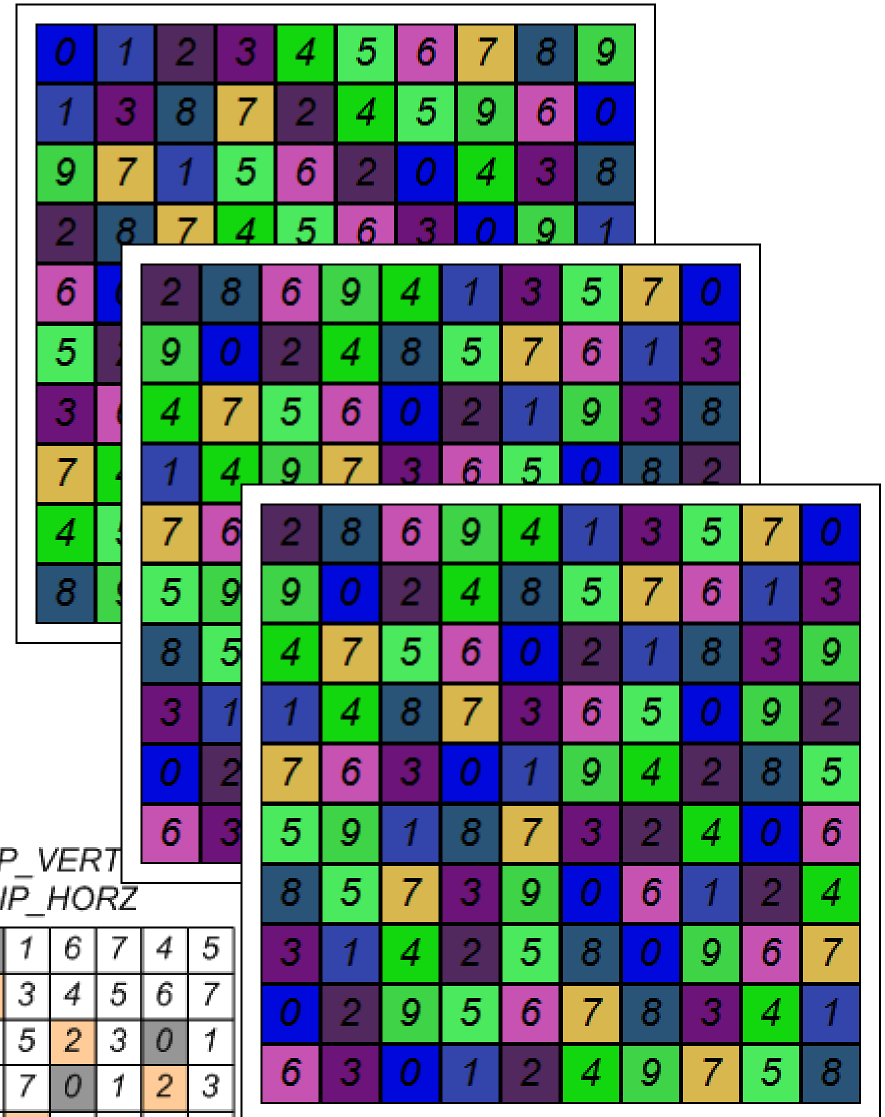
...and at least 13 different (generalized) symmetries for DLS of order 10!!!

Generalized symmetries example



very very rare:
~1000 **1:1** vs 1 **2:1**!





2:1 structure

Torus movements

Square 1

0	1	2	3	4	5	6	7	8	9
1	3	8	7	2	4	5	9	6	0
9	7	1	5	6	2	0	4	3	8
2	8	7	4	5	6	3	0	9	1
6	0	4	8	9	7	1	3	5	2
5	2	3	0	1	8	9	6	7	4
3	6	9	2	0	1	7	8	4	5
7	4	0	6	8	9	2	5	1	3
4	5	6	9	3	0	8	1	2	7
8	9	5	1	7	3	4	2	0	6

FLIP_HORZ

Square 2

9	8	7	6	5	4	3	2	1	0
0	6	9	5	4	2	7	8	3	1
8	3	4	0	2	6	5	1	7	9
1	9	0	3	6	5	4	7	8	2
2	5	3	1	7	9	8	4	0	6
4	7	6	9	8	1	0	3	2	5
5	4	8	7	1	0	2	9	6	3
3	1	5	2	9	8	6	0	4	7
7	2	1	8	0	3	9	6	5	4
6	0	2	4	3	7	1	5	9	8

THORUS_UP(5)

Square 3

4	7	6	9	8	1	0	3	2	5
5	4	8	7	1	0	2	9	6	3
3	1	5	2	9	8	6	0	4	7
7	2	1	8	0	3	9	6	5	4
6	0	2	4	3	7	1	5	9	8
9	8	7	6	5	4	3	2	1	0
0	6	9	5	4	2	7	8	3	1
8	3	4	0	2	6	5	1	7	9
1	9	0	3	6	5	4	7	8	2
2	5	3	1	7	9	8	4	0	6

COPY

Square 3

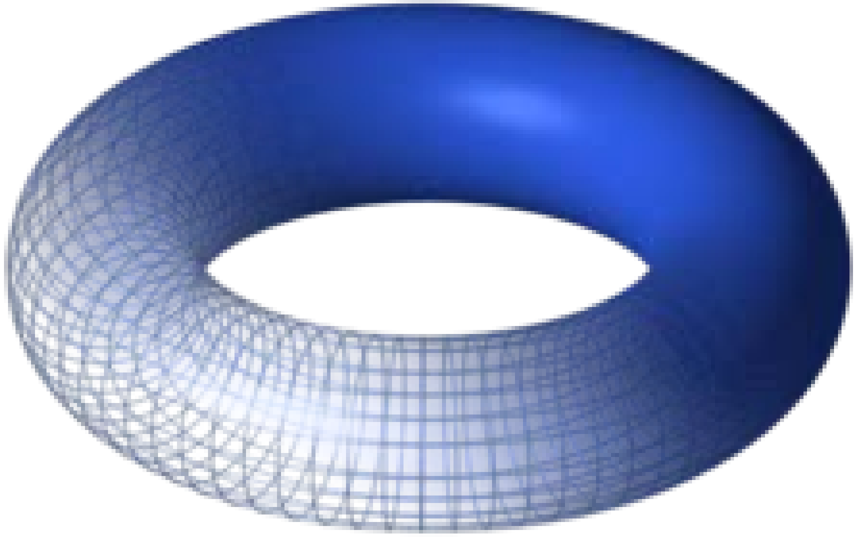
4	7	6	9	8	1	0	3	2	5
5	4	8	7	1	0	2	9	6	3
3	1	5	2	9	8	6	0	4	7
7	2	1	8	0	3	9	6	5	4
6	0	2	4	3	7	1	5	9	8
9	8	7	6	5	4	3	2	1	0
0	6	9	5	4	2	7	8	3	1
8	3	4	0	2	6	5	1	7	9
1	9	0	3	6	5	4	7	8	2
2	5	3	1	7	9	8	4	0	6

THORUS_LEFT(5)

Square 4

1	0	3	2	5	4	7	6	9	8
0	2	9	6	3	5	4	8	7	1
8	6	0	4	7	3	1	5	2	9
3	9	6	5	4	7	2	1	8	0
7	1	5	9	8	6	0	2	4	3
4	3	2	1	0	9	8	7	6	5
2	7	8	3	1	0	6	9	5	4
6	5	1	7	9	8	3	4	0	2
5	4	7	8	2	1	9	0	3	6
9	8	4	0	6	2	5	3	1	7

SYMMETRY



Formulas and permutations

Simple way: using formulas like f(i, j) = – i + 9 (mod 10)

f(0) = 9	0 -> 9 -> 0	loop with length 2
f(1) = 8	1 -> 8 -> 1	length 2
f(2) = 7	2 -> 7 -> 2	2
f(3) = 6	3 -> 6 -> 3	2
f(4) = 5	4 -> 5 -> 4	2
f(5) = 4		
f(6) = 3		S = {2, 2, 2, 2, 2}
f(7) = 2		
f(8) = 1		
f(9) = 0		

P = [9, 8, 7, 6, 5, 4, 3, 2, 1, 0] – one of 10! permutations

Some formulas gives isomorphic decisions. Why?

Multi set loop length structures for permutations (A.D. Belyshev)

1: P = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9] S = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
2: P = [0, 1, 2, 3, 4, 5, 6, 7, 9, 8] S = {1, 1, 1, 1, 1, 1, 1, 1, 2}
3: P = [0, 1, 2, 3, 4, 5, 6, 8, 9, 7] S = {1, 1, 1, 1, 1, 1, 1, 3}
4: P = [0, 1, 2, 3, 4, 5, 7, 6, 9, 8] S = {1, 1, 1, 1, 1, 1, 2, 2}
...
43: P = [1, 2, 3, 4, 5, 6, 7, 8, 9, 0] S = {10}

(Px, Py, Pv) 43x43x43 = 79 507 combinations (up bound)

Not of them exists!
Not of them provides interesting combinatorial structures.

Sx = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1} Sy = {2, 2, 2, 2, 2} – plane symmetry

Exploration: **(1, 31)**; (2, 31); (3, 31); **(4, 31)**; (8, 31); (31, 31); ...
~5 days per one pair (Sx, Sy) within Gerasim@Home project
~1–2 years per all combinations

Well known combinations of generalized symmetries

Symmetry (1,31,31) — 948
Symmetry (2,31,31) — 34
Symmetry (4,31,31) — 2201
Symmetry (8,8,8) — 1
Symmetry (8,31,31) — 598
Symmetry (16,16,16) — 13
Symmetry (16,31,31) — 2022
Symmetry (21,21,21) — 1
Symmetry (21,36,36) — 1
Symmetry (27,27,27) — 19
Symmetry (41,41,41) — 1
Symmetry (41,42,42) — 1

Total: 5840 LSs with 12 different generalized symmetries

Partial generalized symmetries

M cells corresponds to some generalized symmetry

N^2 – M cells filled using Brute Force approach

Current computing experiments within Gerasim@Home project:

- some symmetry (X, Y, V)
- M in {80, 70, 60, 50} (cardinality of dominating set)

Different M values for different (X, Y, V) are preferable:

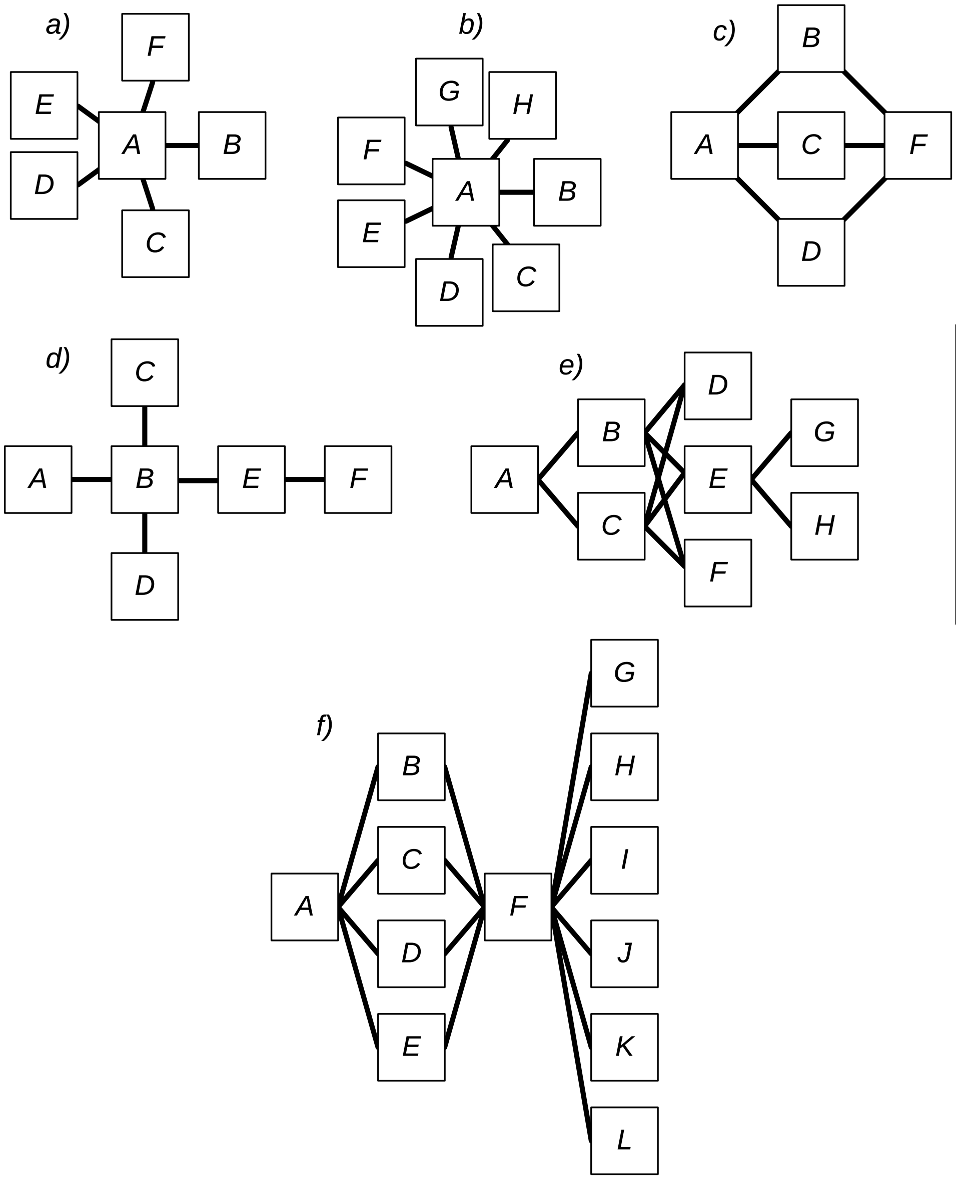
- pseudo central symmetric squares, (1, 31) generalized symmetry – M=60...70
- pseudo (4, 31) generalized symmetry – M = 80

			6	5	3		0	8	
	9	2	1				5		
						6			
		4			0				
									9
7					1		8	4	
	0		8						7
6	8	3	5						1
9	1					2	7		
		9	4		8		3		0

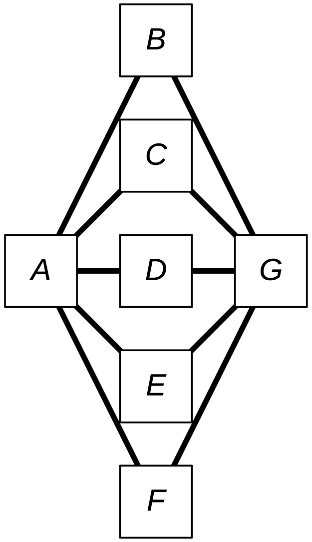
→

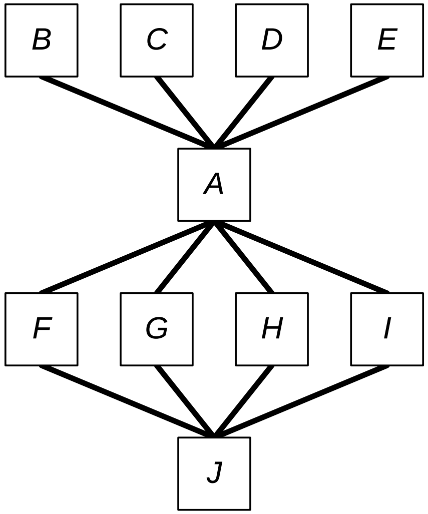
4	7	1	6	5	3	9	0	8	2
3	9	2	1	6	7	8	5	0	4
0	5	7	9	3	2	6	4	1	8
8	2	4	3	1	0	7	6	9	5
2	4	0	7	8	6	3	1	5	9
7	3	5	2	9	1	0	8	4	6
1	0	6	8	2	4	5	9	3	7
6	8	3	5	0	9	4	2	7	1
9	1	8	0	4	5	2	7	6	3
5	6	9	4	7	8	1	3	2	0

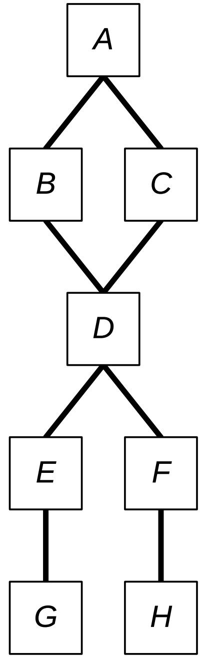
Combinatorial structures for partial central symmetry

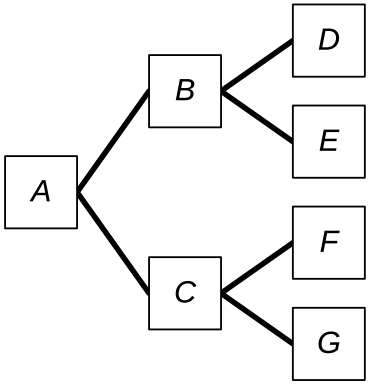


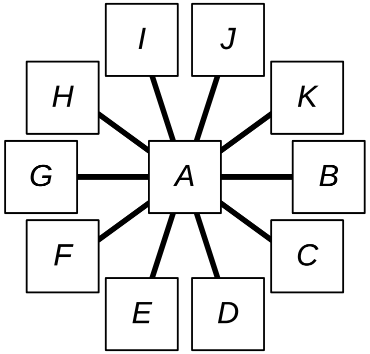
Combinatorial structures for partial (4, 31) symmetry

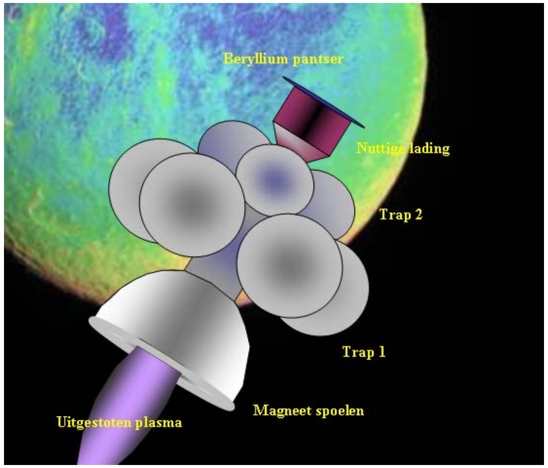








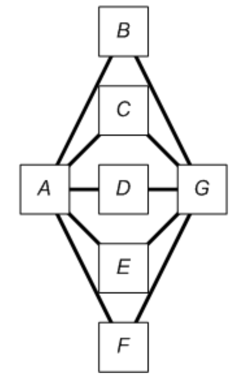




Daedalus

Venus

Combinatorial structure information (example)



0 1 2 3 4 5 6 7 8 9
1 2 3 0 7 9 8 6 5 4
6 8 7 5 9 0 3 2 4 1
5 3 1 4 0 2 7 8 9 6
9 5 8 7 6 1 4 3 2 0
4 6 9 2 8 3 1 0 7 5
8 4 5 6 3 7 9 1 0 2
2 9 4 8 1 6 0 5 3 7
7 0 6 9 2 8 5 4 1 3
3 7 0 1 5 4 2 9 6 8

DLS 0: 0123456789123079865468759032415314027896958761432046928310758456379102294816053770692854133701542968
DLS 1: 0123456789270658934153670981246985342017325190746884706152931098234675963472185045128709367849163502
DLS 2: 0123456789279658034153670981246085342917325190746884706152931908234675963472185045128790367849163502
DLS 3: 0123456789123079865468759032415314027896958761432046923810753456879102294816053770692354188701542963
DLS 4: 0123456789127039865468359072415714023896958761432046927810353456879102294816057370692354188301542967
DLS 5: 0123456789187039265462359078415714083296958761432046987210353456879102894216057370692354182301548967
DLS 6: 0123456789183079265462759038415314087296958761432046983210753456879102894216053770692354182701548963

Unique CFs for combinatorial structure:
DLS 0: 0123456789123079865468759032415314027896958761432046928310758456379102294816053770692854133701542968
DLS 1: 0123456789123406957848529703168367514902790862543165901438272716398054948573216030412876955679801243
DLS 2: 0123456789123079865468759032415314027896958761432046923810753456879102294816053770692354188701542963
DLS 3: 0123456789123068954794572318606879540213894136705246927183053085974126751802369453068924712764105938
DLS 4: 0123456789123486790587459102636517028394509867243179065318424371289056986214357024893056173650794128

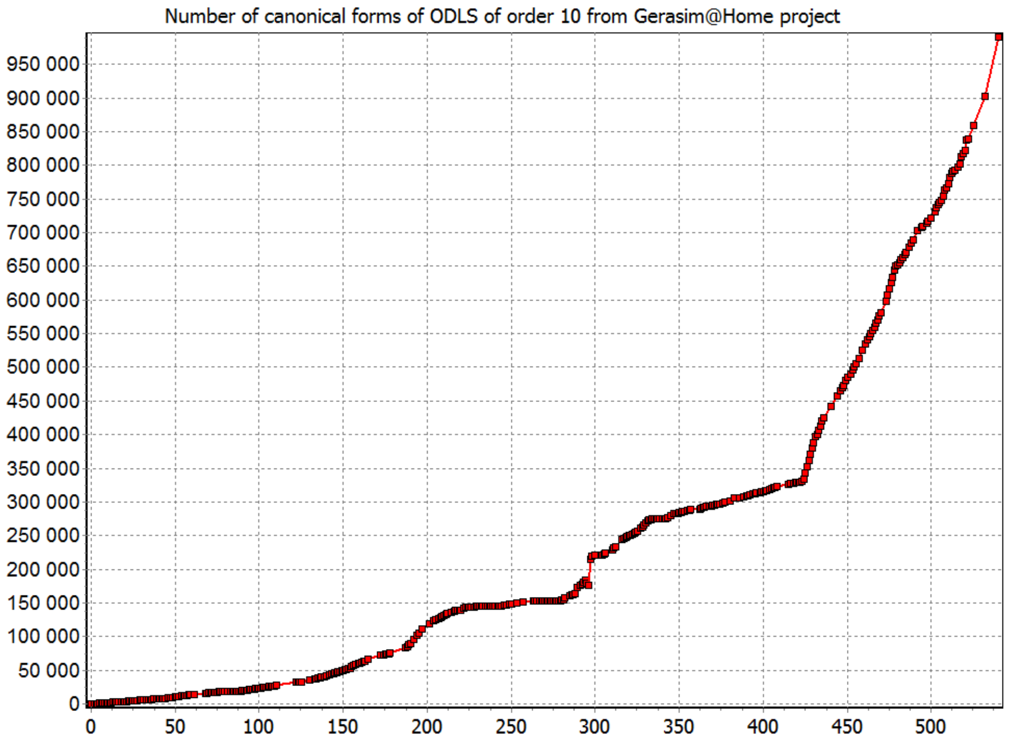
Adjacency matrix:
0110000
1001111
1001111
0110000
0110000
0110000
0110000

Vertexes powers: 2 2 2 2 2 5 5

WU'шки: wu_e69_80_34_106.dat, wu_e69_80_78_611.dat, посчитаны пользователями okes (команда **Russia**) и Robert

Getting ODLS CFs within Gerasim@Home project

Strategy of search: getting source square (random generator, symmetric random generator), try to get orthogonal square, add the unique CF to collection



- Brute Force with bits arithmetic (03.2017)
- DLX v1, array (04.2017)
- DLX v2, pointers (05.2017)
- SN DLS (SCFs) (08.2017)
- horizontal symmetry (10.2017)
- different canonization strategy (04.2018)

Combinatorial structures brief classification

ONCE (A):1 - 947458 (+241030)
LINE3 (B):1 - 22112 (+8717)
LINE3 (B):2 - 17390 (+7943)
LINE4 (C):1 - 38 (+4)
LINE4 (C):2 - 38 (+4)
LINE5 (D):1 - 13 (+6)
LINE5 (D):2 - 26 (+12)
LOOP4 (E):2 - 1630 (+882)
1TO3 (F):1 - 228 (+9)
1TO3 (F):3 - 76 (+3)
1TO4 (G):1 - 952 (+220)
1TO4 (G):4 - 413 (+106)
1TO5 (k):1 - 10
1TO5 (k):5 - 2
1TO6 (H):1 - 42 (+6)
1TO6 (H):6 - 11 (+1)
1TO7 (h):1 - 7
1TO7 (h):7 - 1
1TO8 (I):1 - 48 (+8)
1TO8 (I):8 - 10 (+2)

RHOMBUS3 (J):2 - 6
RHOMBUS3 (J):3 - 4
RHOMBUS4 (K):2 - 68 (+26)
RHOMBUS4 (K):4 - 32 (+10)
FISH (N):1 - 5 (+2)
FISH (N):2 - 8 (+3)
FISH (N):4 - 3 (+1)
TREE1 (V):1 - 2 NEW!!!
TREE1 (V):2 - 1 NEW!!!
TREE1 (V):3 - 1 NEW!!!
CROSS (X):1 - 12
CROSS (X):2 - 3
CROSS (X):4 - 3
N10-1TO10BASED (i):1 - 6
N10-1TO10BASED (i):2 - 4
N10-1TO10BASED (i):4 - 1
N10-1TO10BASED (i):10 - 1
FLYER (j):1 - 2
FLYER (j):2 - 3
FLYER (j):4 - 3
VENUS (l):1 - 1 NEW!!!
VENUS (l):2 - 3 NEW!!!
VENUS (l):4 - 1 NEW!!!
DAEDALUS (m):1 - 2 NEW!!!
DAEDALUS (m):2 - 2 NEW!!!
DAEDALUS (m):4 - 1 NEW!!!
DAEDALUS (m):8 - 1 NEW!!!
RHOMBUS5 (n):2 - 4 NEW!!!
RHOMBUS5 (n):5 - 1 NEW!!!

One experiment results example (exp71, (8, 31)-symmetry, M=80)

ONCE (A):1 - 9552, where:
  2 CFs - 9552

LINE3 (B):1 - 36, where:
  2 CFs - 4
  3 CFs - 32

LINE3 (B):2 - 20, 478:1, where:
  2 CFs - 4
  3 CFs - 16

I have some additional minutes? :)

Related works...

Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (is it exist?)
- different structures?

GPU implementation of transversal and cover algorithms?

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (string-inverse, symmetric, ...) and its CFs

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details

Gerasim@Home
Distributed computing project
Managed by BOINC.RU community
With support of Southwest State University, Department of Computer Sciences (kvt)

Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

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