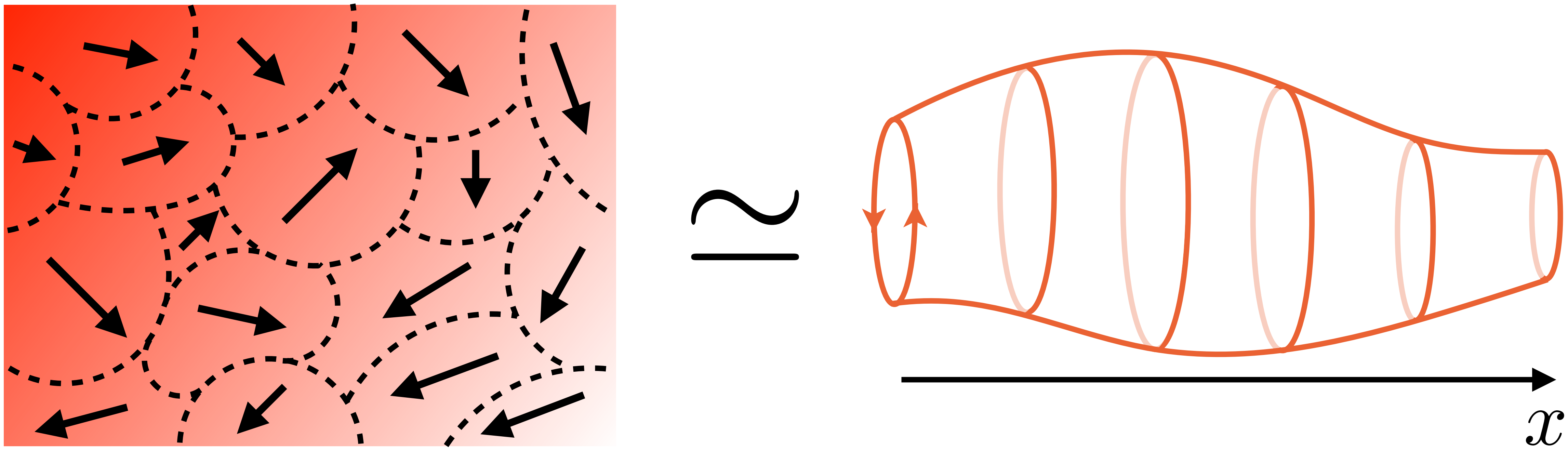
# Revisiting hydrodynamics from quatum field theory



**Masaru Hongo**

**RIKEN, iTHEMS Program**

*Colloquium on Nonequilibrium phenomena in strongly correlated systems at JINR,* 2018 4/18

Based on
Hayata-Hidaka-MH-Noumi PRD(2015), MH Ann. Phys. (2017), MH arXiv: 1801.06520

# Zubarev’s method in Japan (1976)

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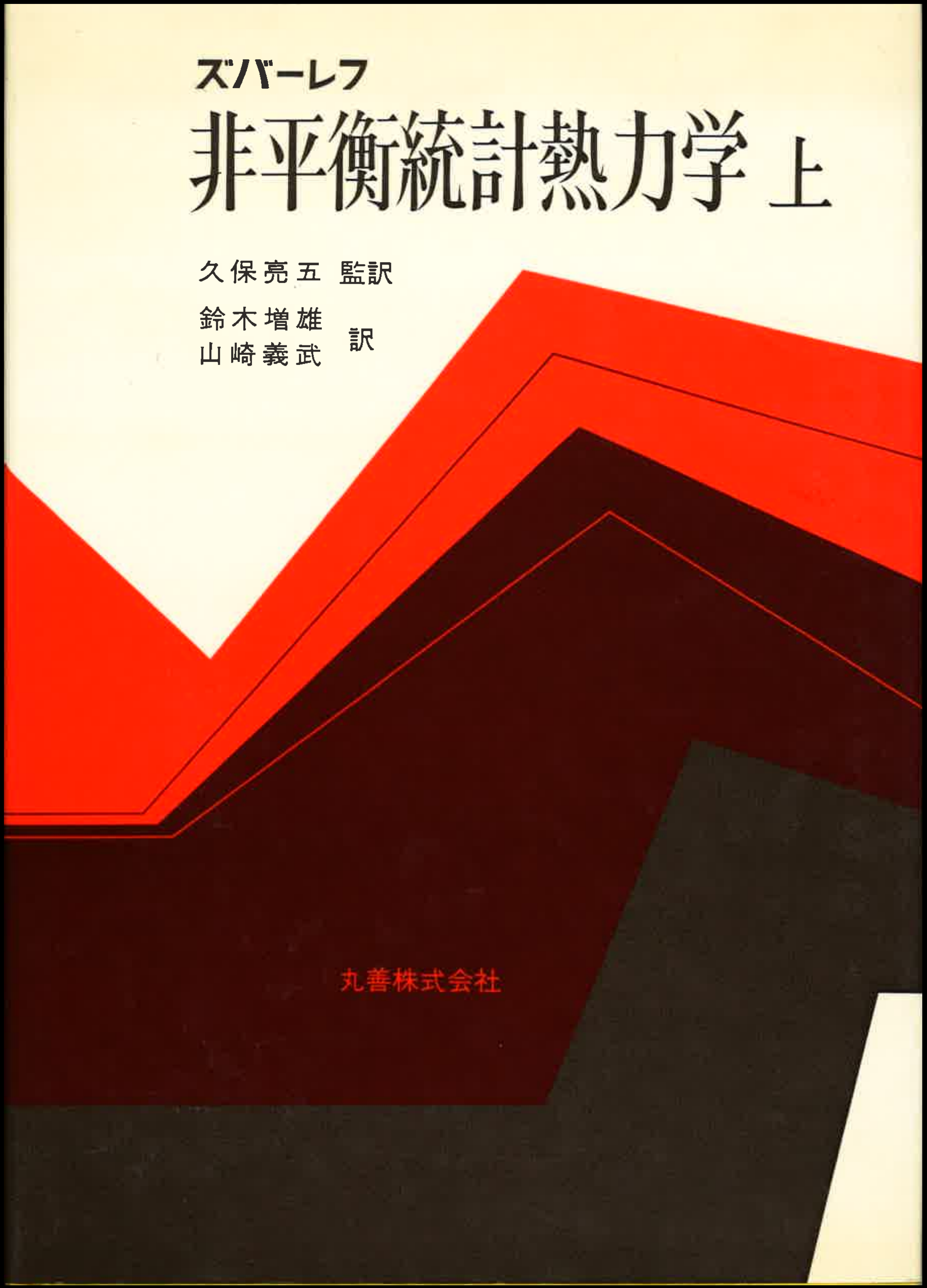

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非平衡統計熱力学 上

久保亮五 監訳

鈴木増雄
山崎義武 訳

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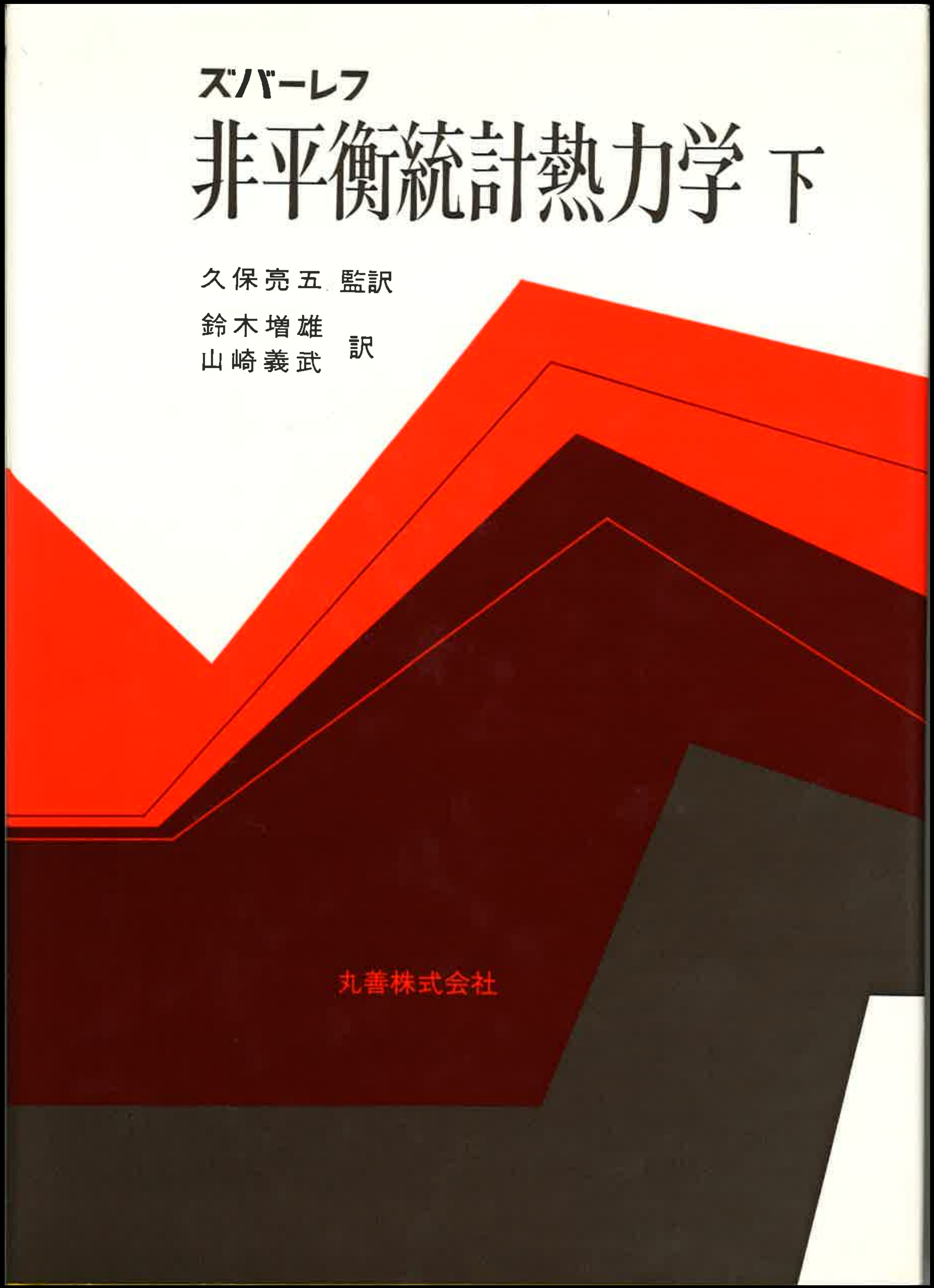

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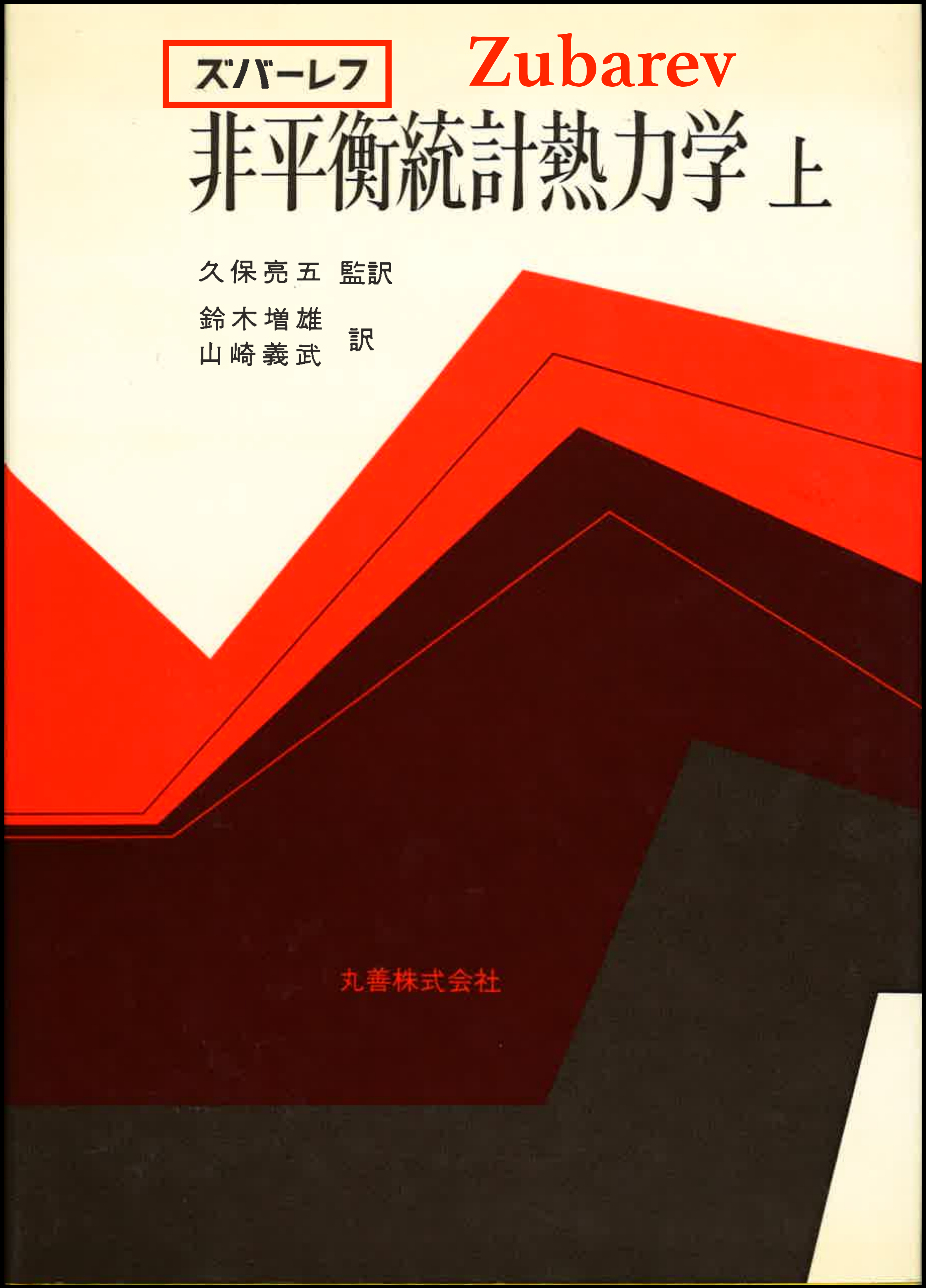

Zubarev

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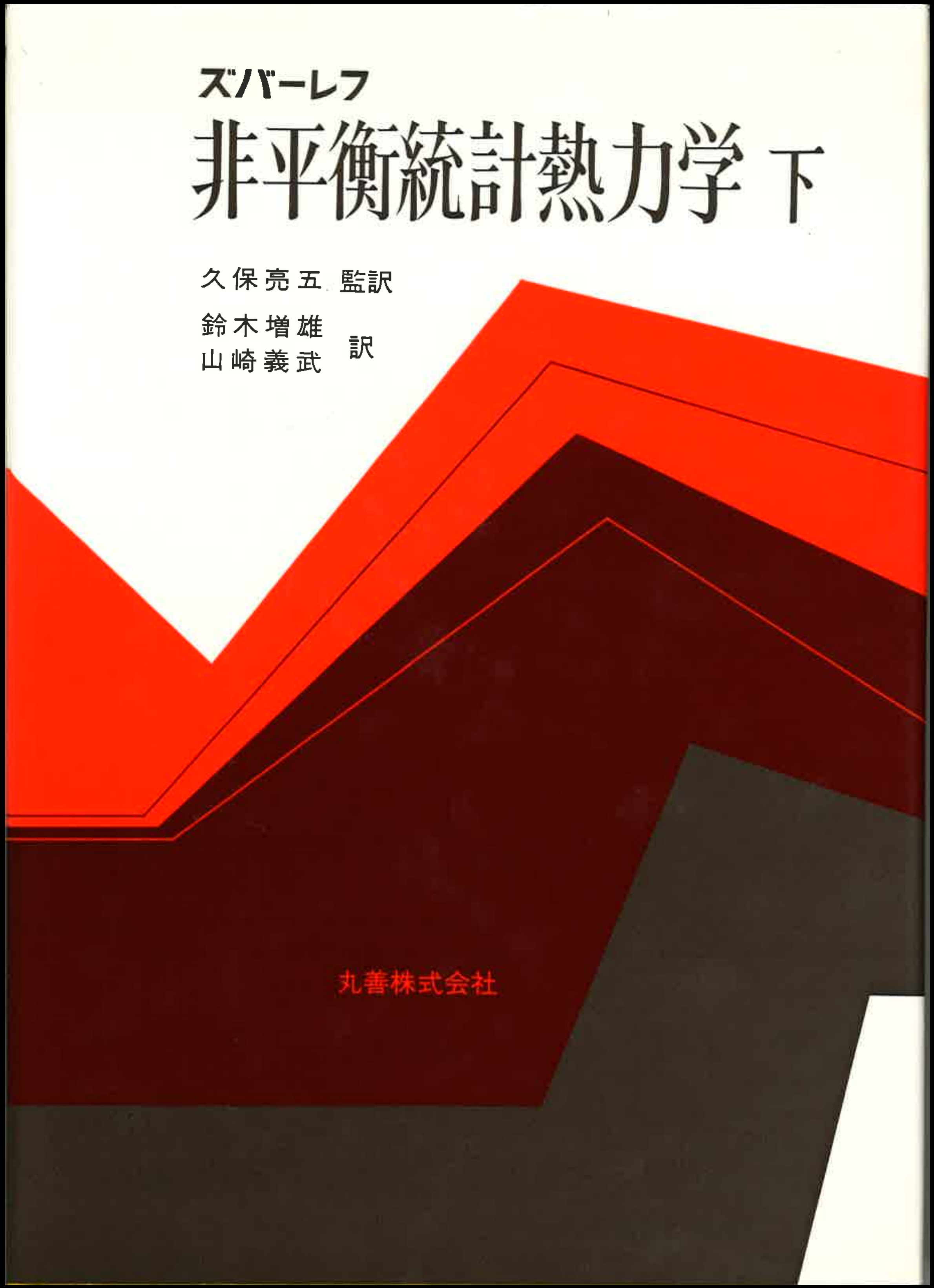

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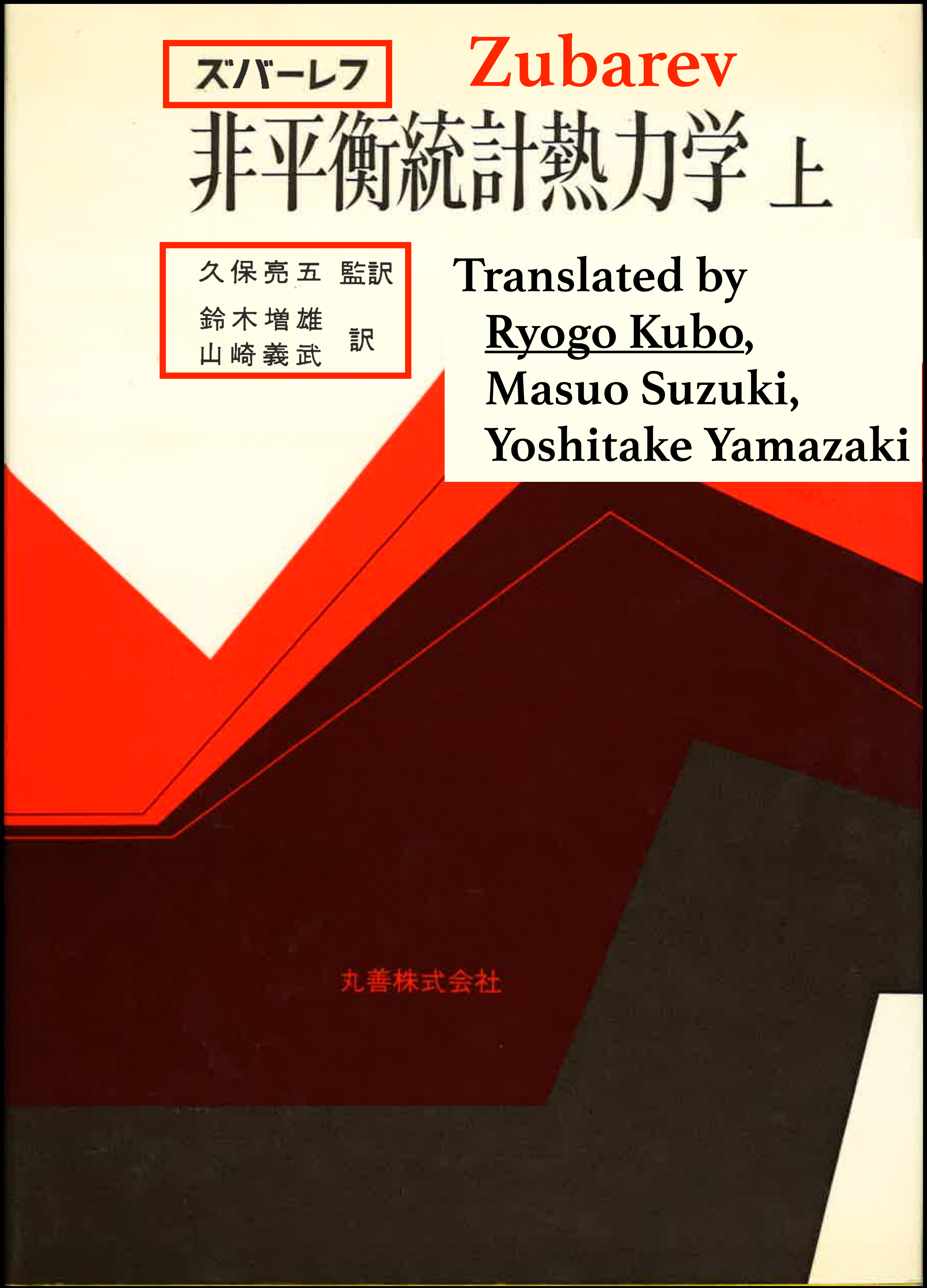

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Zubarev

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Translated by
Ryogo Kubo,
Masuo Suzuki,
Yoshitake Yamazaki

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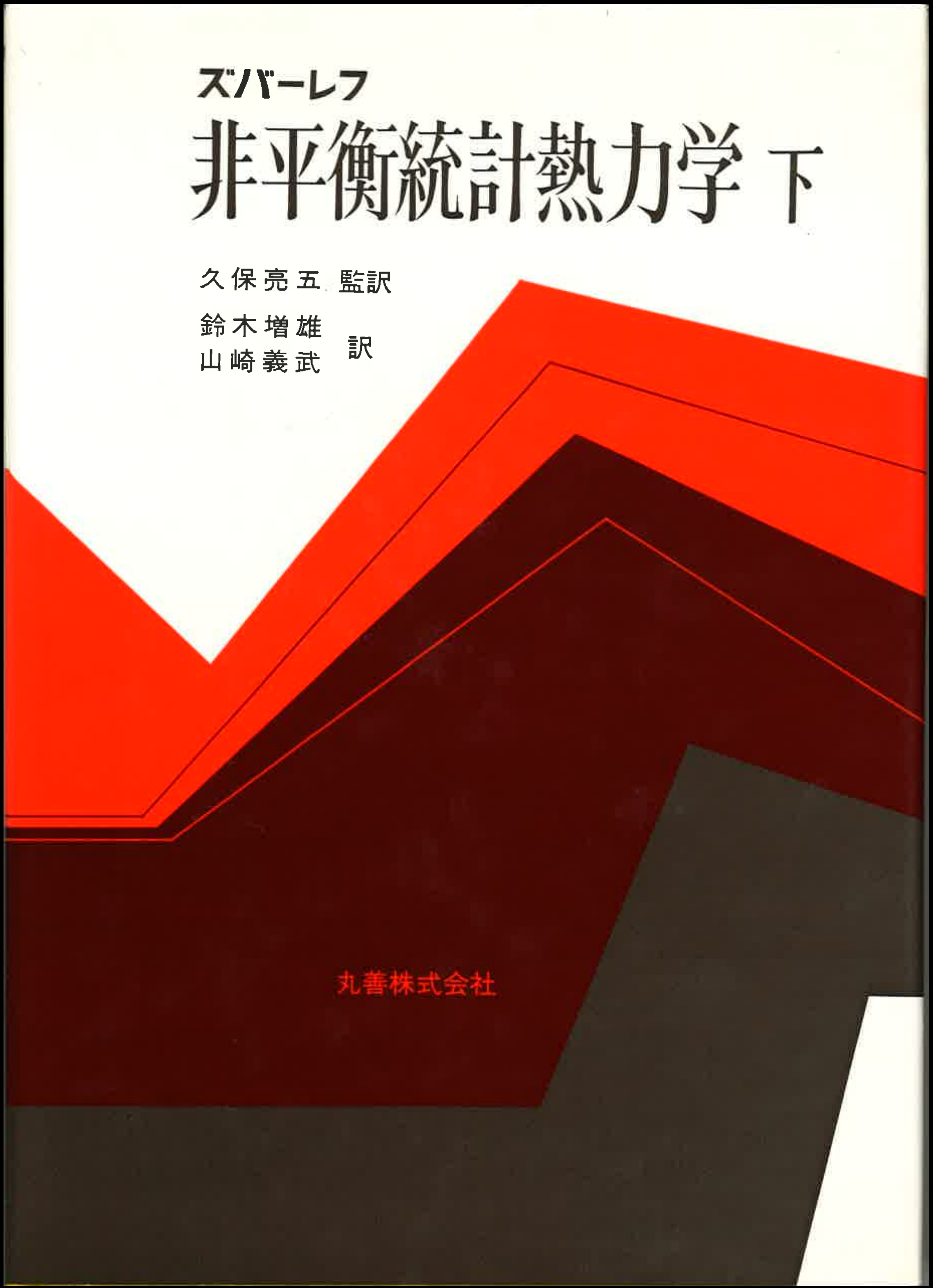

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# Zubarev’s method in the 21st century

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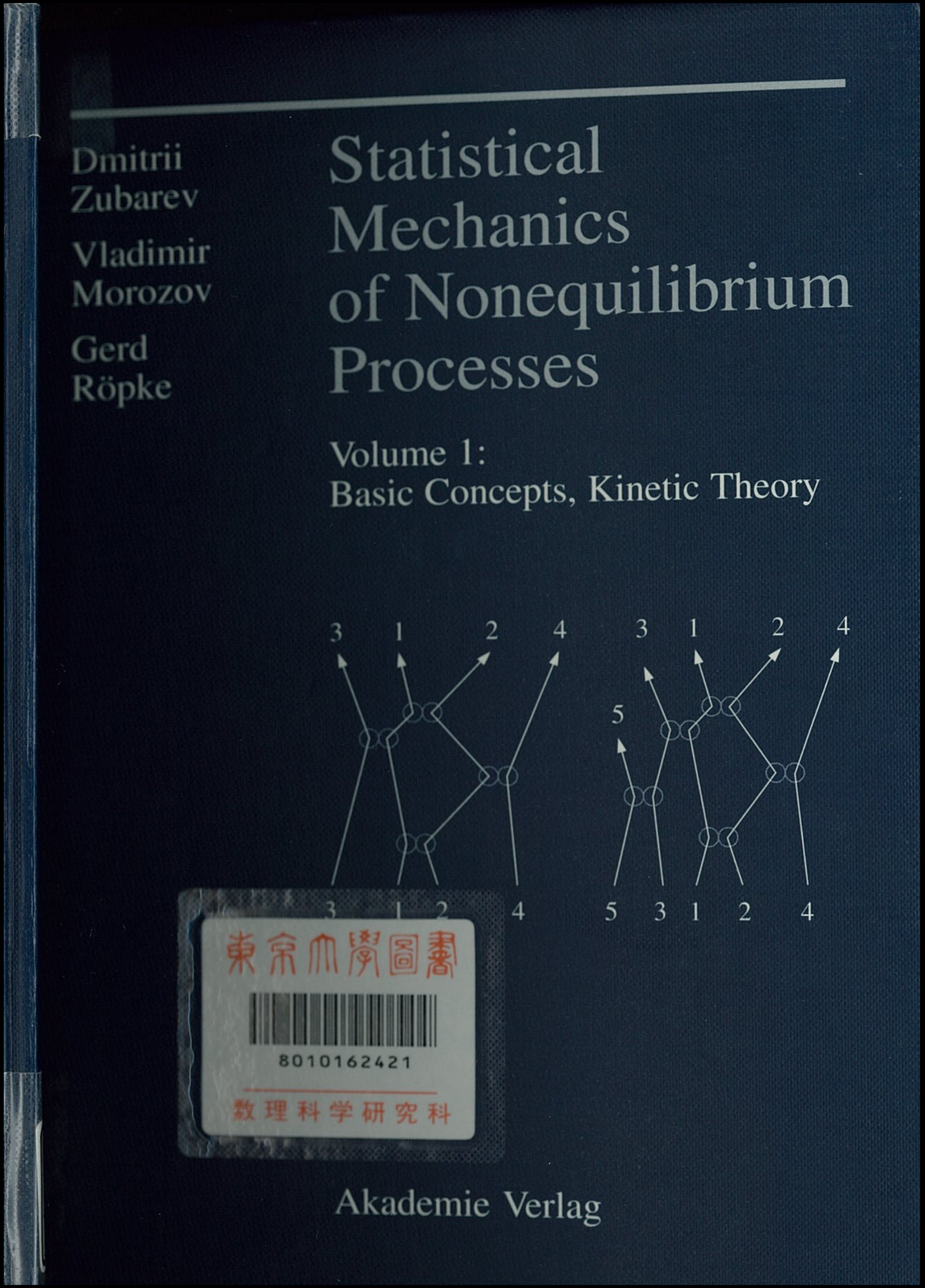

Dmitrii
Zubarev
Vladimir
Morozov
Gerd
Röpke

## Statistical Mechanics of Nonequilibrium Processes

Volume 1:
Basic Concepts, Kinetic Theory

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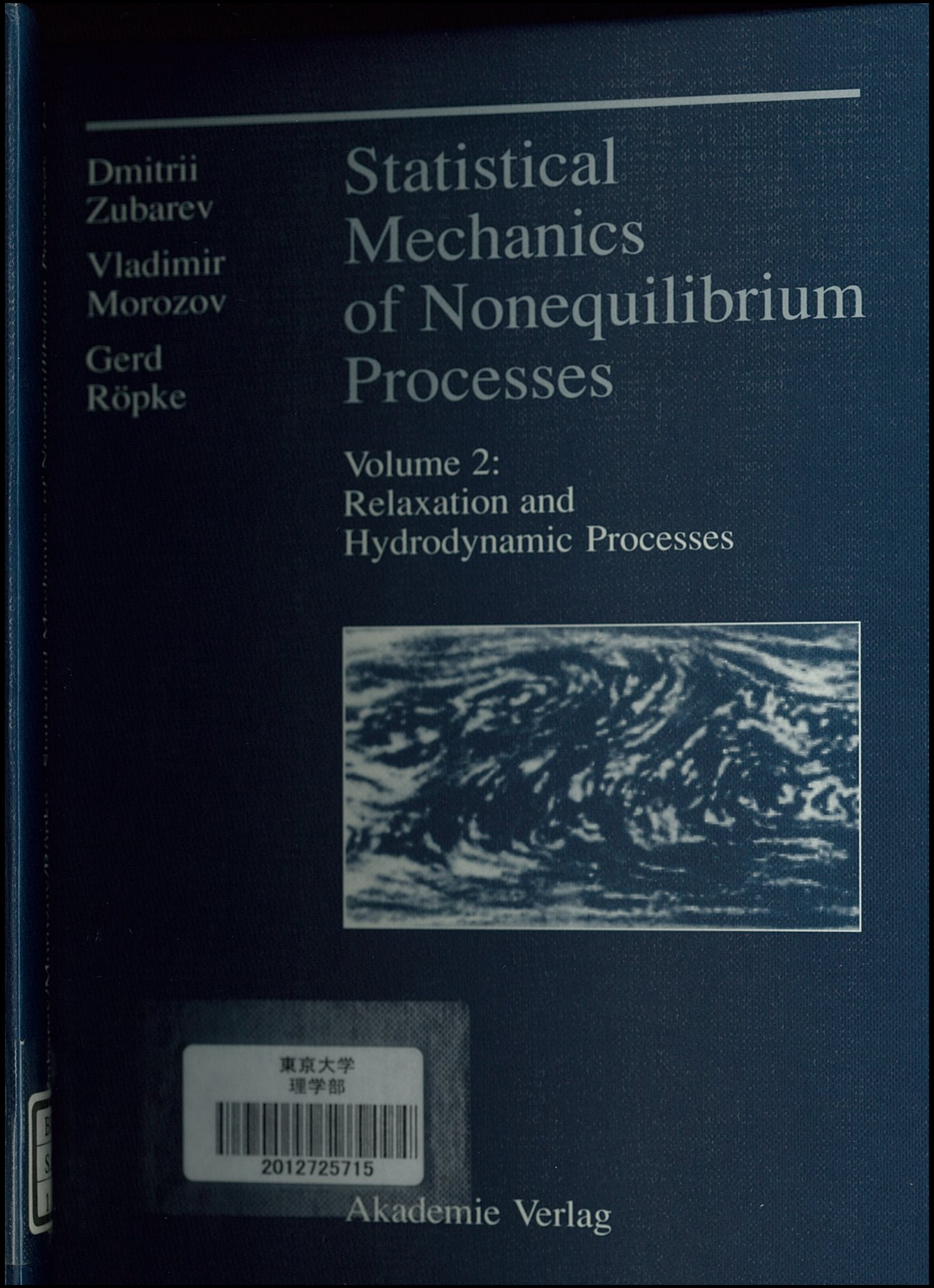

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## Statistical Mechanics of Nonequilibrium Processes

Volume 2:
Relaxation and
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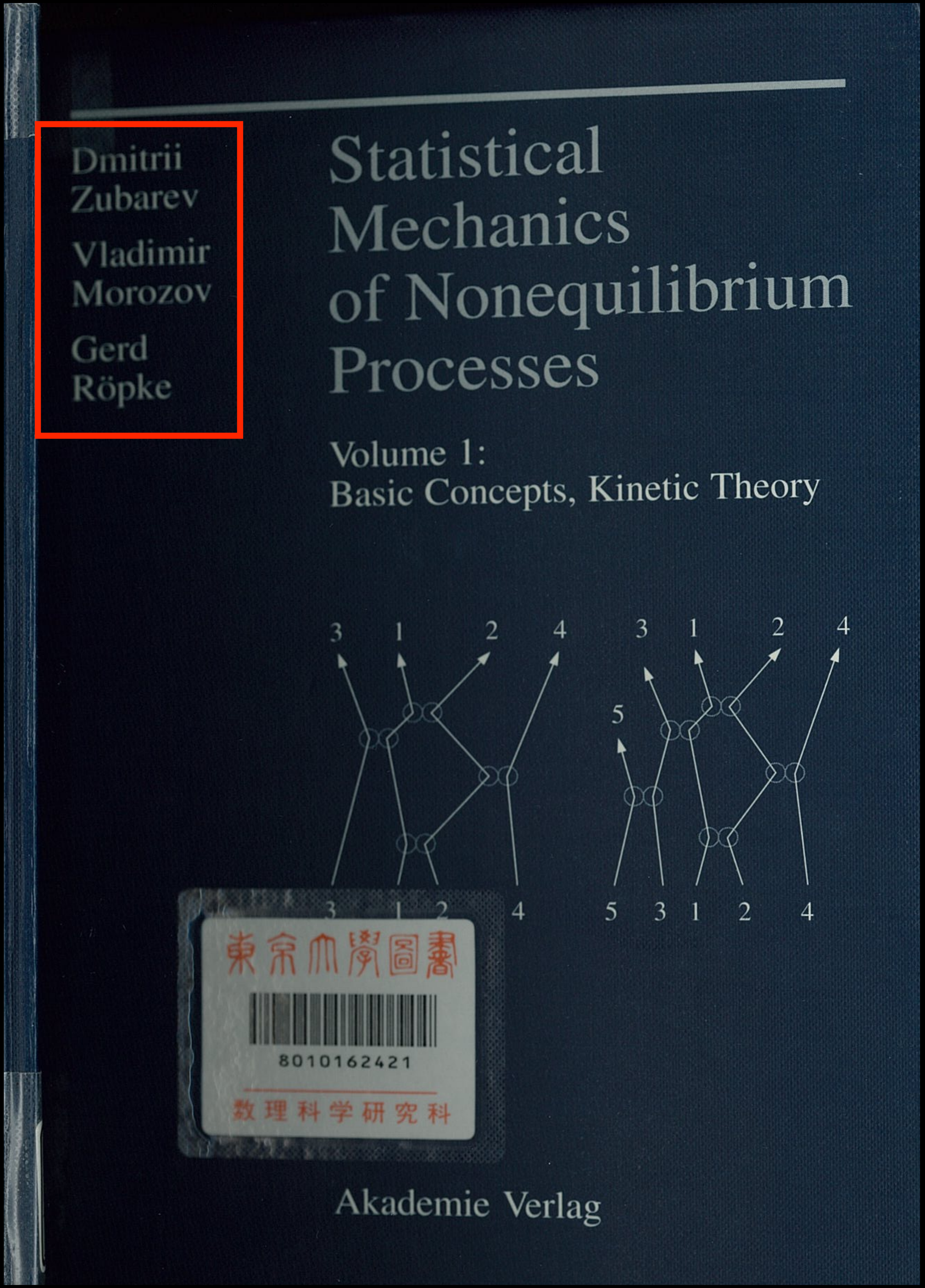

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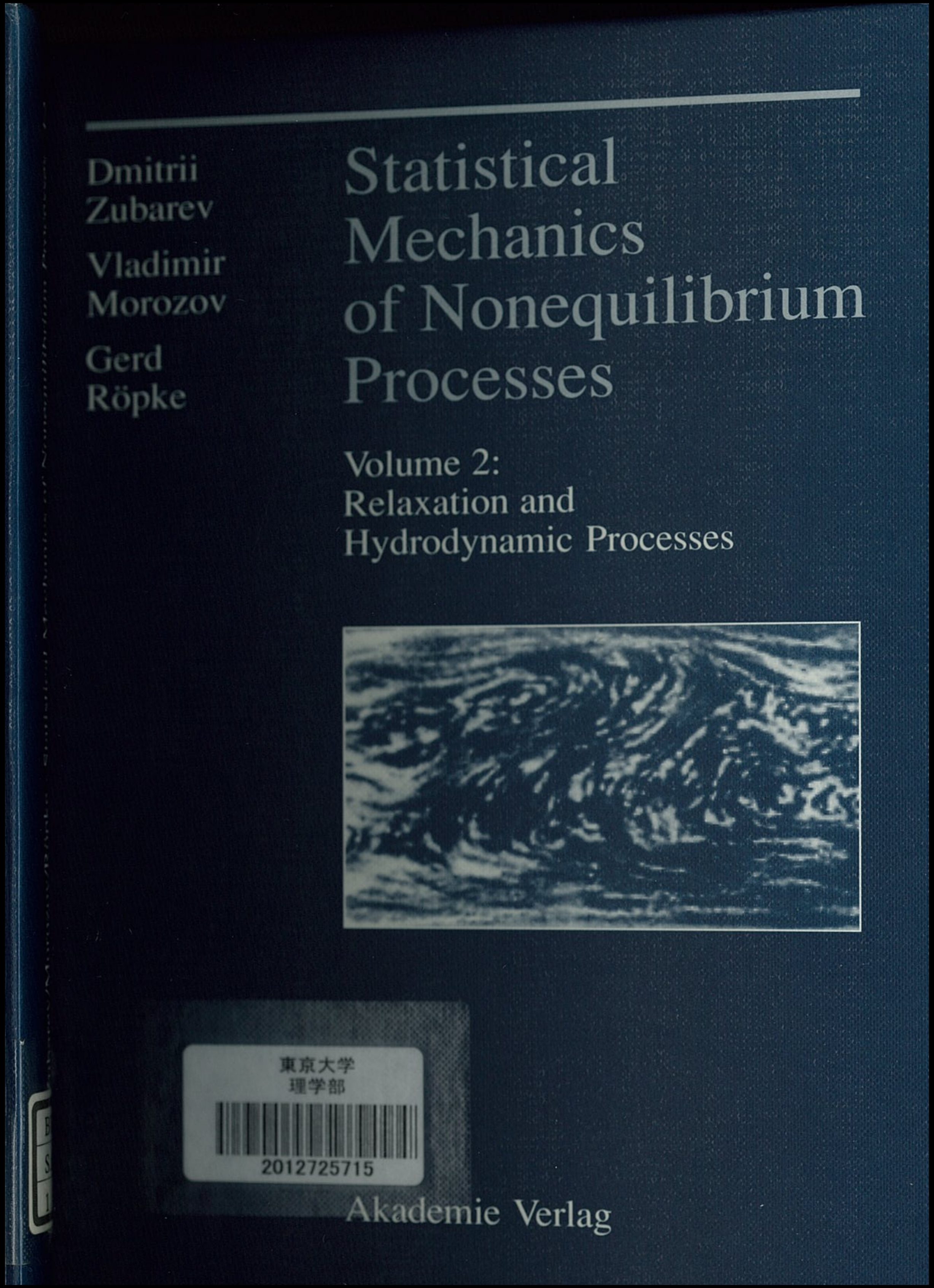

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# Recent development after Zubarev

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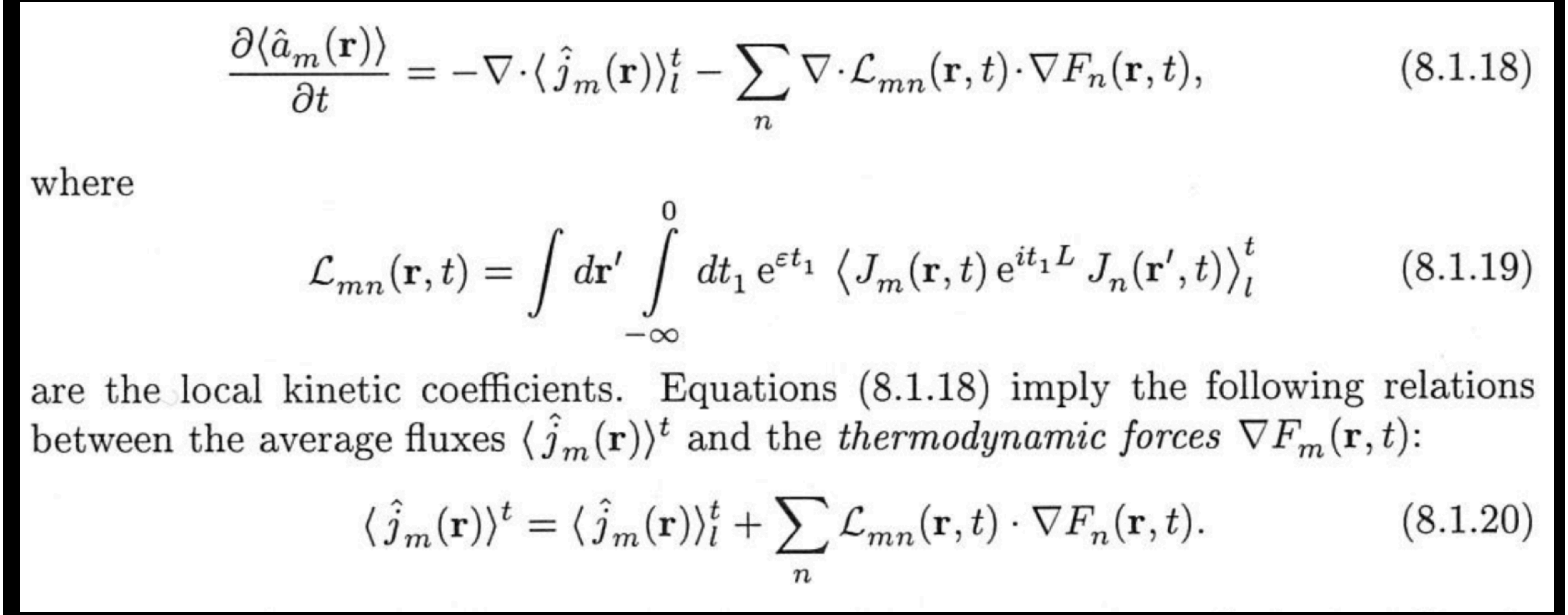
◆ **From “Statistical mechanics of Nonequilibrium Processes Vol.2”**



$$\frac{\partial\langle \hat{a}_m(\mathbf{r})\rangle}{\partial t} = -\nabla\cdot\langle \hat{j}_m(\mathbf{r})\rangle_l^t - \sum_n \nabla\cdot\mathcal{L}_{mn}(\mathbf{r},t)\cdot\nabla F_n(\mathbf{r},t), \tag{8.1.18}$$

where

$$\mathcal{L}_{mn}(\mathbf{r},t) = \int d\mathbf{r}' \int_{-\infty}^{0} dt_1\, \mathrm{e}^{\varepsilon t_1}\, \left\langle J_m(\mathbf{r},t)\, \mathrm{e}^{i t_1 L}\, J_n(\mathbf{r}',t)\right\rangle_l^t \tag{8.1.19}$$

are the local kinetic coefficients. Equations (8.1.18) imply the following relations between the average fluxes $\langle \hat{j}_m(\mathbf{r})\rangle^t$ and the *thermodynamic forces* $\nabla F_m(\mathbf{r},t)$:

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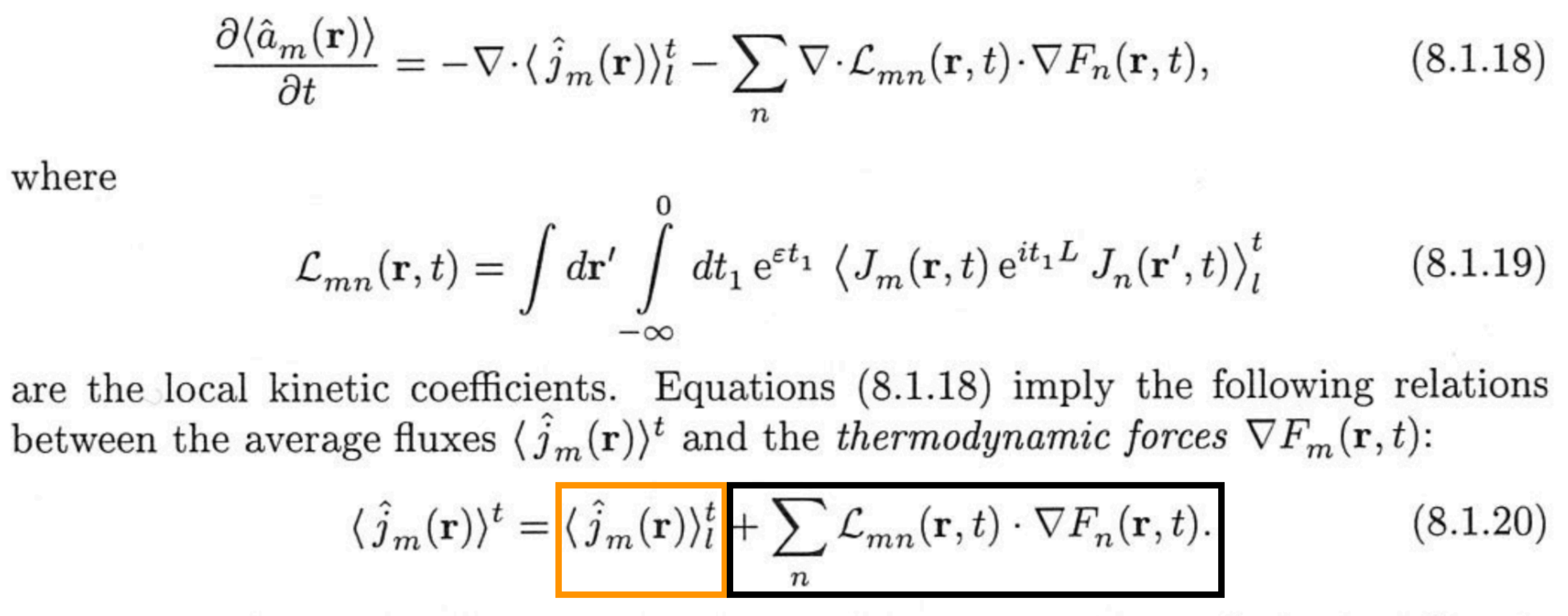

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Nondissipative & dissipative transport

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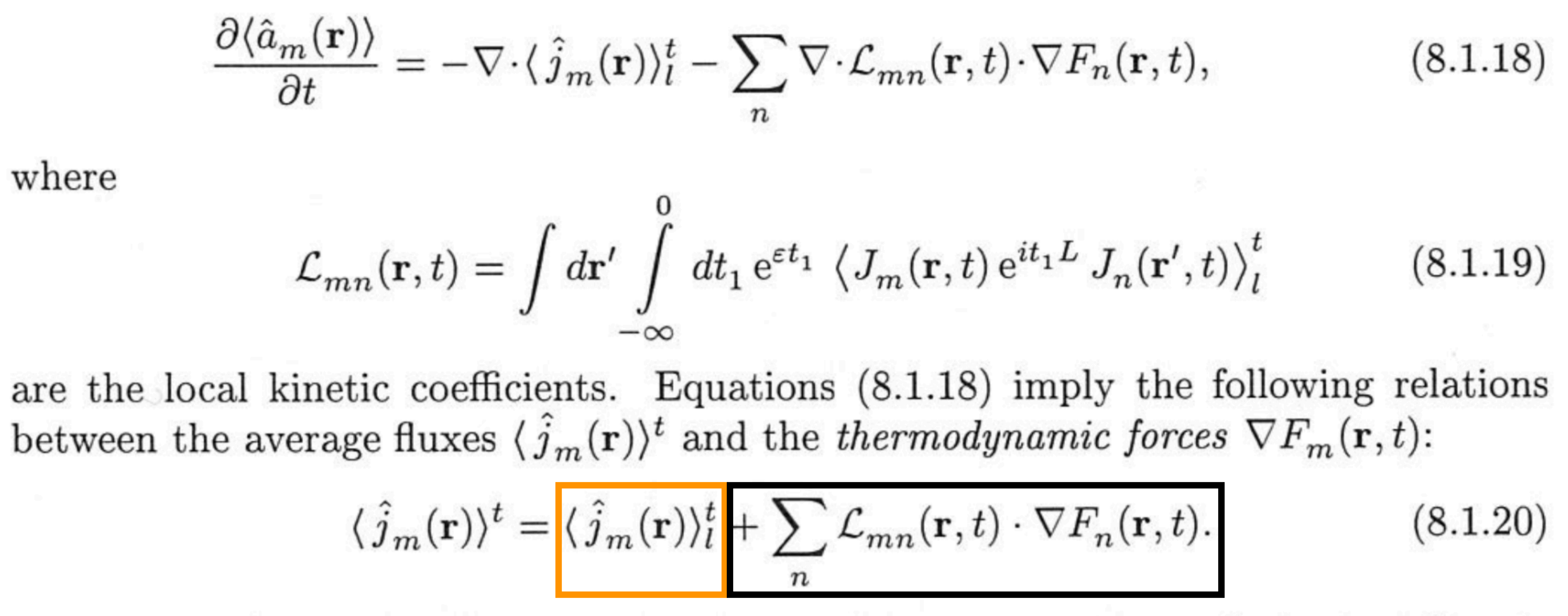

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Nondissipative & dissipative transport

(Perfect fluid)

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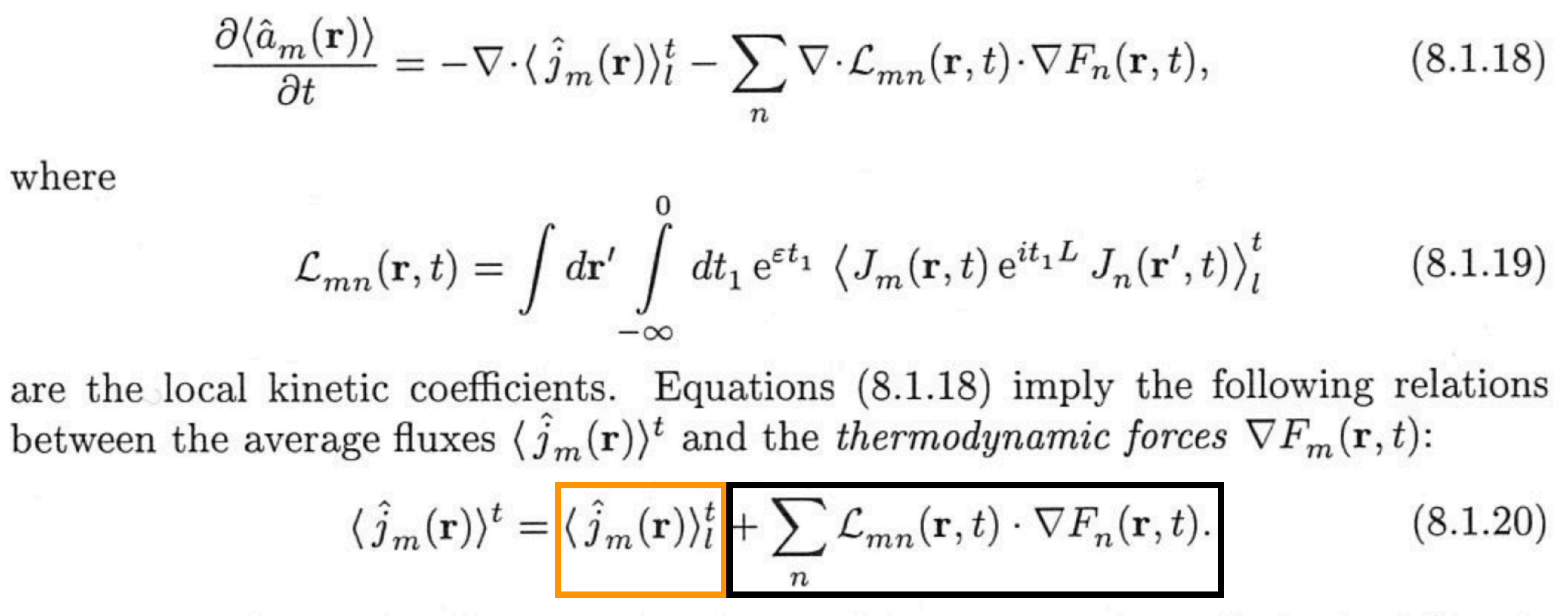

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Nondissipative & dissipative transport

(Perfect fluid) (Navier-Stokes fluid)

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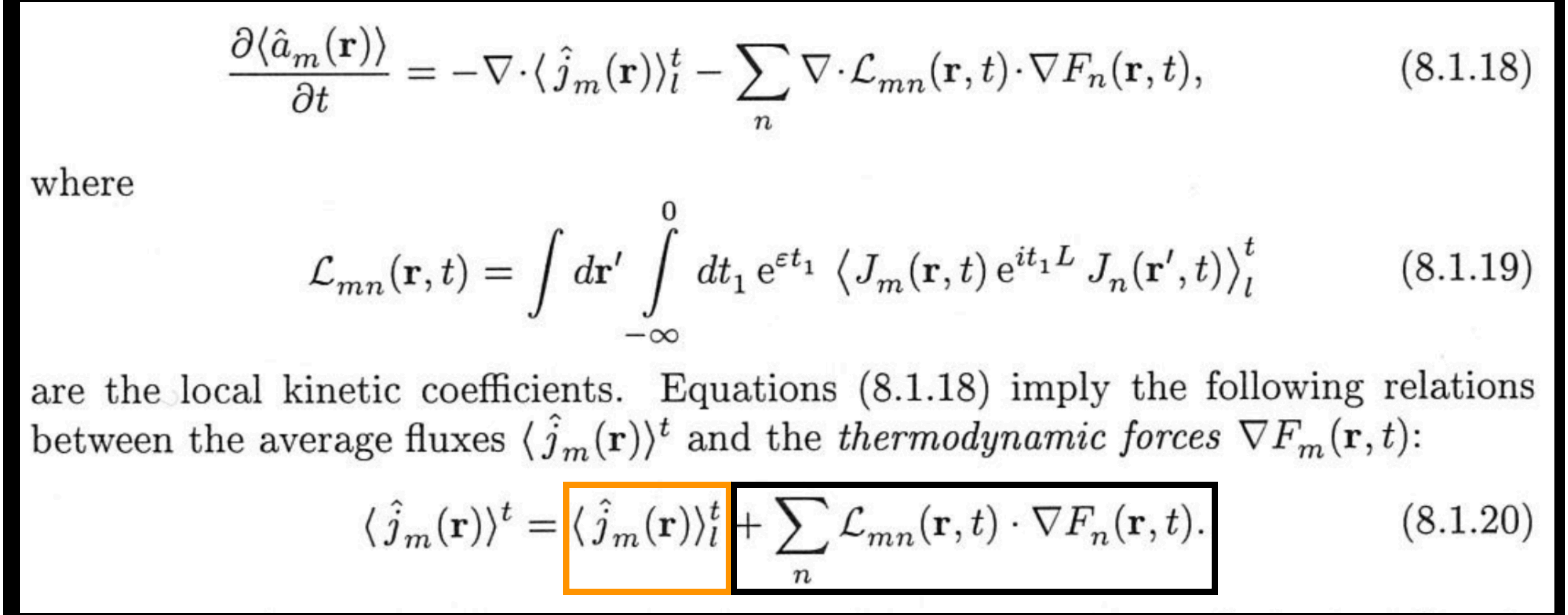

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Nondissipative & dissipative transport

(Perfect fluid) (Navier-Stokes fluid)

⇒ Nondissipative part has interesting & rich structure!

# Outline

**MOTIVATION:**

Quantum field theory under
local thermal equilibrium?

**APPROACH:**

QFT for Local Gibbs distribution

**APPLICATION:**

Derivation of
Anomalous hydrodynamics

# Hydrodynamics is

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- **Effective theory** for **macroscopic dynamics**

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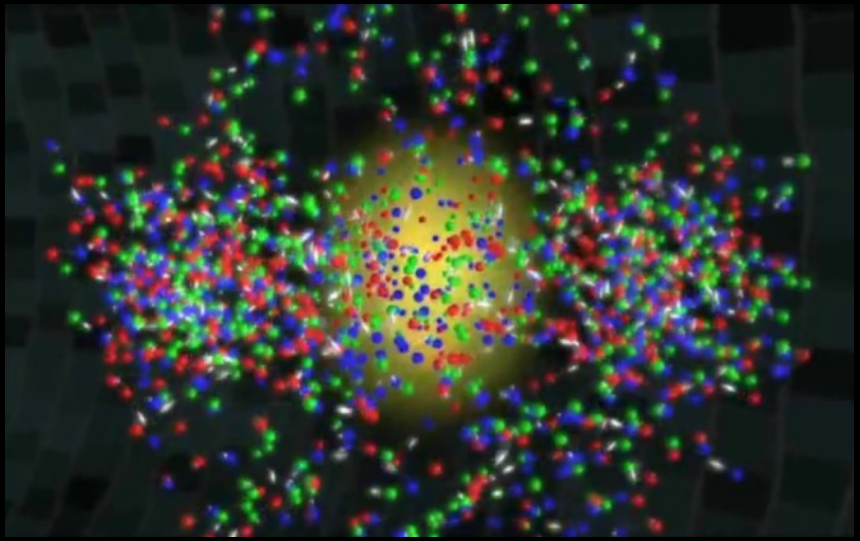
**Quark-Gluon Plasma**



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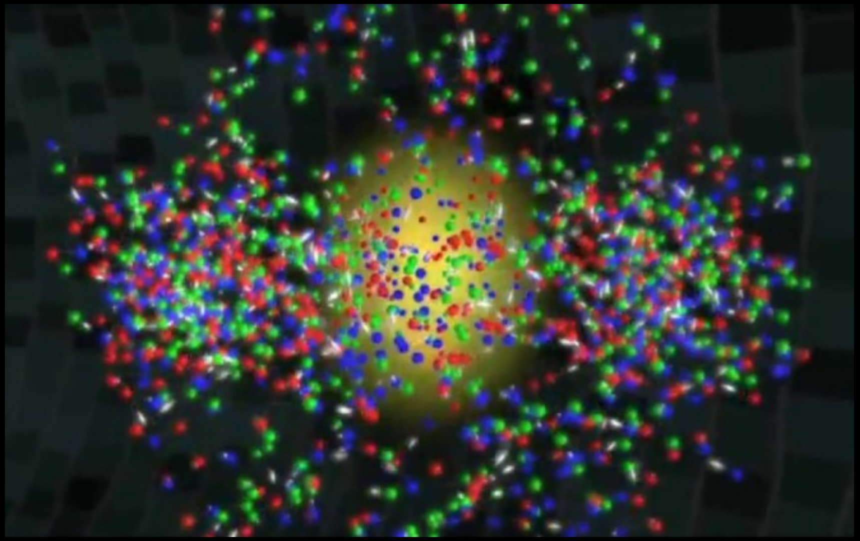

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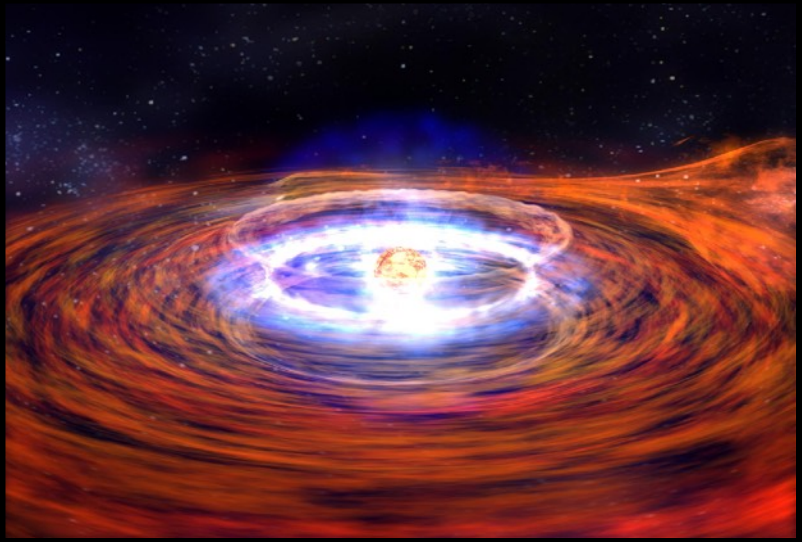
**Neutron Star**



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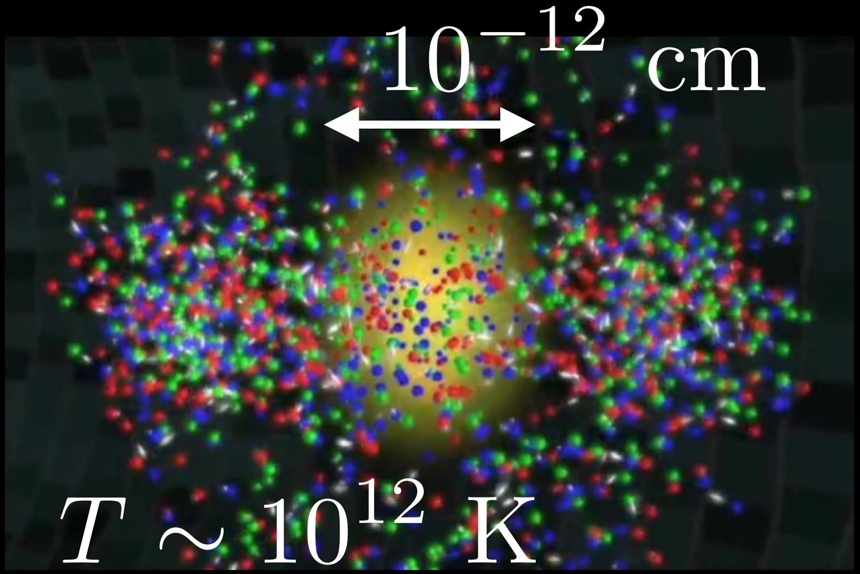

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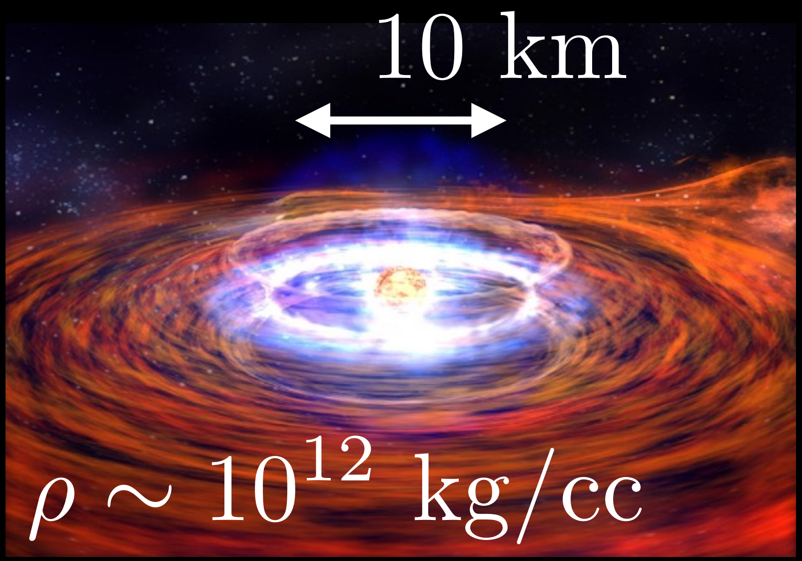

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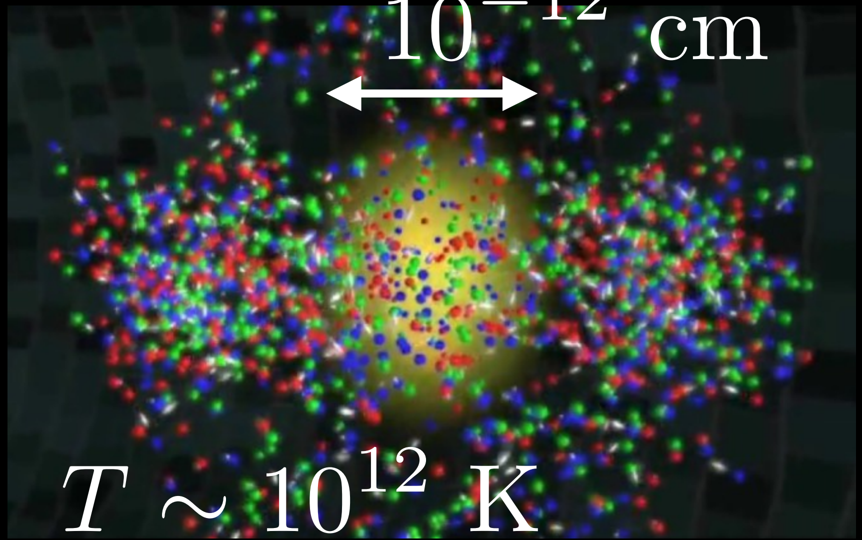

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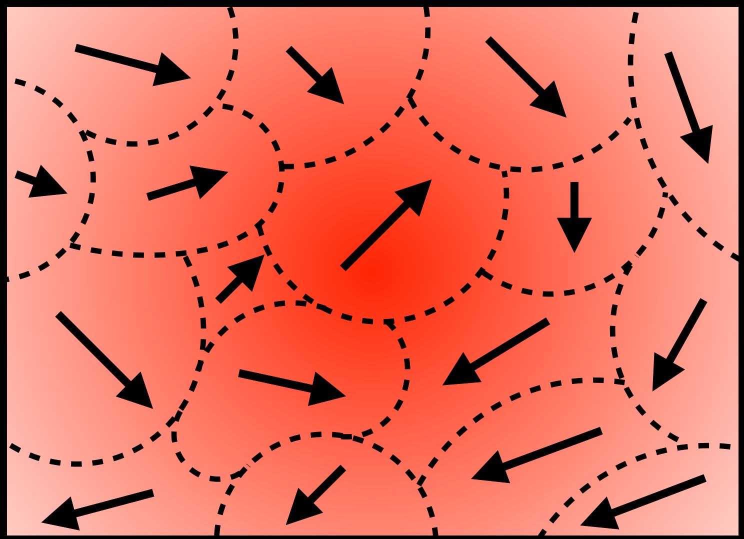
$\simeq$

**Hydro:** $\{\beta(x),\ \vec{v}(x)\}$



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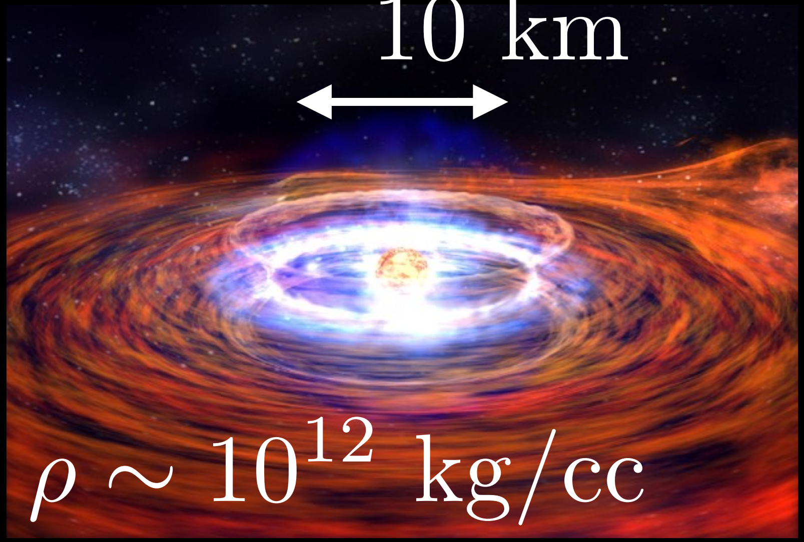

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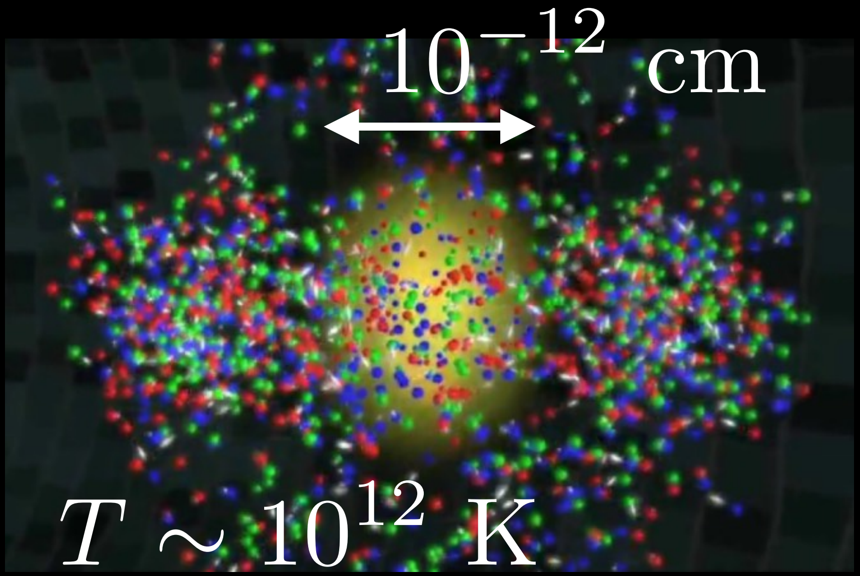
# Hydrodynamics is

- **Effective theory** for **macroscopic dynamics**
- **Universal description**, not depending on details
- Only **conserved quantity**

**Quark-Gluon Plasma**



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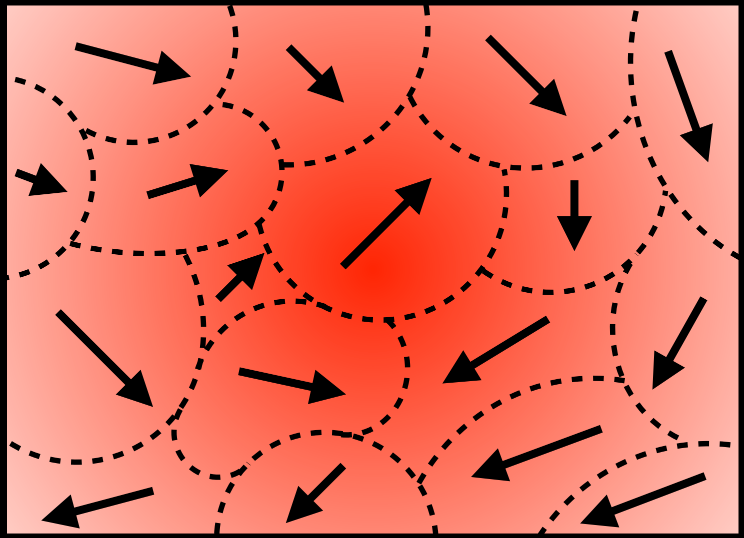

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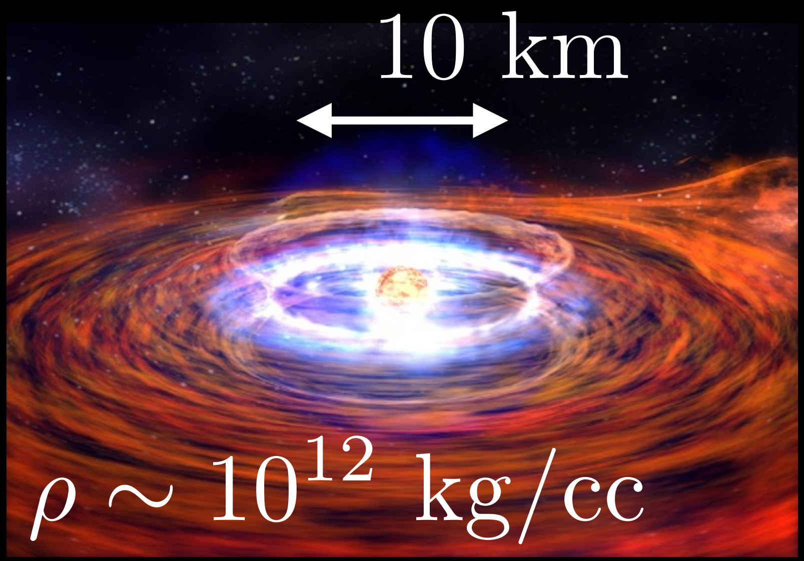

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- **Only conserved quantity ~ symmetry of system**

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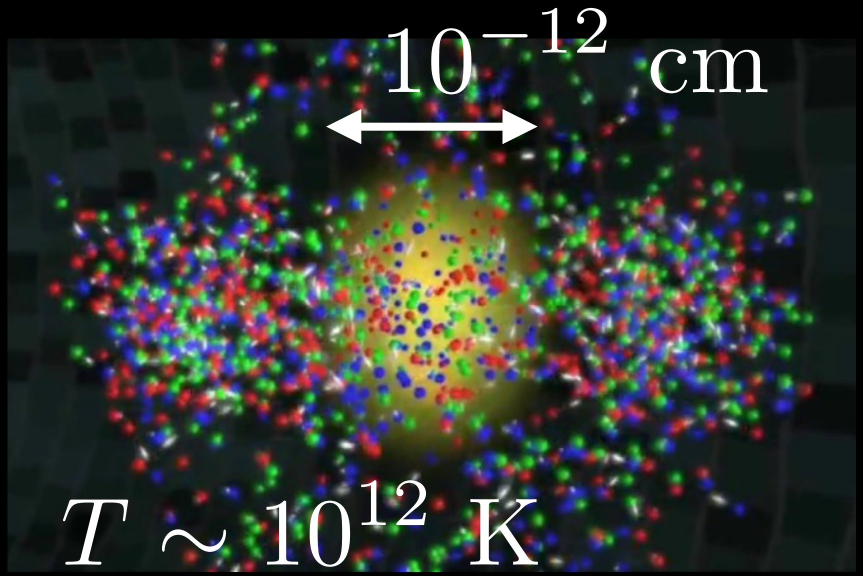

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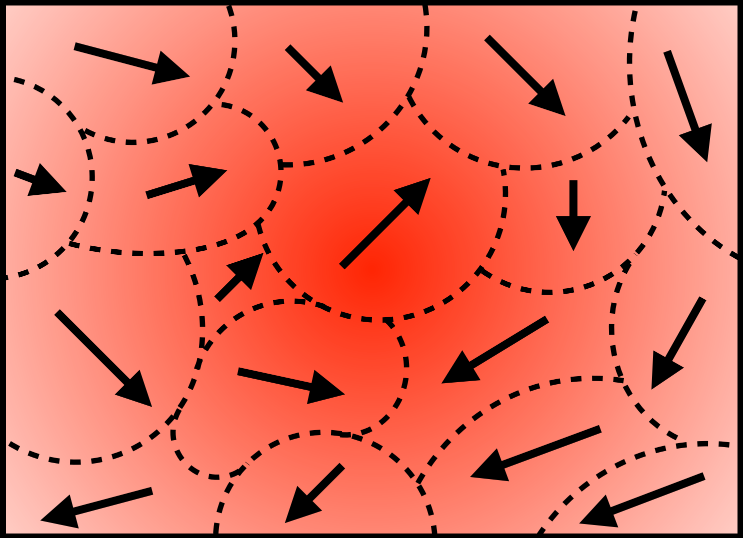

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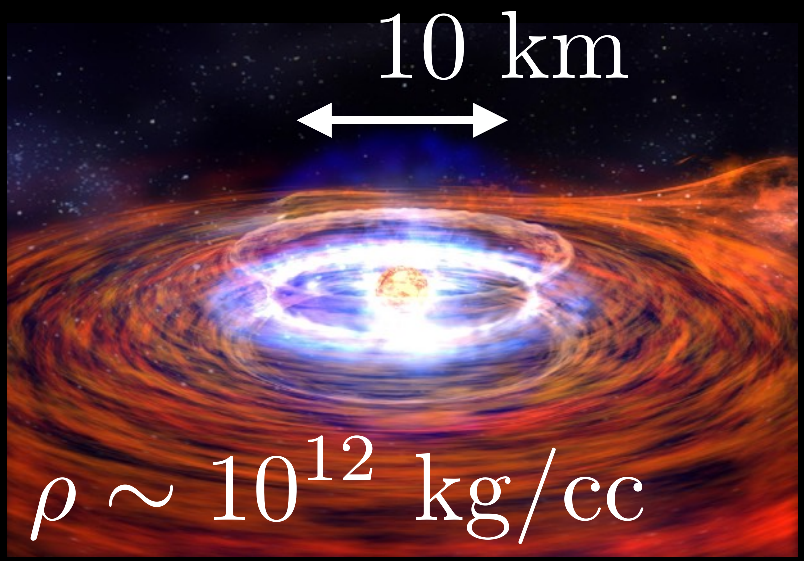

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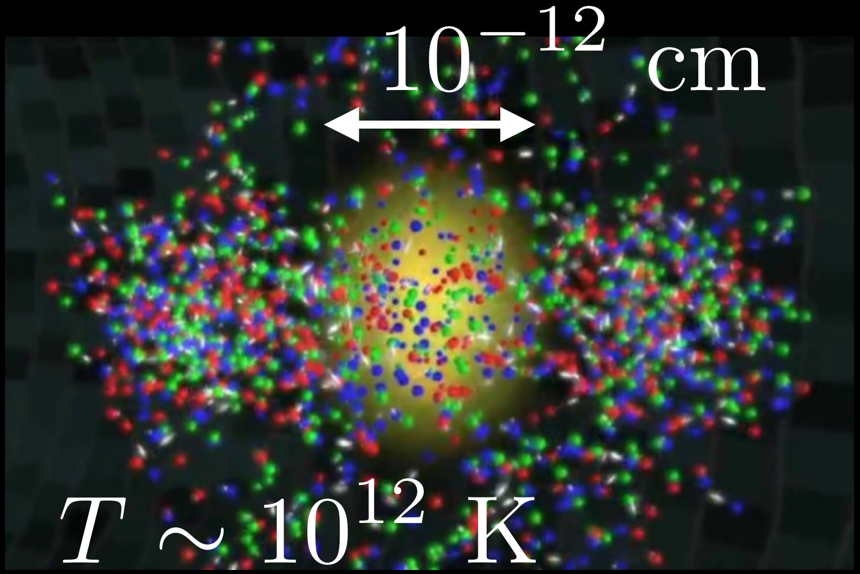

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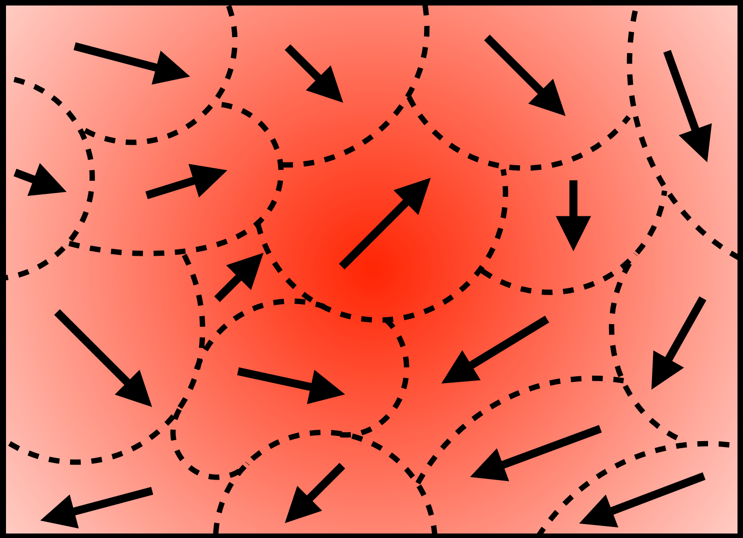

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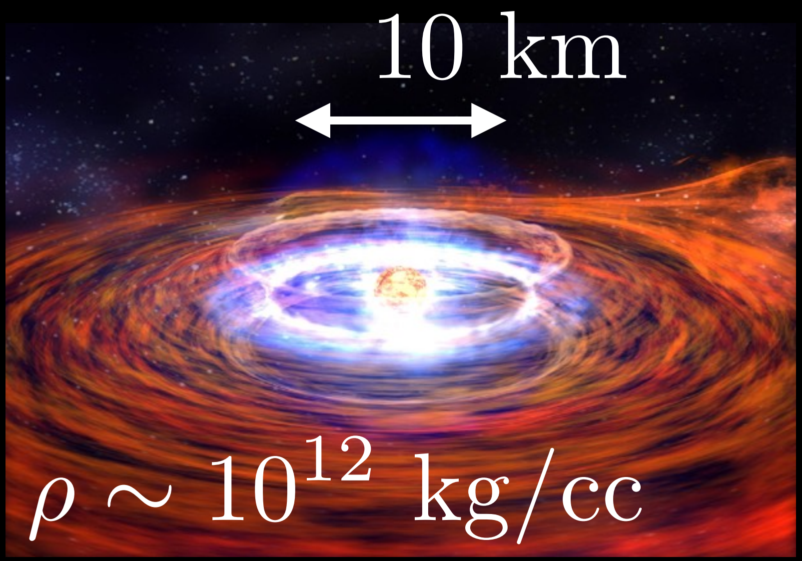

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# Symmetry breaking & Hydro

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- Spontaneous symmetry breaking

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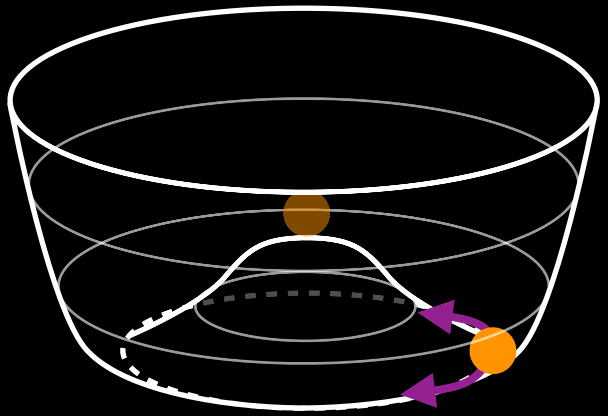
- **Spontaneous symmetry breaking**

  **Micro：Selecting vacuum**



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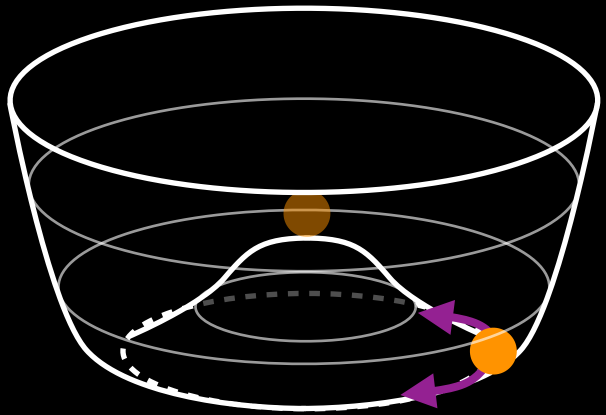
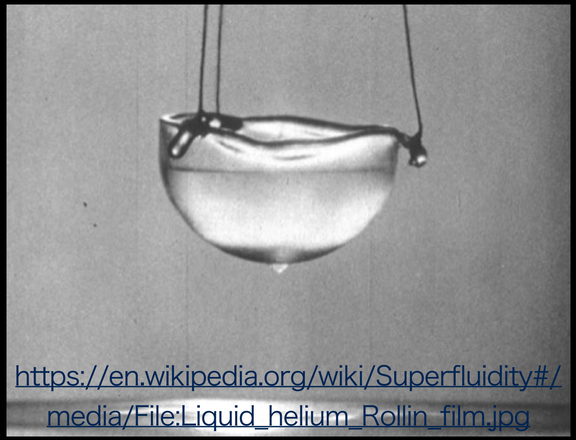


Macro：Superfluid



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# Symmetry breaking & Hydro

◆ Spontaneous symmetry breaking

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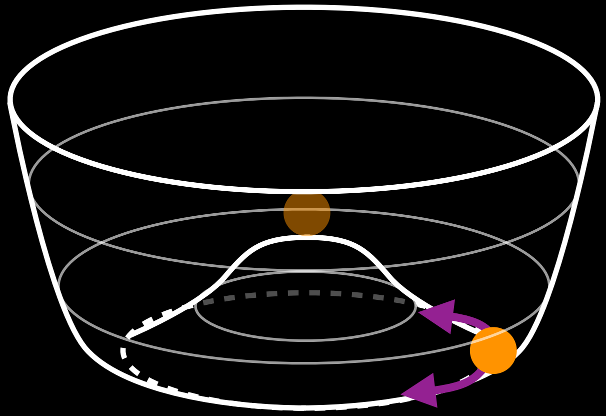

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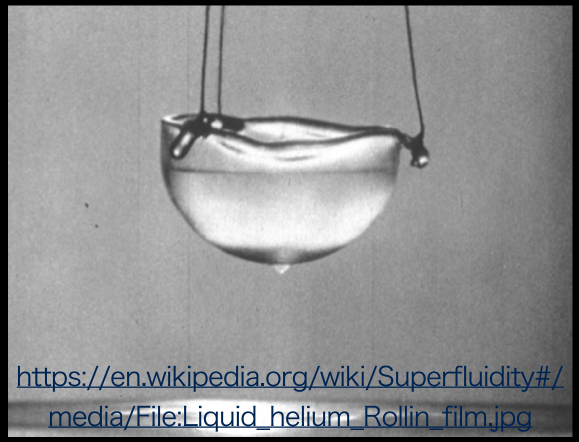

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◆ Symmetry breaking by quantum anomaly

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- Spontaneous symmetry breaking

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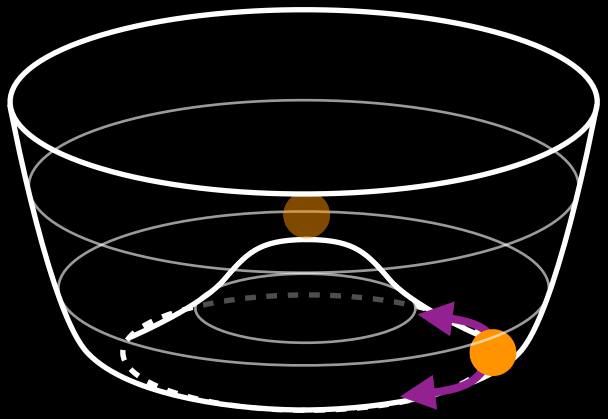
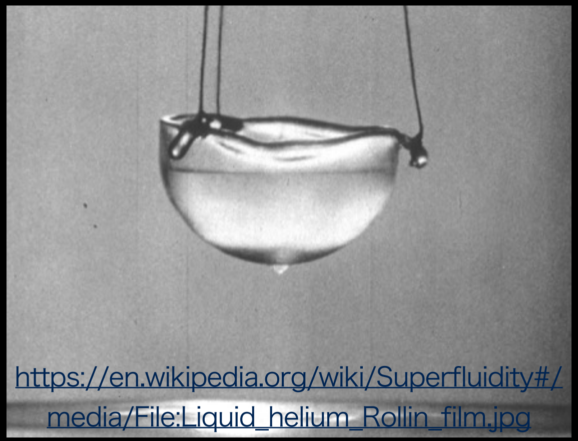


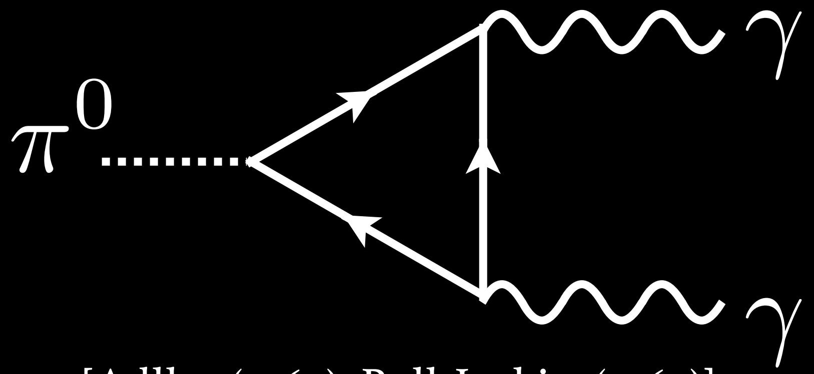
Macro：Superfluid



https://en.wikipedia.org/wiki/Superfluidity#/media/File:Liquid_helium_Rollin_film.jpg

- **Symmetry breaking by quantum anomaly**

Micro：$\pi^0$ decay

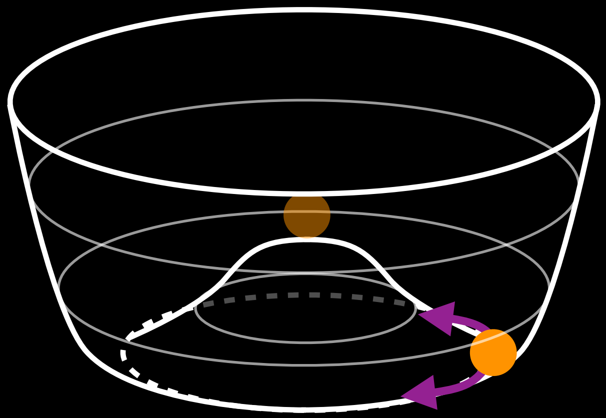


[Adller (1969), Bell-Jackiw (1969)]

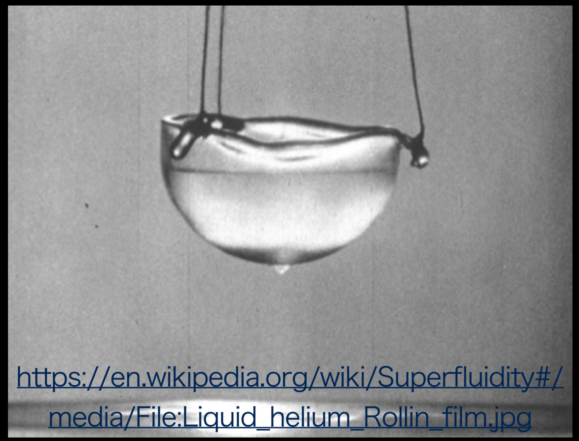
# Symmetry breaking & Hydro

- Spontaneous symmetry breaking

Micro : Selecting vacuum



Macro : Superfluid



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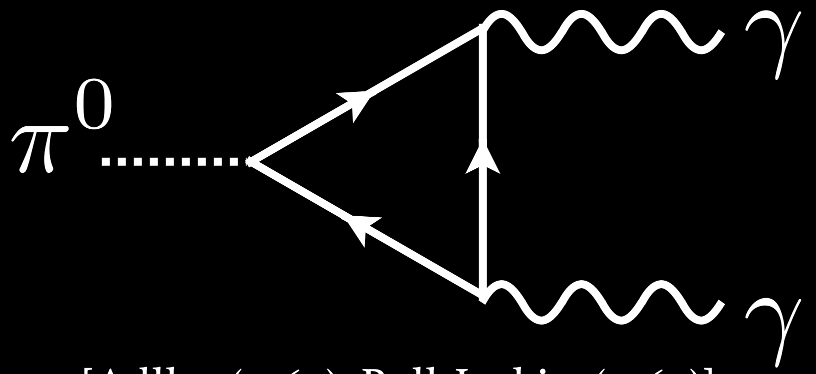
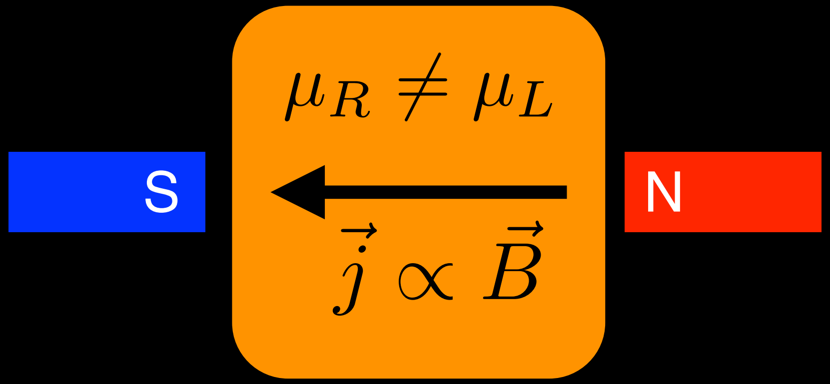


[Adller (1969), Bell-Jackiw (1969)]

Macro : Anomalous transport



[Erdmenger et al. (2008), Son-Surowka (2009)]

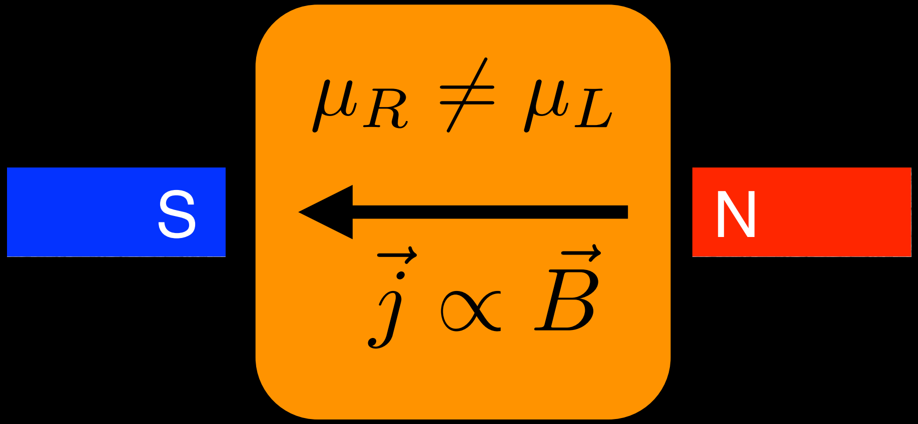
# Anomaly-induced transport

# Anomaly-induced transport

- **Chiral Magnetic Effect (CME)**

[Fukushima et al.(2008), Vilenkin (1980)]

$$\vec{j} = \frac{e\mu_5}{2\pi^2}\vec{B}$$



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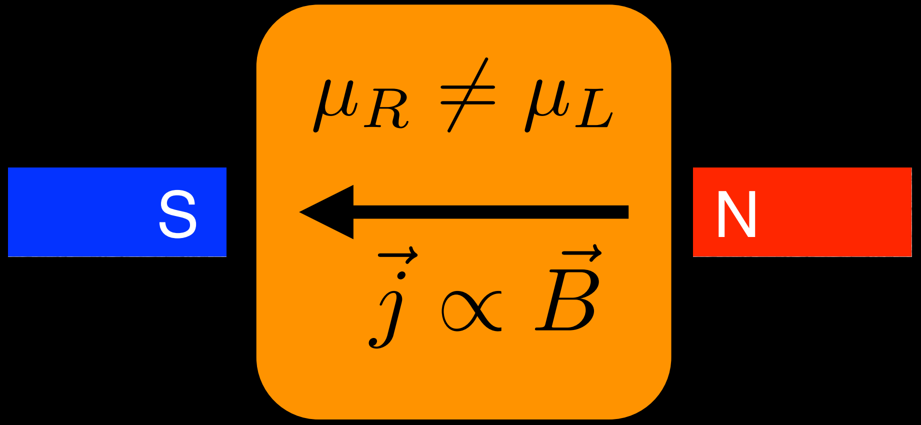
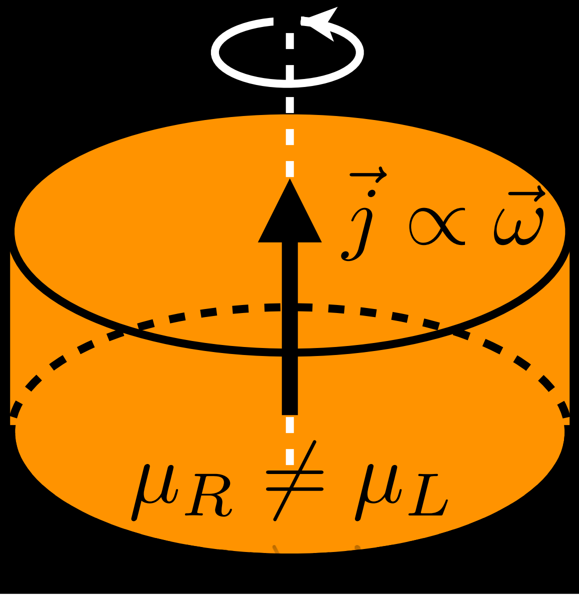
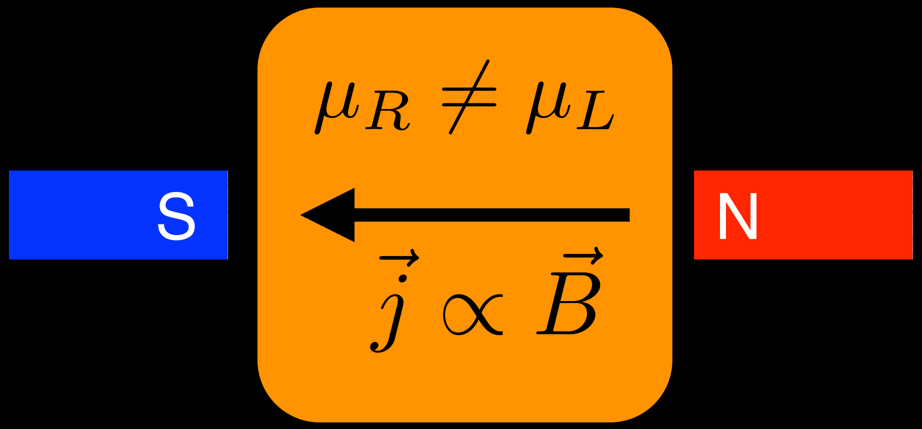


◆ **Chiral Vertical Effect (CVE)**

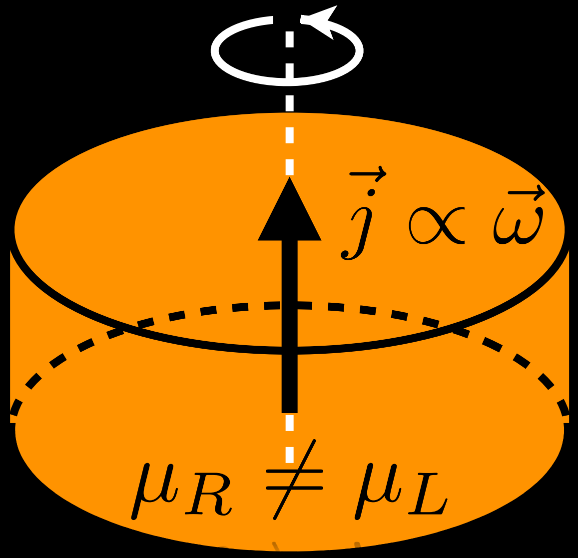
[Erdmenger et al. (2008), Son-Surowka (2009)]

$$\vec{j} = \frac{\mu\mu_5}{2\pi^2}\vec{\omega}$$



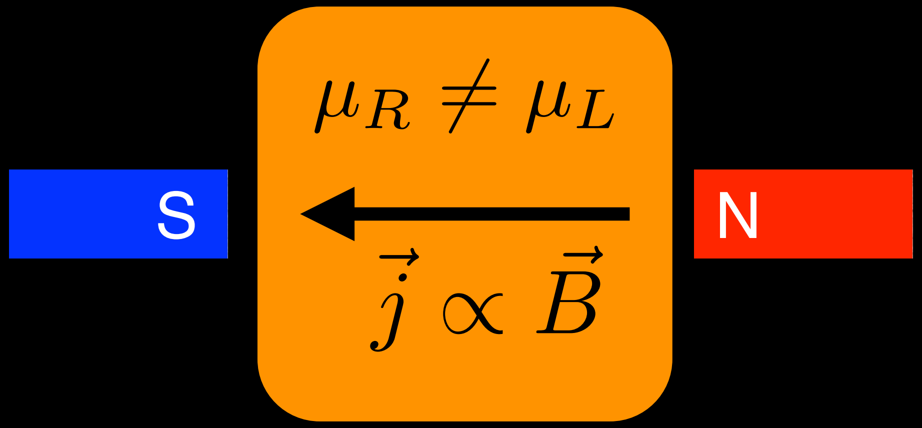


Chiral Magnetic Effect

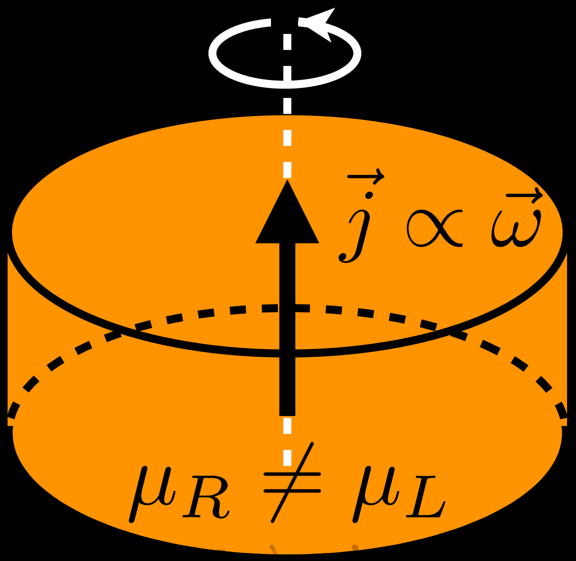


Chiral Vortical Effect

# Are these new?



Chiral Magnetic Effect



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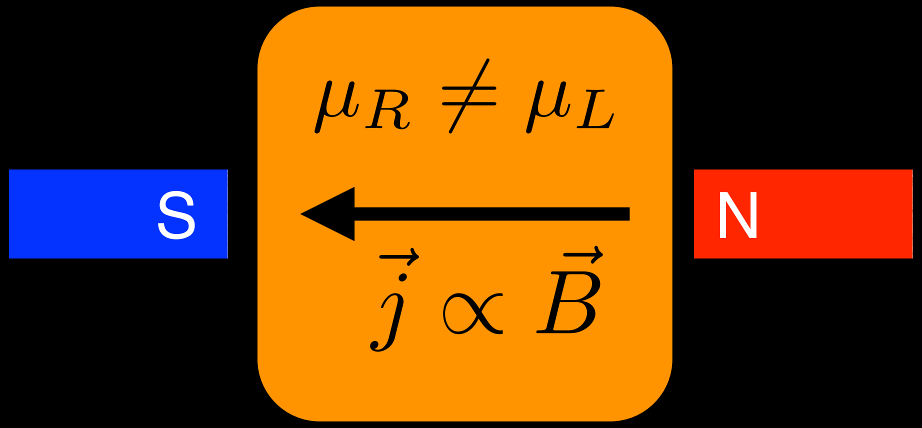

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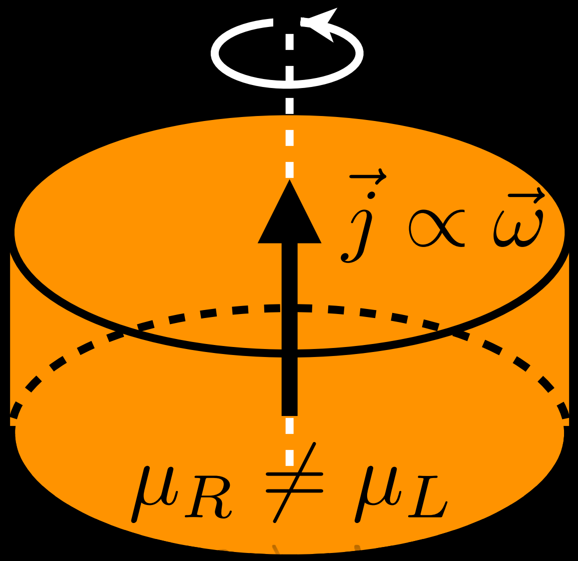

Chiral Vortical Effect

**YES!**

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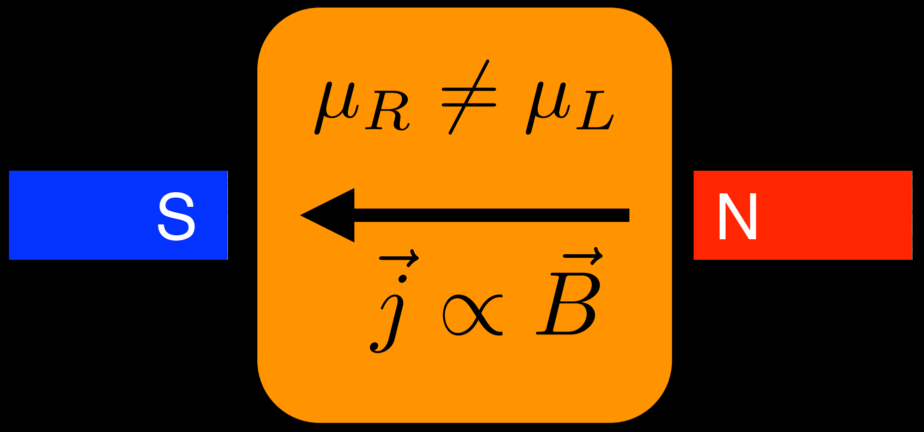
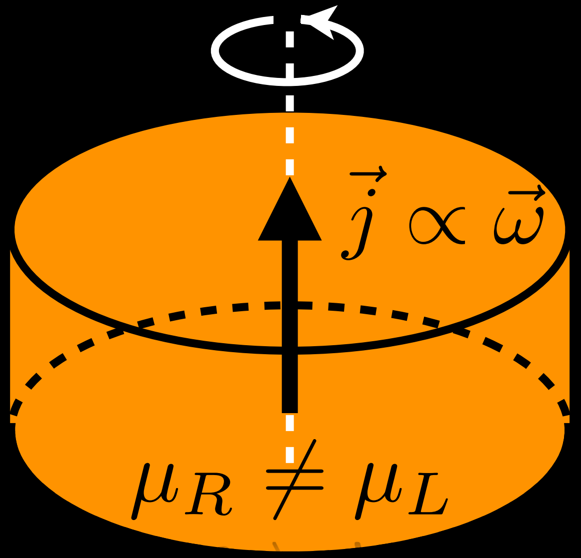


Chiral Magnetic Effect



Chiral Vortical Effect

<u>**YES!**</u>

They are not covered by
famous two textbooks!!

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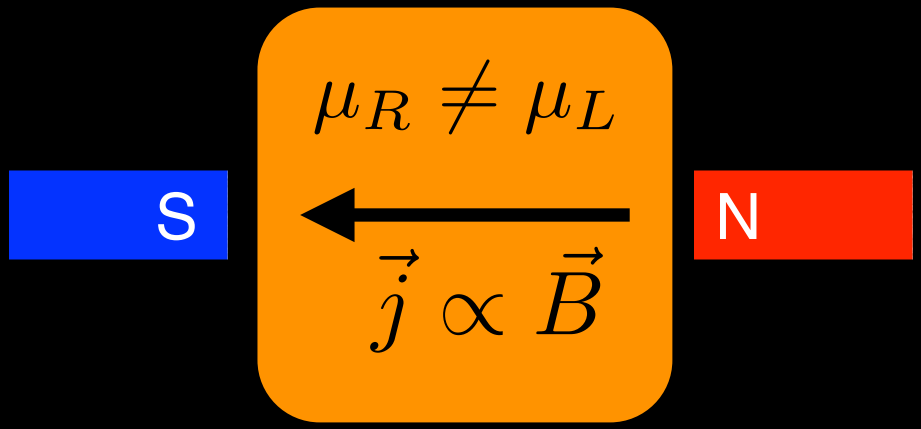

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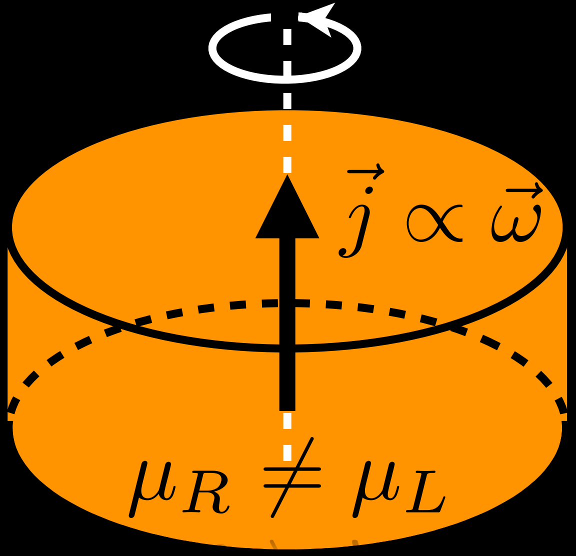

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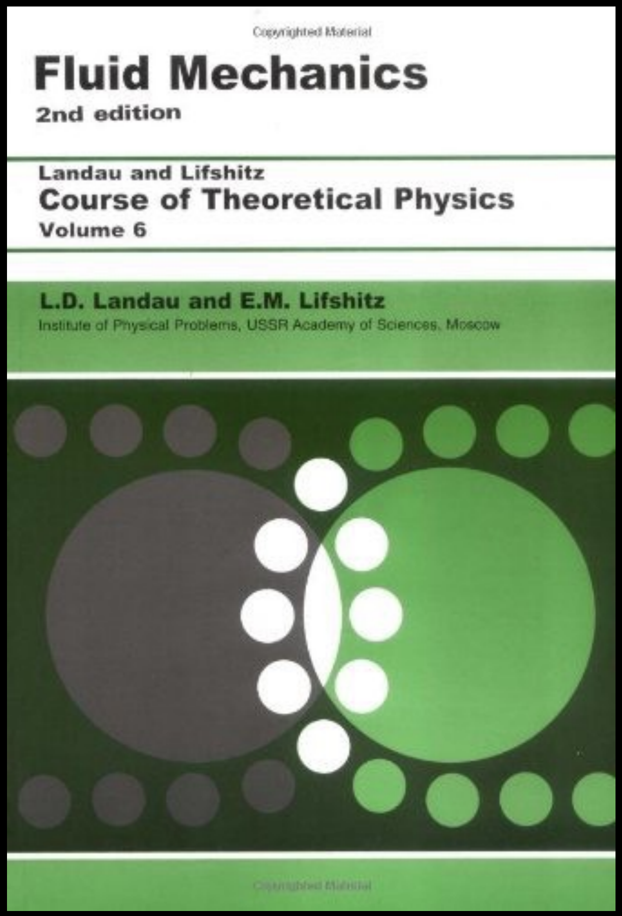

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**Fluid Mechanics**

2nd edition

Landau and Lifshitz

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Volume 6

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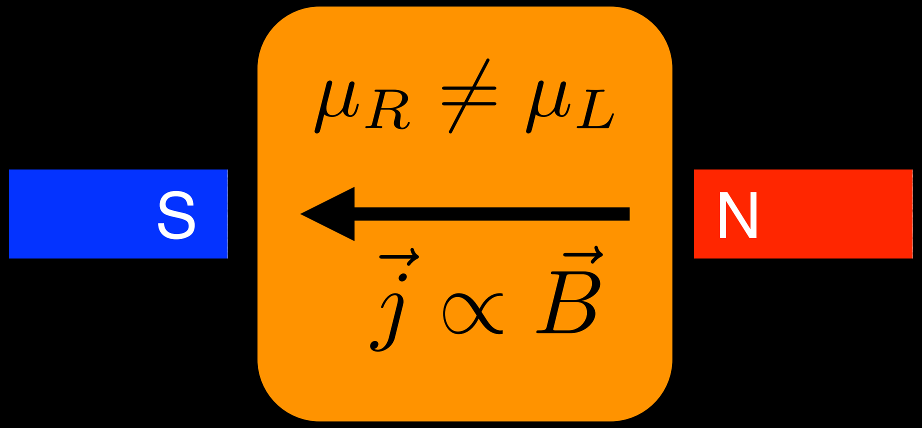

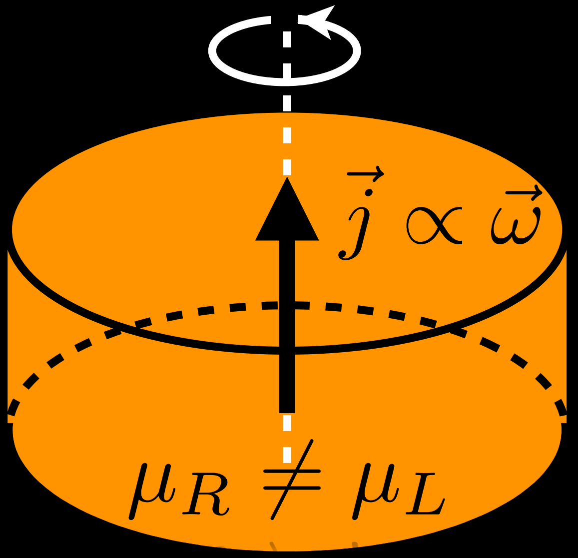

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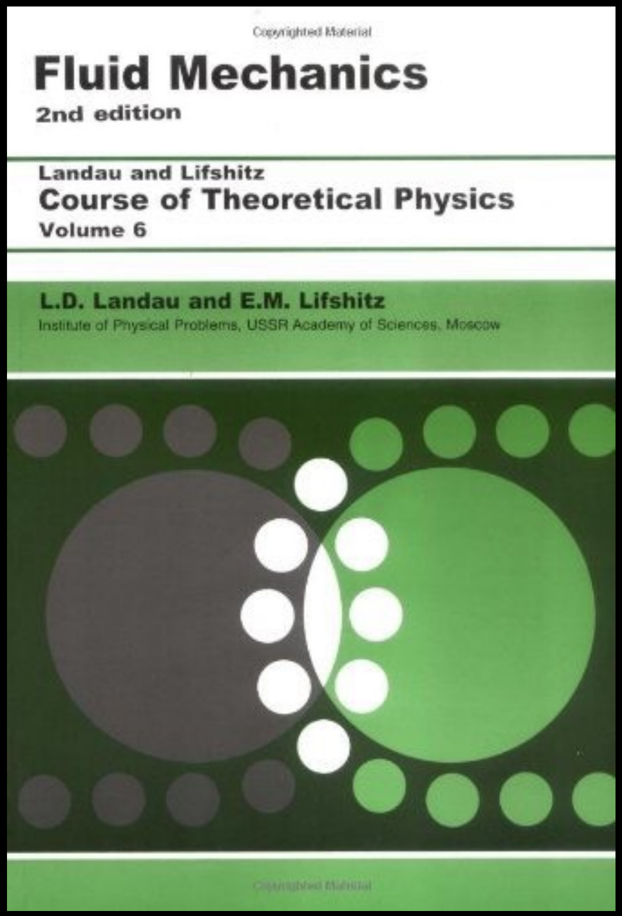

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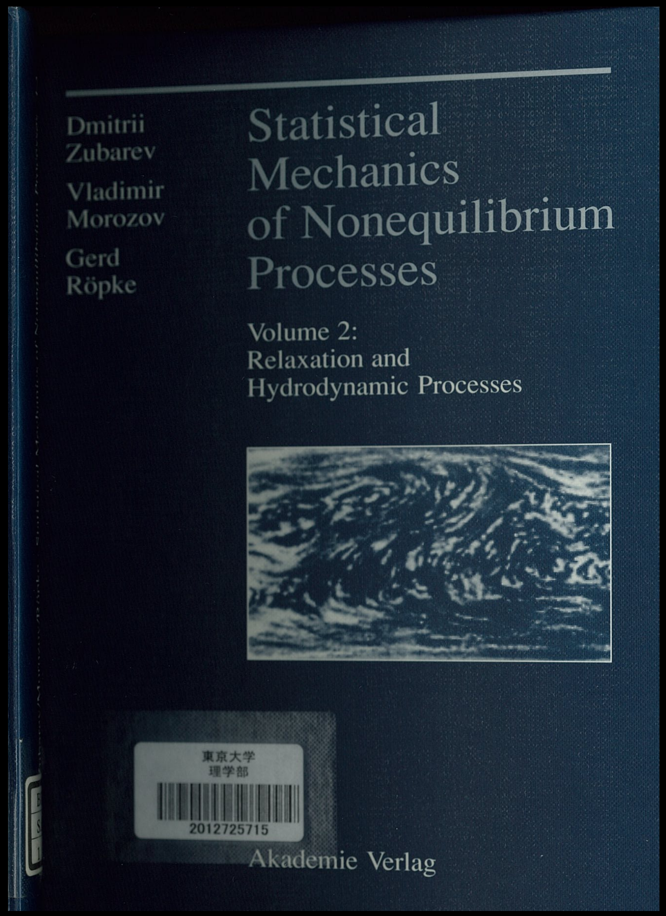


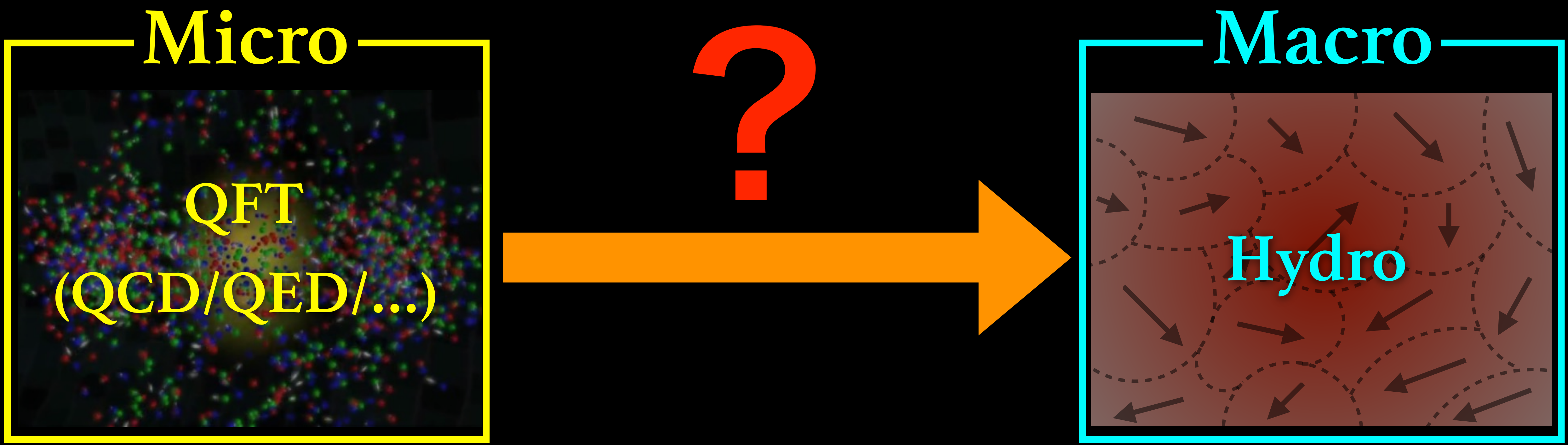
Dmitrii Zubarev
Vladimir Morozov
Gerd Röpke

Statistical Mechanics of Nonequilibrium Processes

Volume 2:
Relaxation and Hydrodynamic Processes

Akademie Verlag

# How to construct hydrodynamics



**Micro**

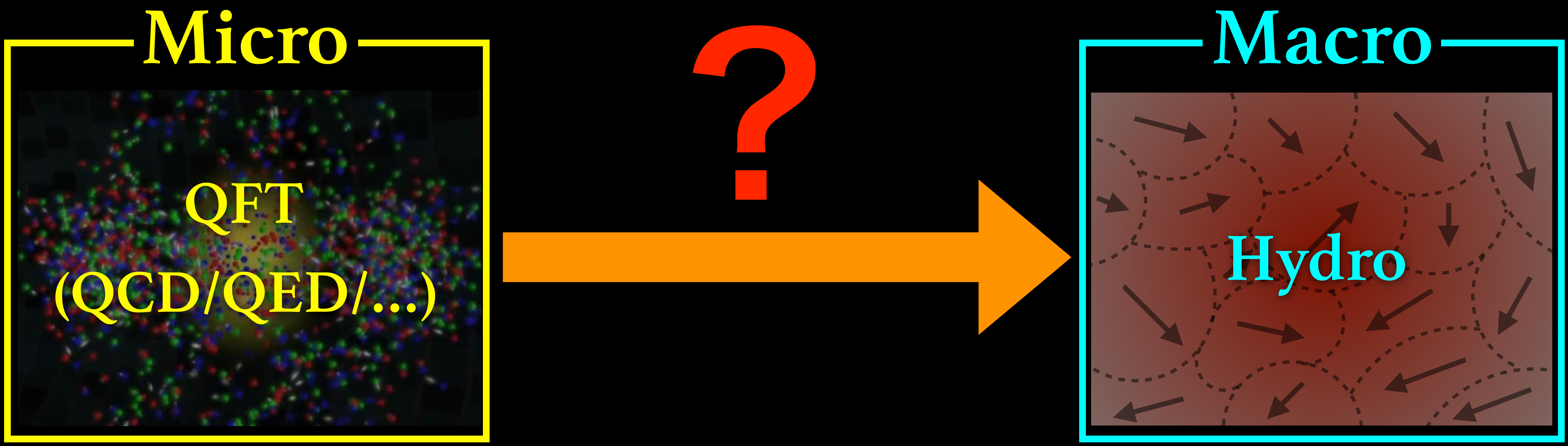
QFT
(QCD/QED/...)

?

**Macro**

Hydro

# How to construct hydrodynamics



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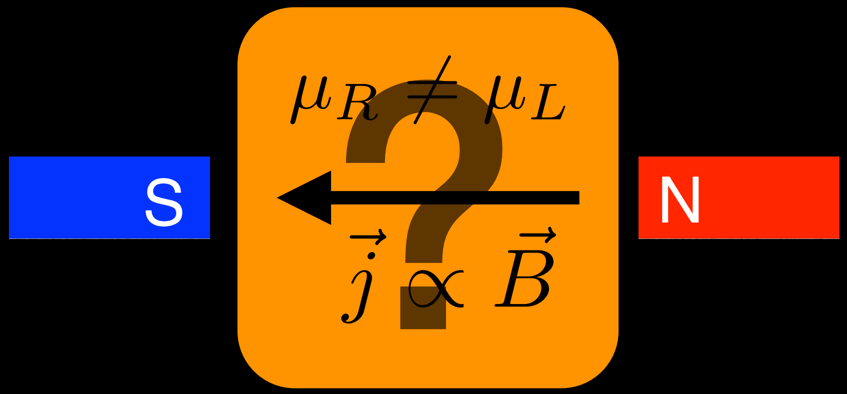
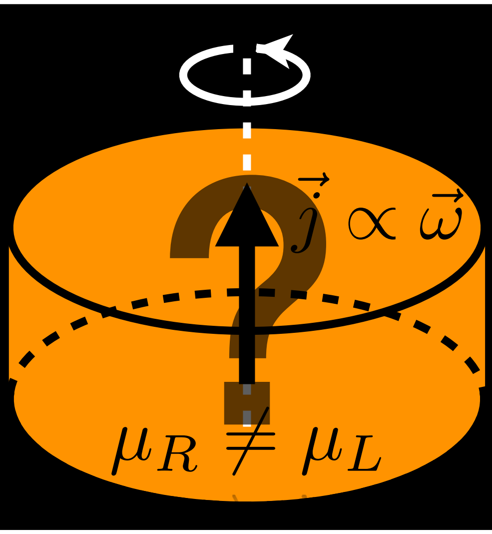


$\mu_R \neq \mu_L$

$\vec{j} \propto \vec{B}$



$\vec{j} \propto \vec{\omega}$

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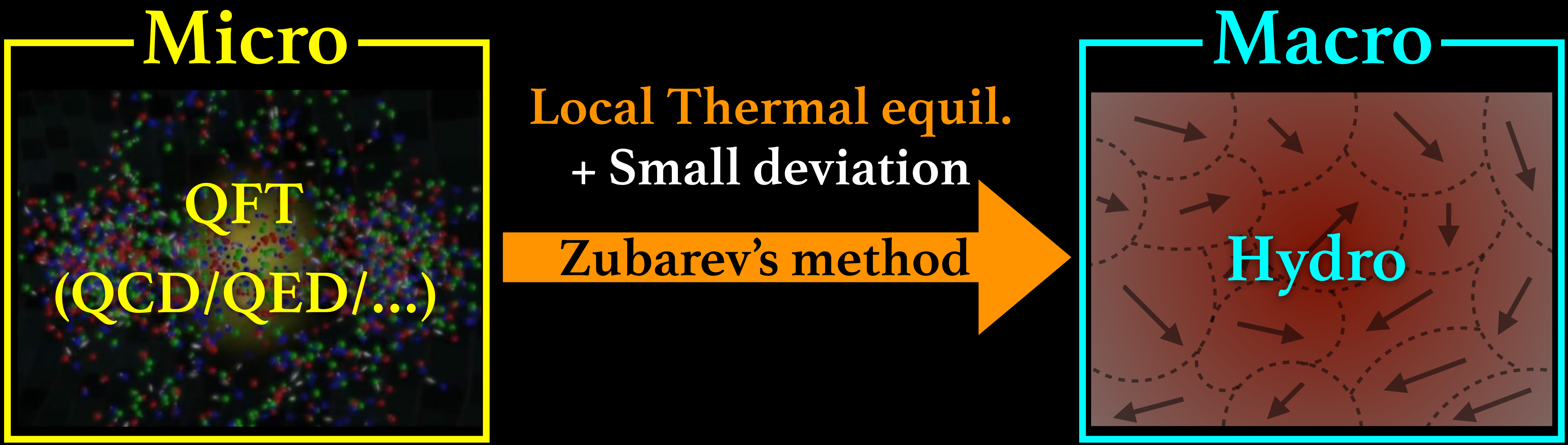

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Local Thermal equil.
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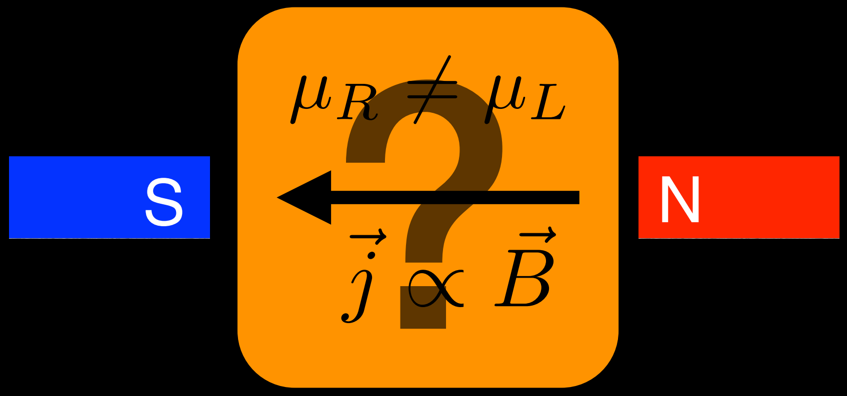

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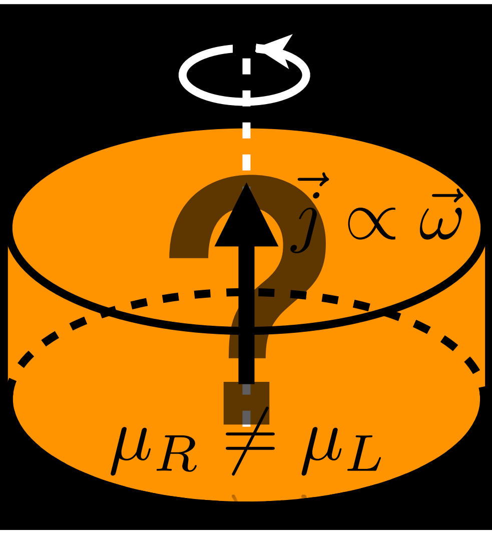

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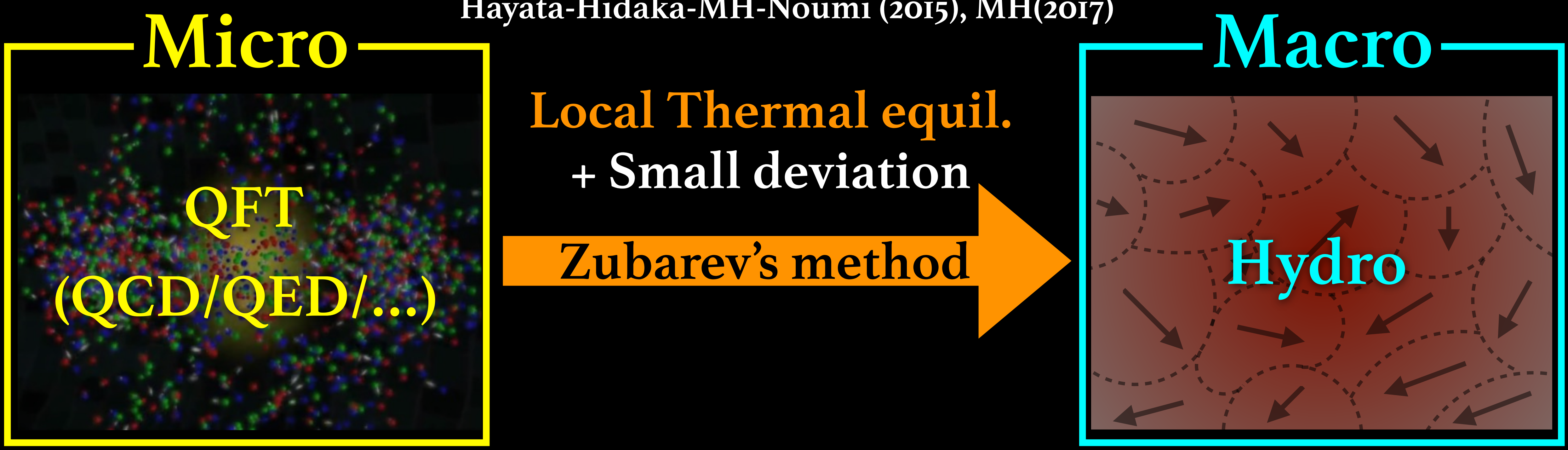

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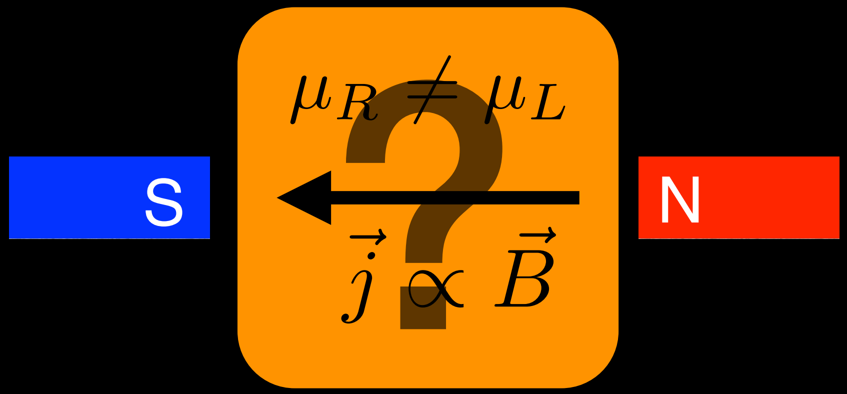

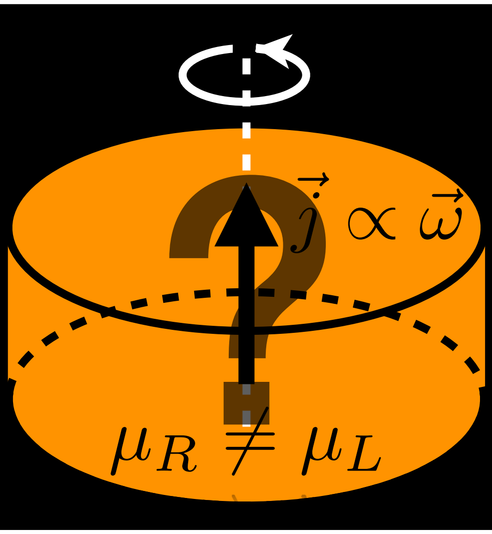

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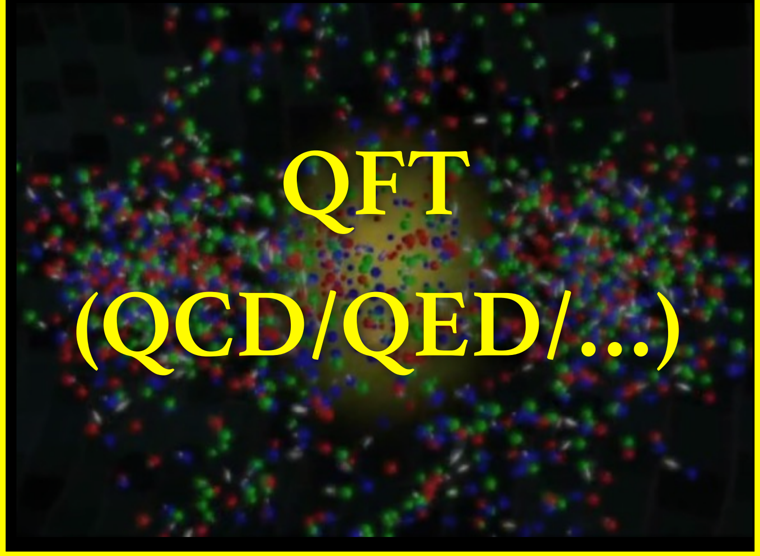

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+ Small deviation

Zubarev's method

Also applicable to
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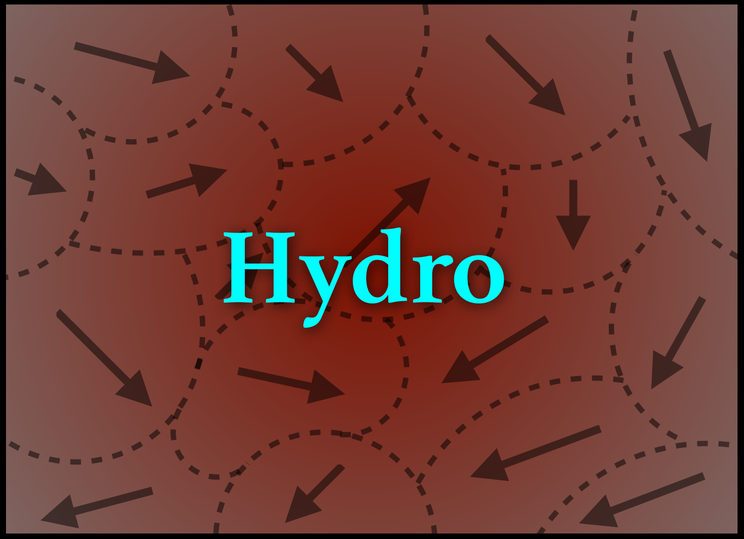

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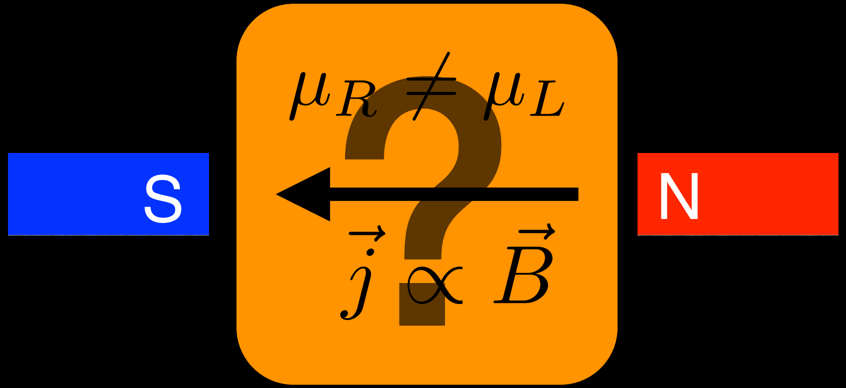

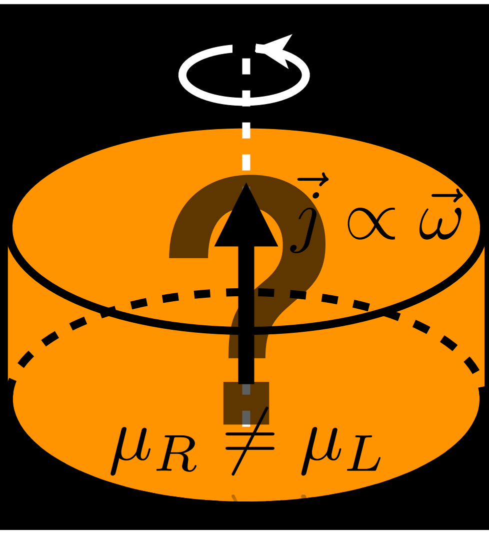

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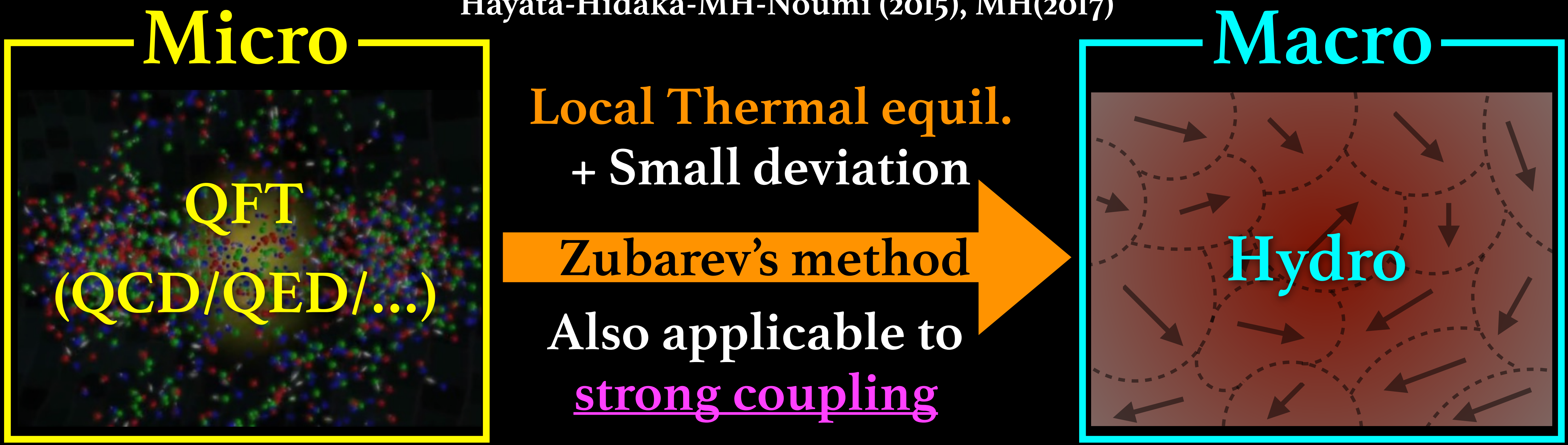

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EOS, Kubo formula, ...

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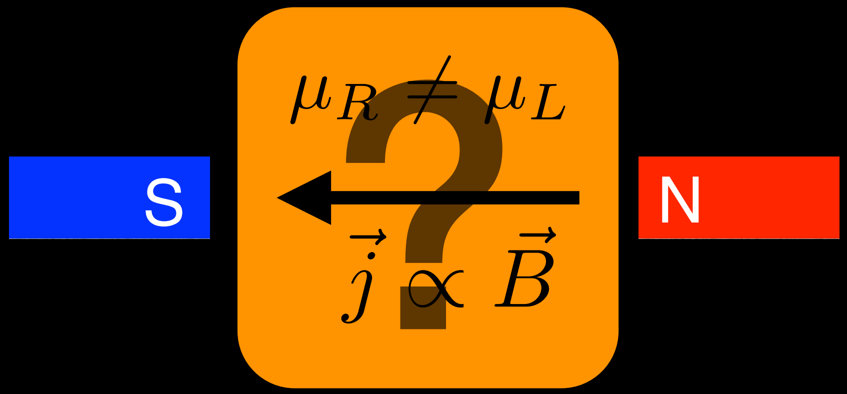

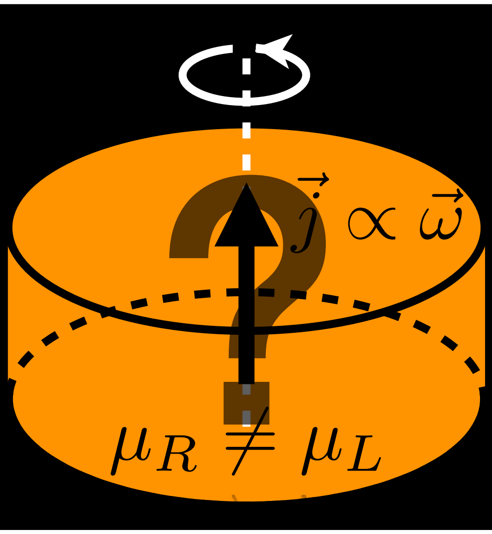

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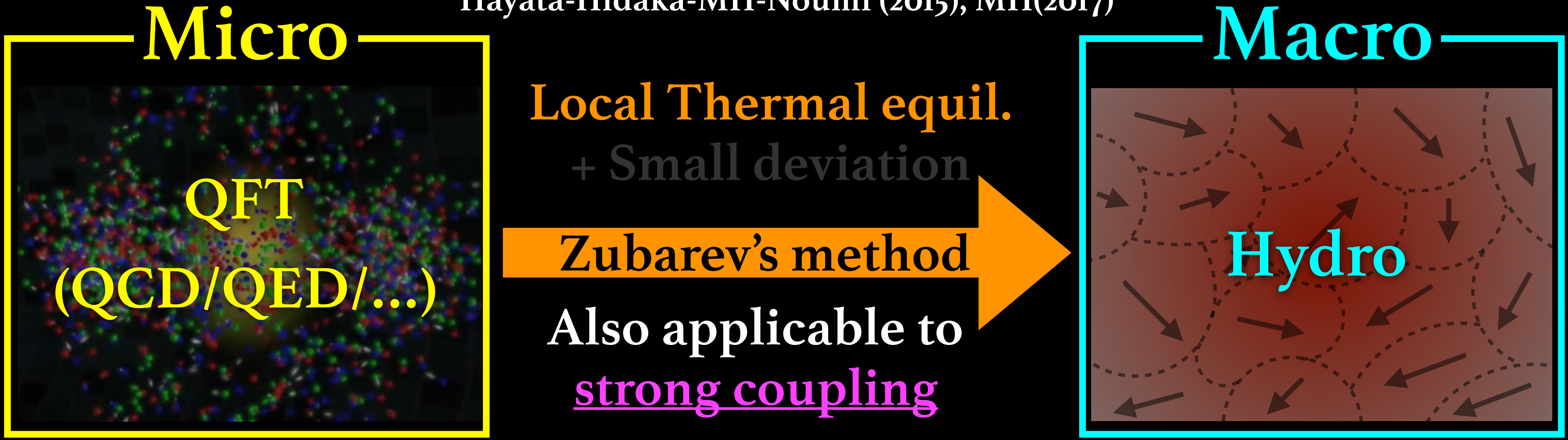

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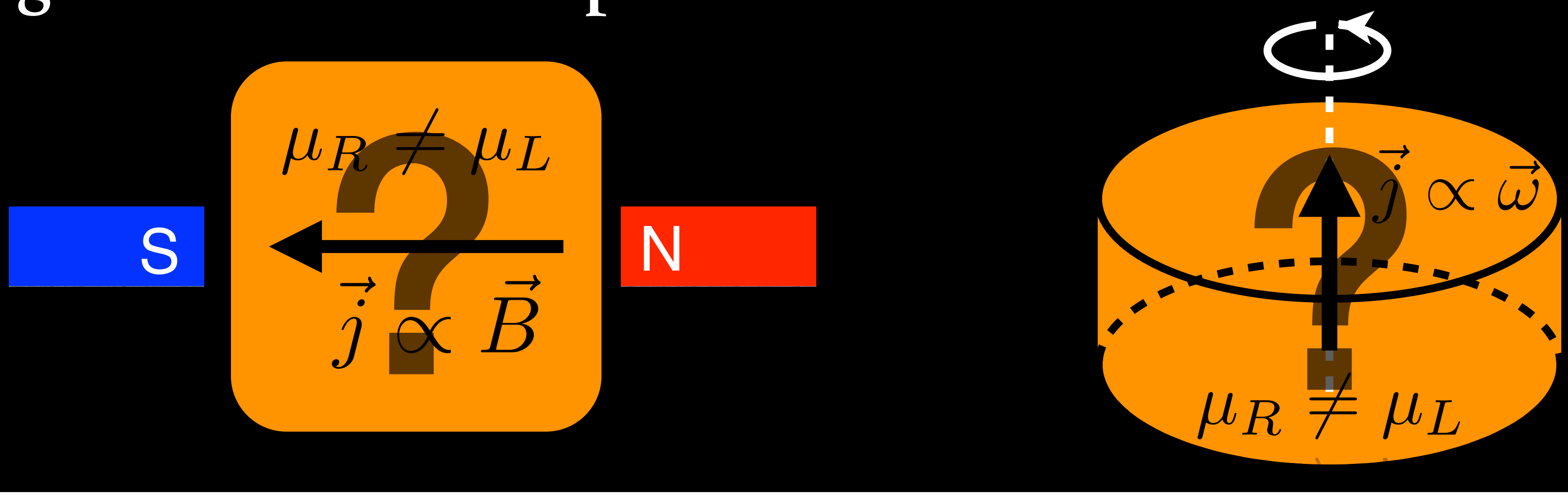
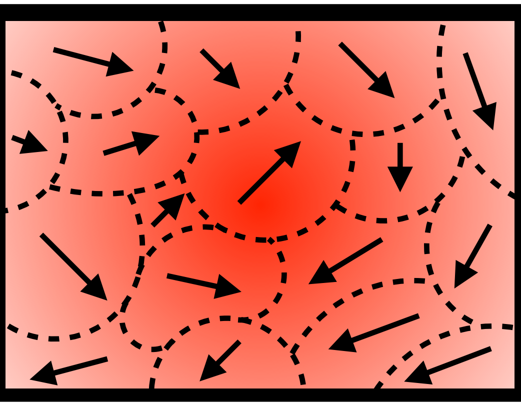
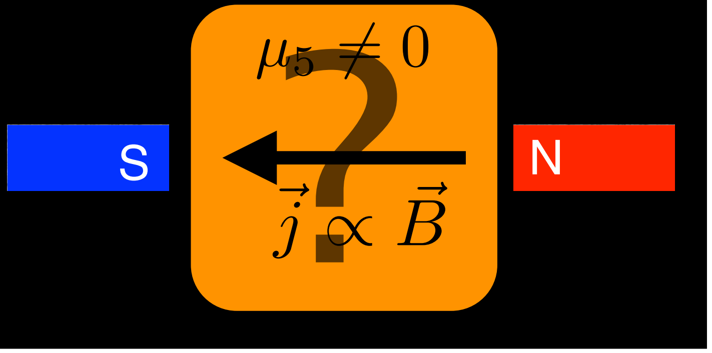


# Outline

## MOTIVATION:

Quantum field theory under
local thermal equilibrium?





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QFT for Local Gibbs distribution

## APPLICATION:

Derivation of
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# Outline

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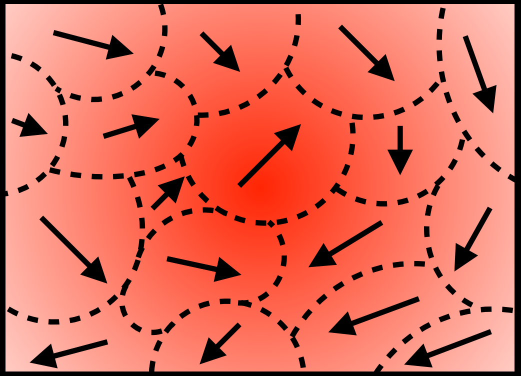

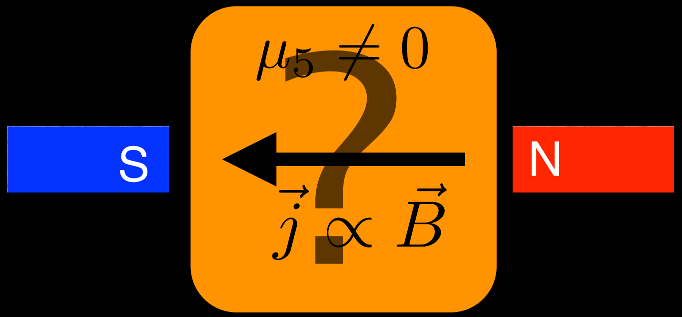

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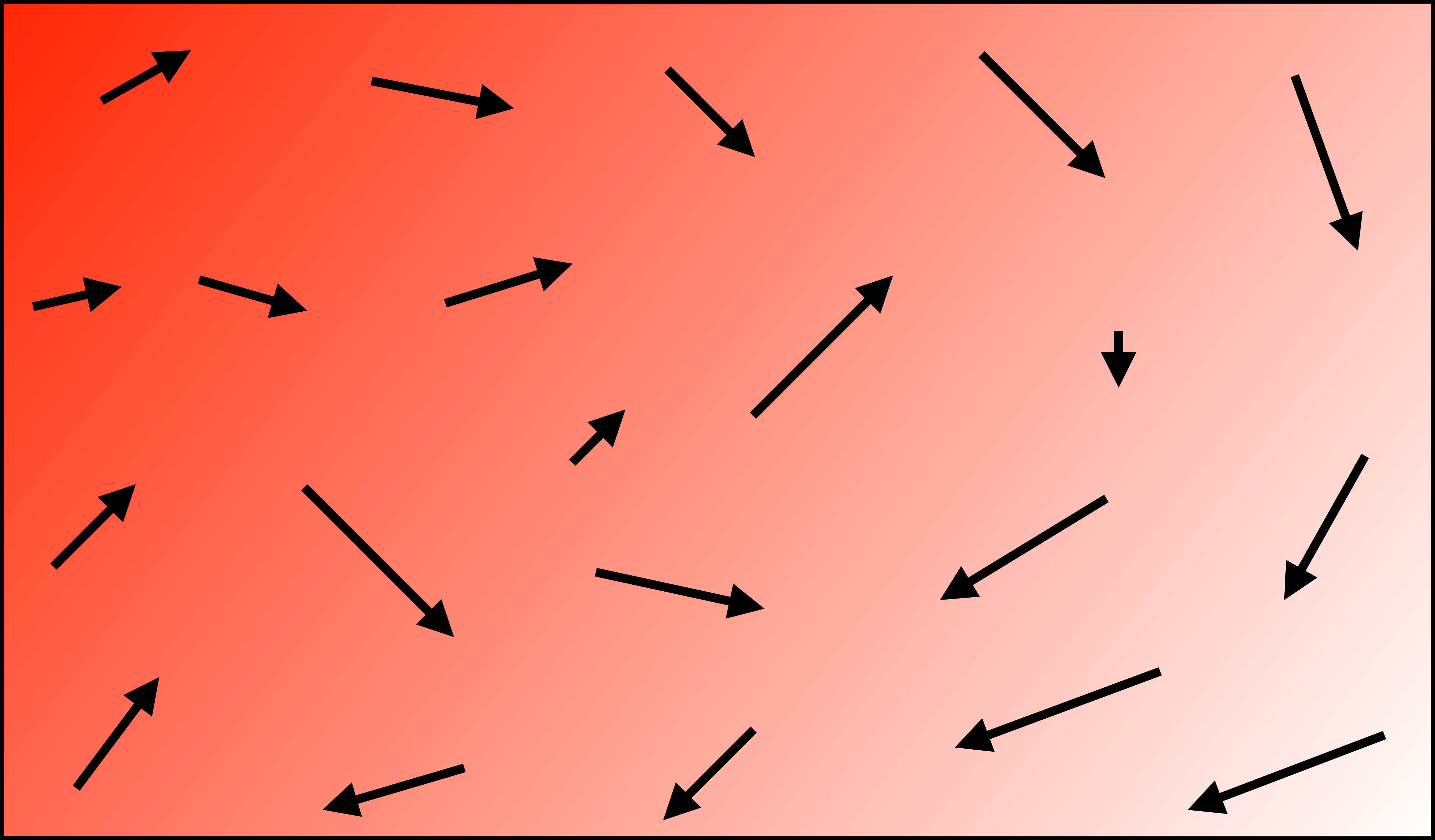
# Local thermal equilibrium



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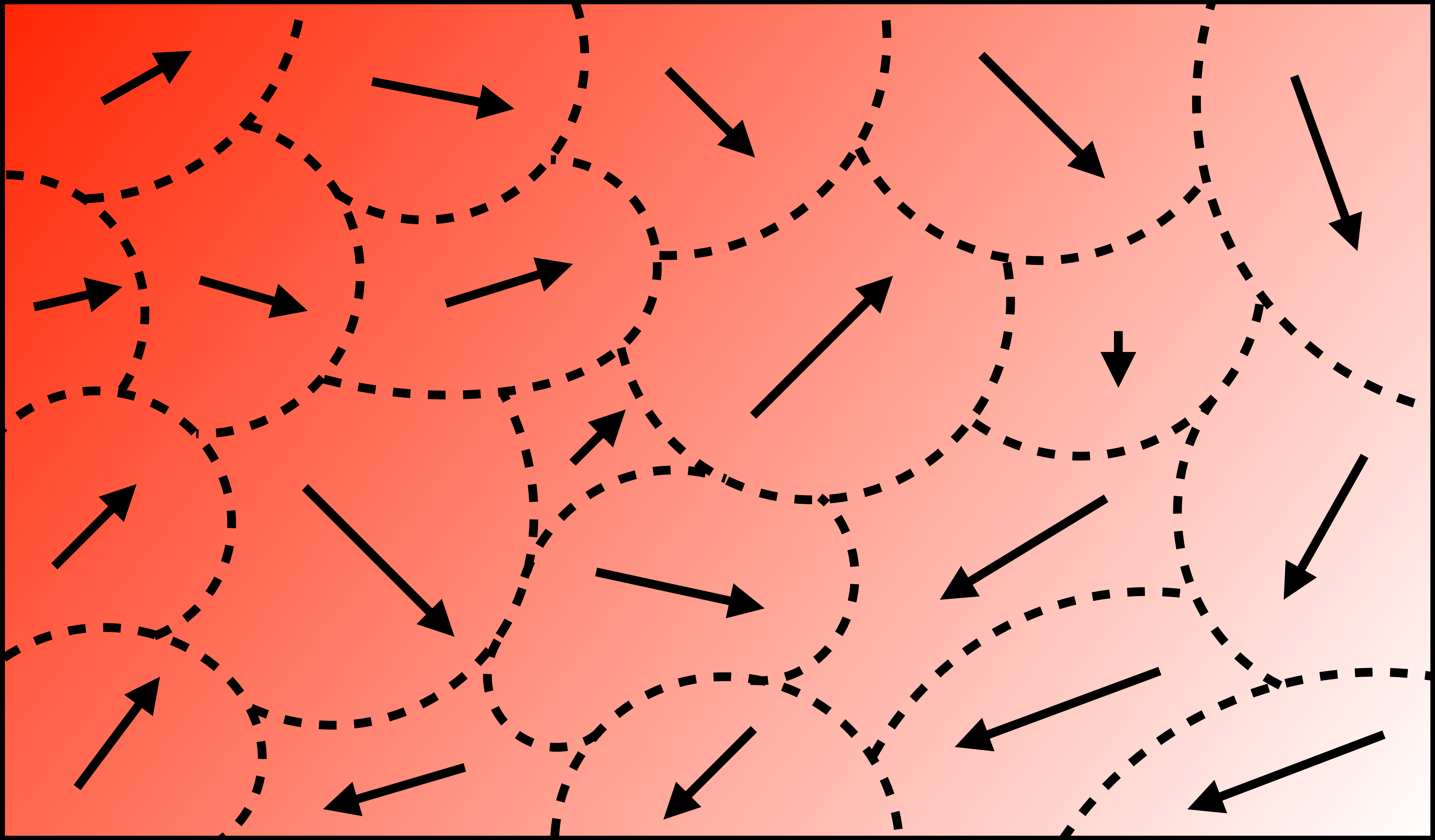

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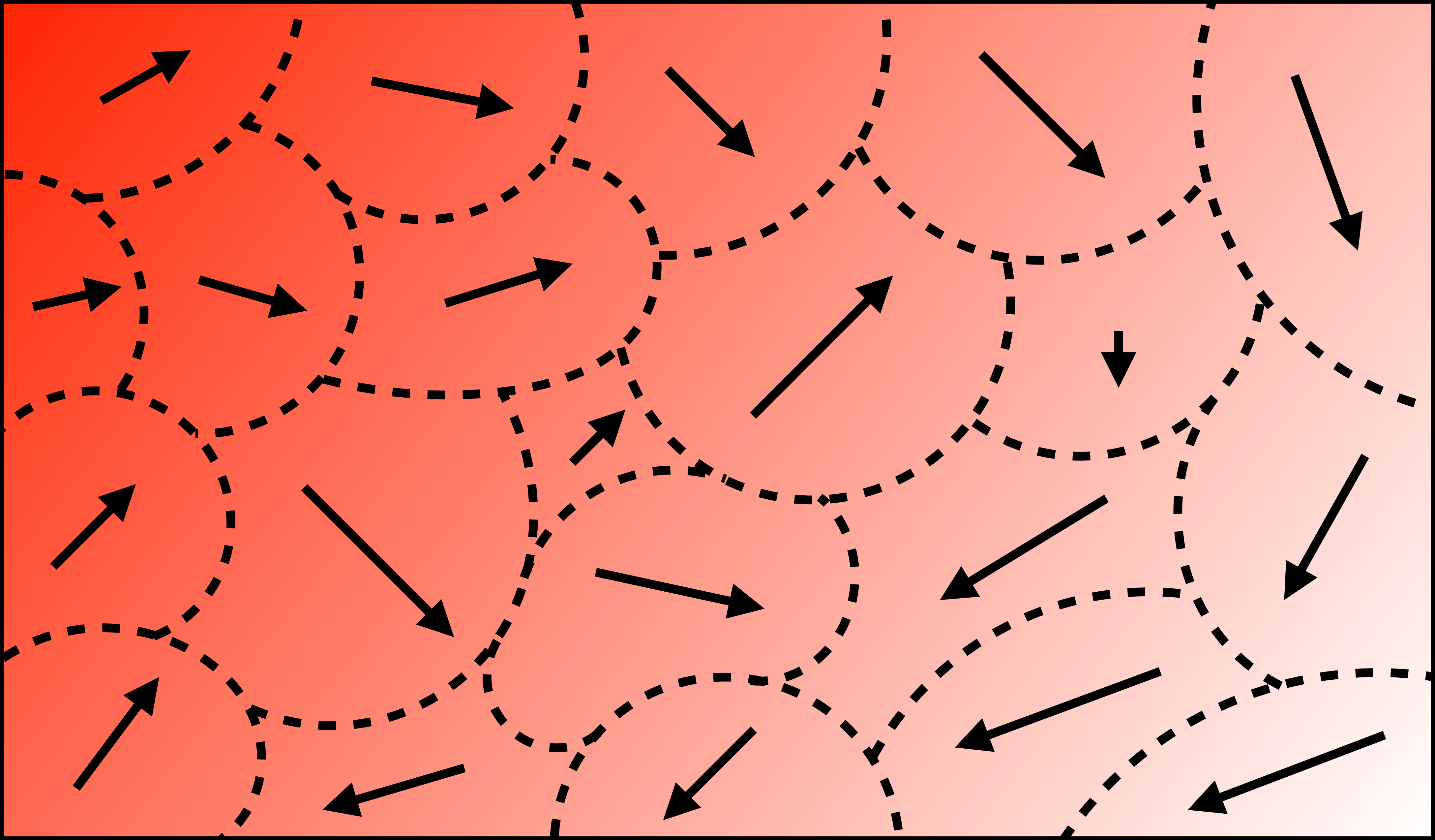


Determined only by **local temperature, local velocity…** at that time

# How to describe local thermal equil.

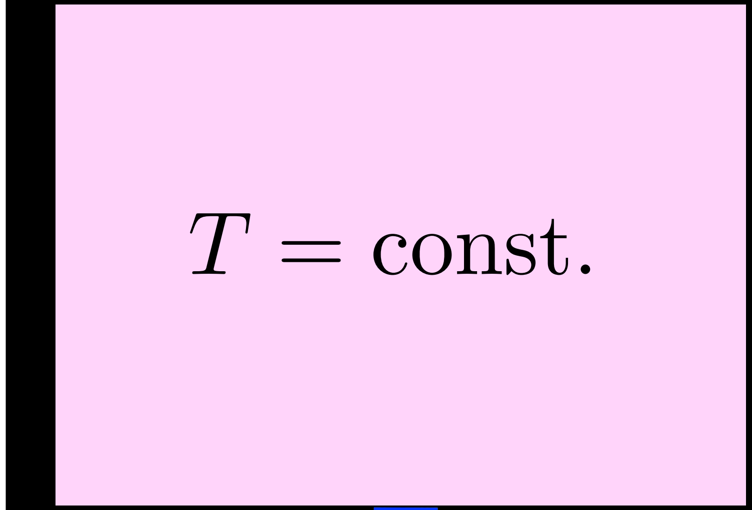
$T = \mathrm{const.}$

**Global thermal equilibrium:**

Gibbs distribution:

$$\hat{\rho}_G = e^{-\beta \hat{H} - \Psi[\beta]}, \quad \Psi[\beta] \equiv \log \mathrm{Tr} e^{-\beta \hat{H}}$$

# How to describe local thermal equil.



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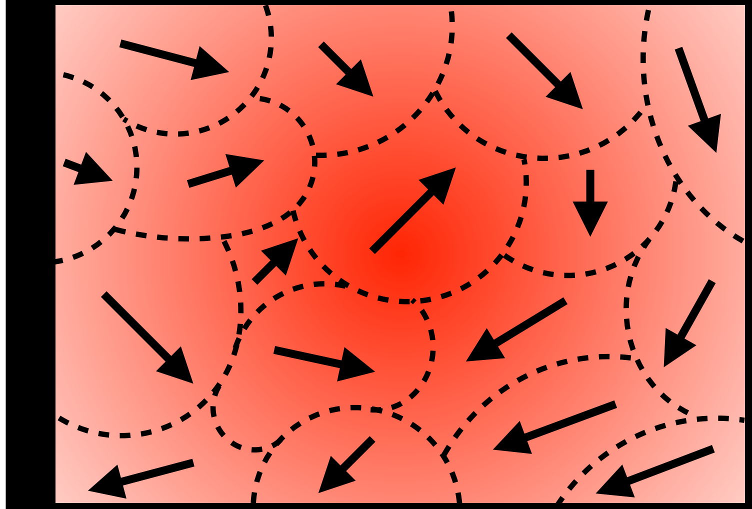
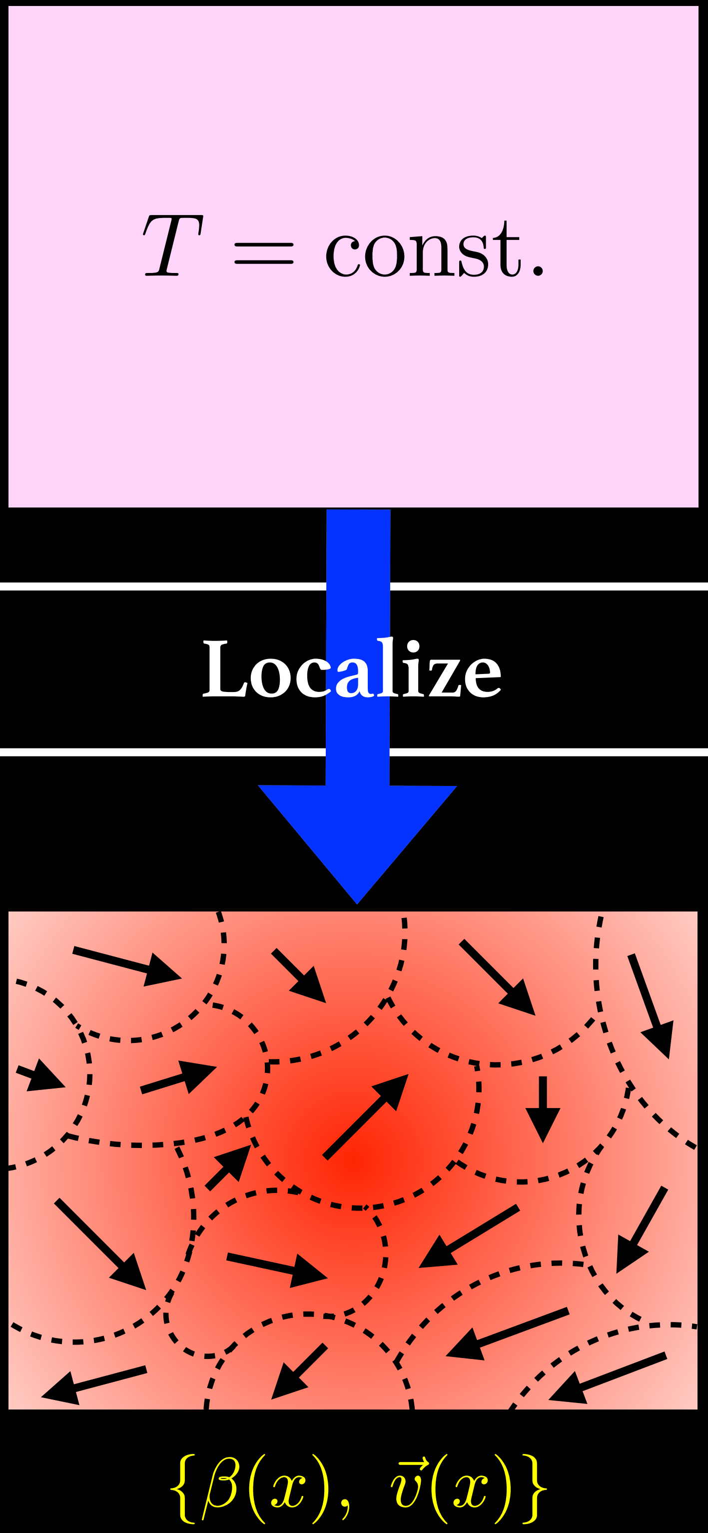
Localize

**Local thermal equilibrium:**



$\{\beta(x),\ \vec{v}(x)\}$

# How to describe local thermal equil.



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**Local thermal equilibrium:**

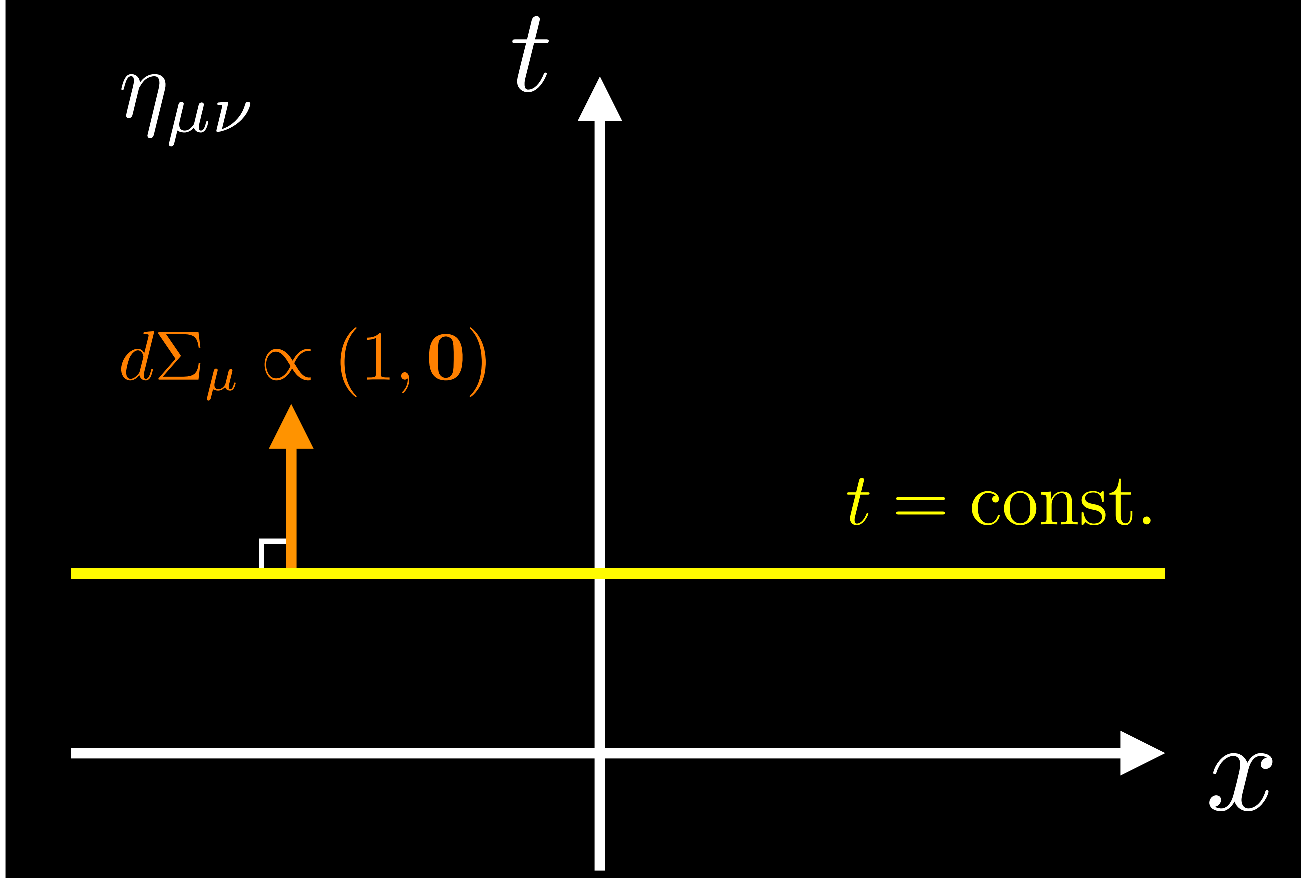
Local Gibbs (LG) distribution:

$$\hat{\rho}_{LG} = e^{-\hat{K} - \Psi[\beta^\mu(x),\nu(x)]}$$

$$\hat{K} = -\int d^3x \left(\beta^\mu(\boldsymbol{x})\hat{T}^0{}_\mu(\boldsymbol{x}) + \nu(\boldsymbol{x})\hat{J}^0(\boldsymbol{x})\right)$$

# Introducing background metric

## Flat spacetime



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# Introducing background metric

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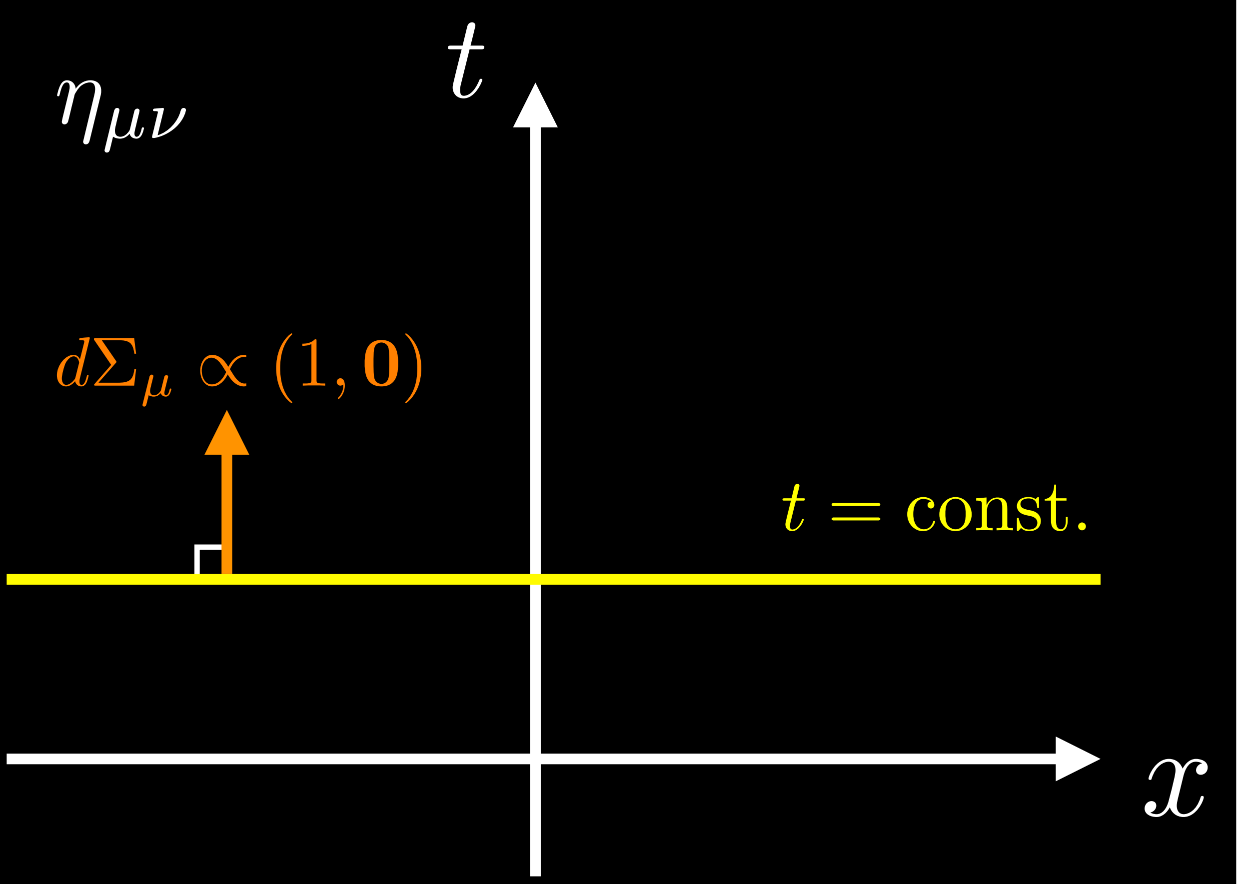
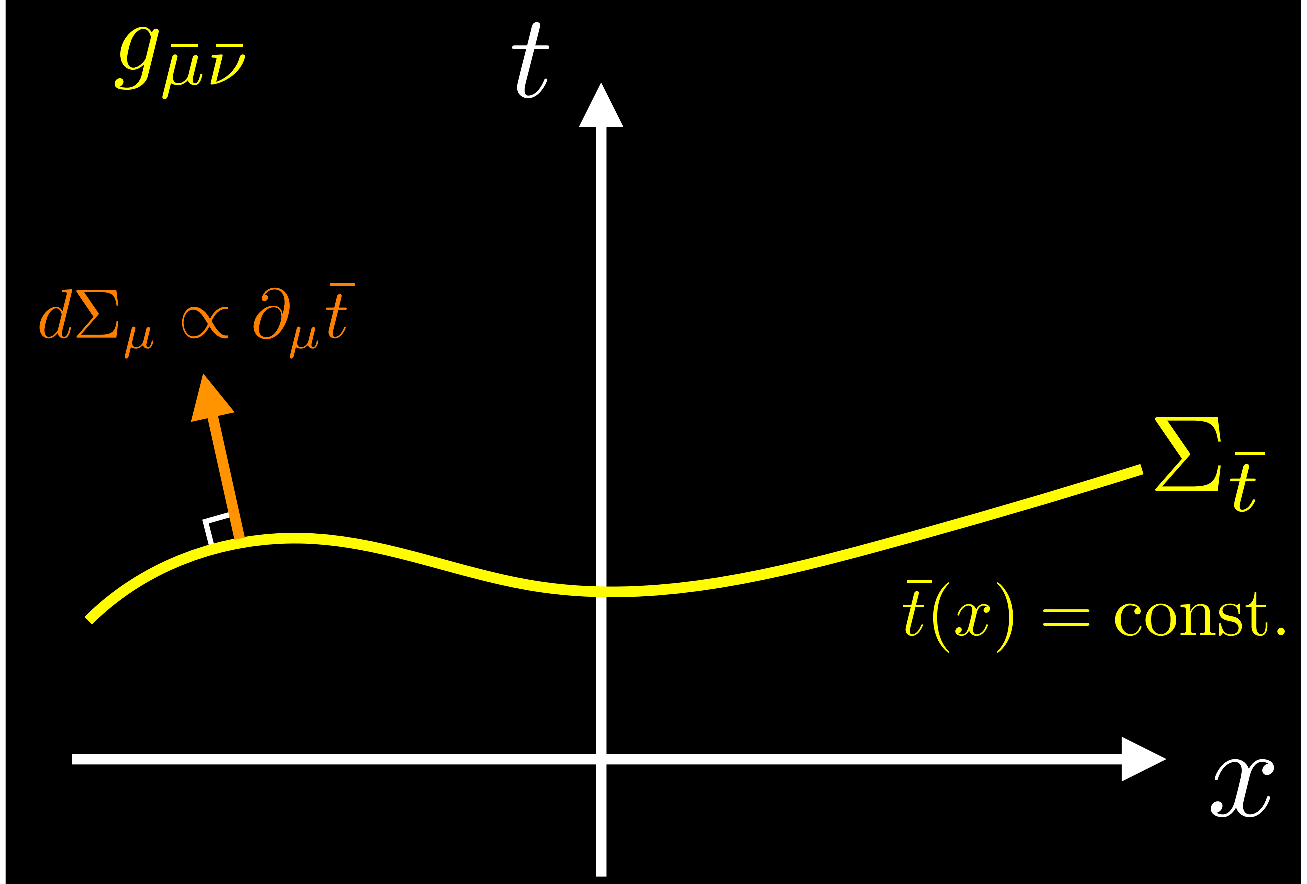


$$\hat{K} = -\int d^3x \left(\beta^\mu(\boldsymbol{x})\hat{T}^0{}_\mu(\boldsymbol{x}) + \nu(\boldsymbol{x})\hat{J}^0(\boldsymbol{x})\right)$$

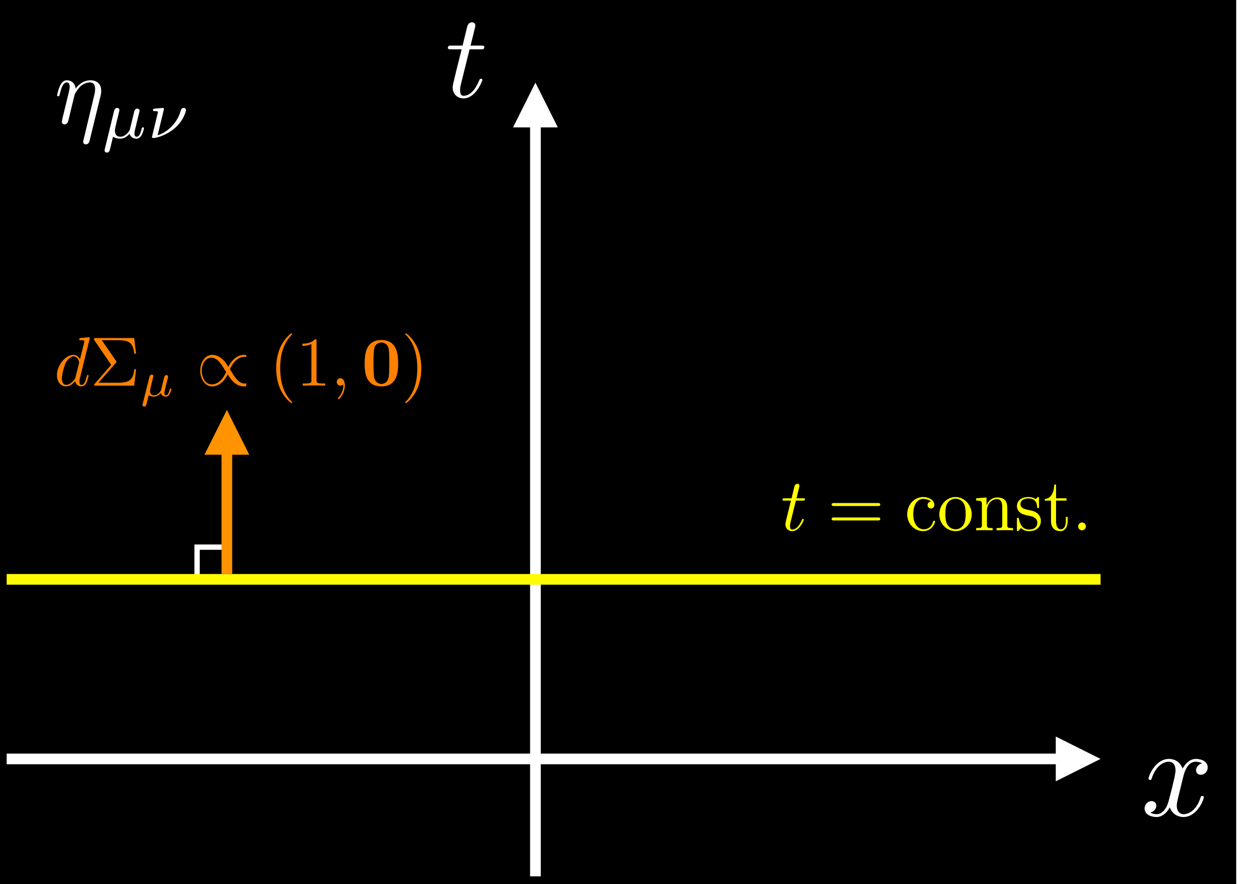
## Curved spacetime



$$\hat{K} = -\int d\Sigma_{\bar{t}\nu} \left(\beta^\mu(x)\hat{T}^\nu{}_\mu(x) + \nu(x)\hat{J}^\nu(x)\right)$$

# Introducing background metric

## Flat spacetime



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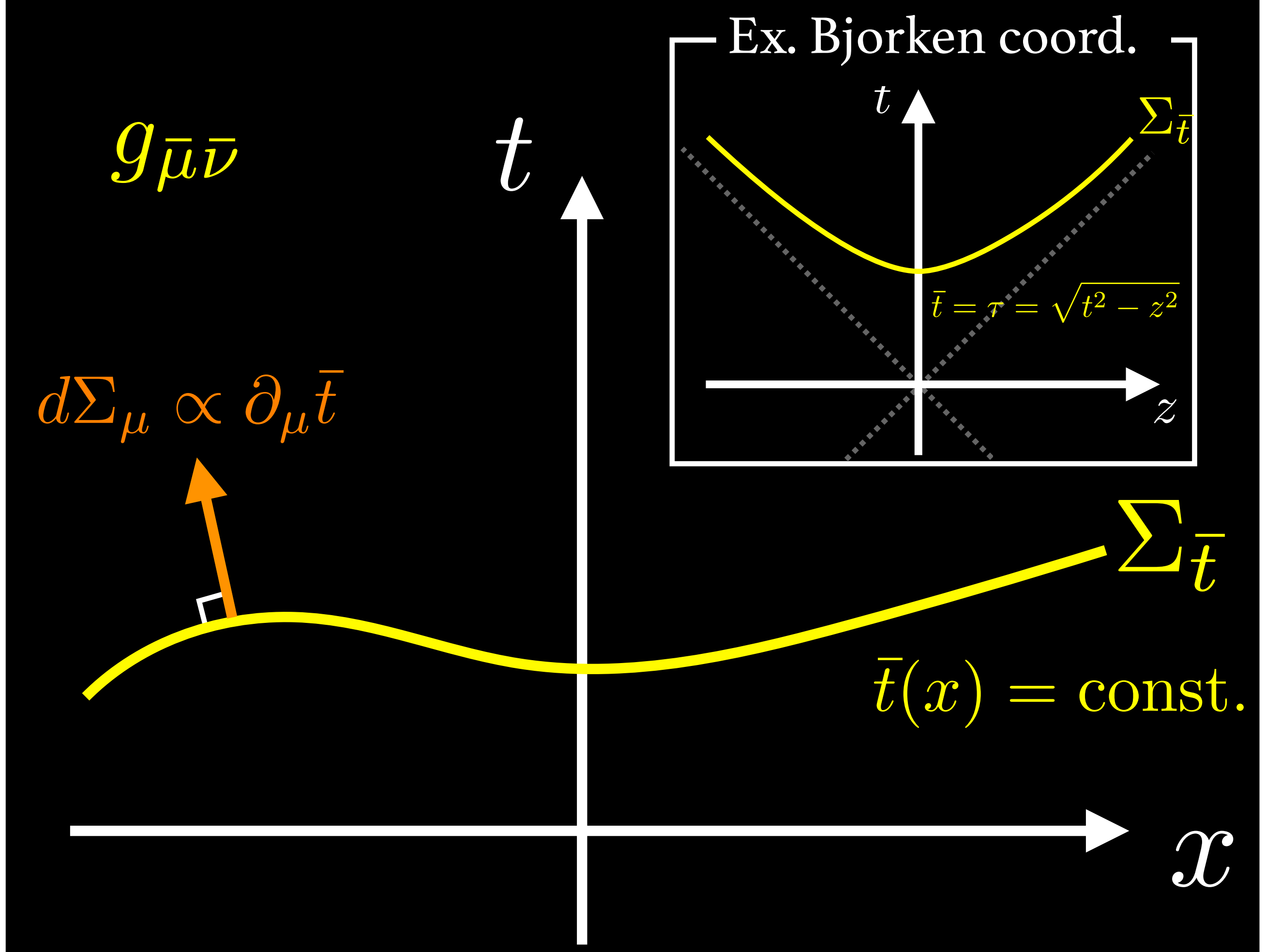

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# Introducing background metric

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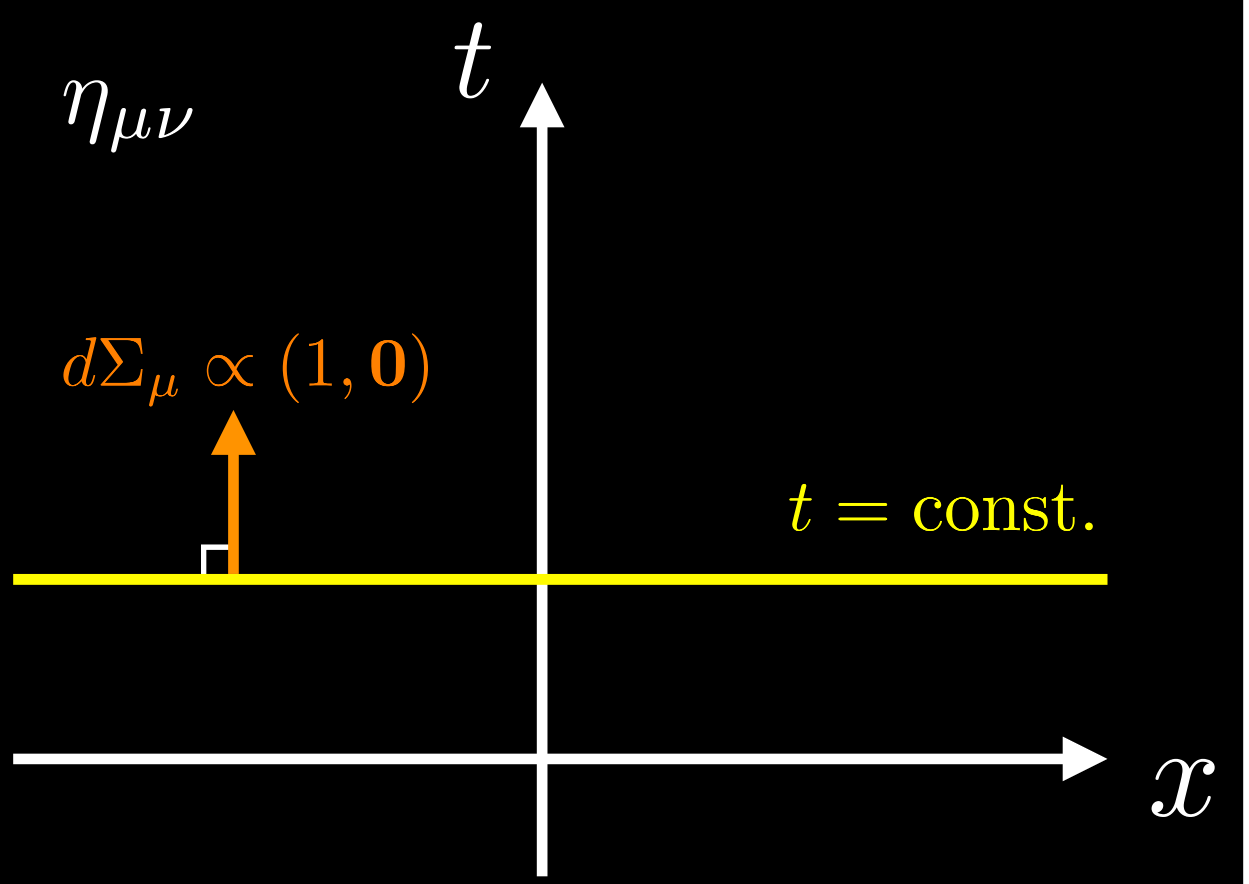

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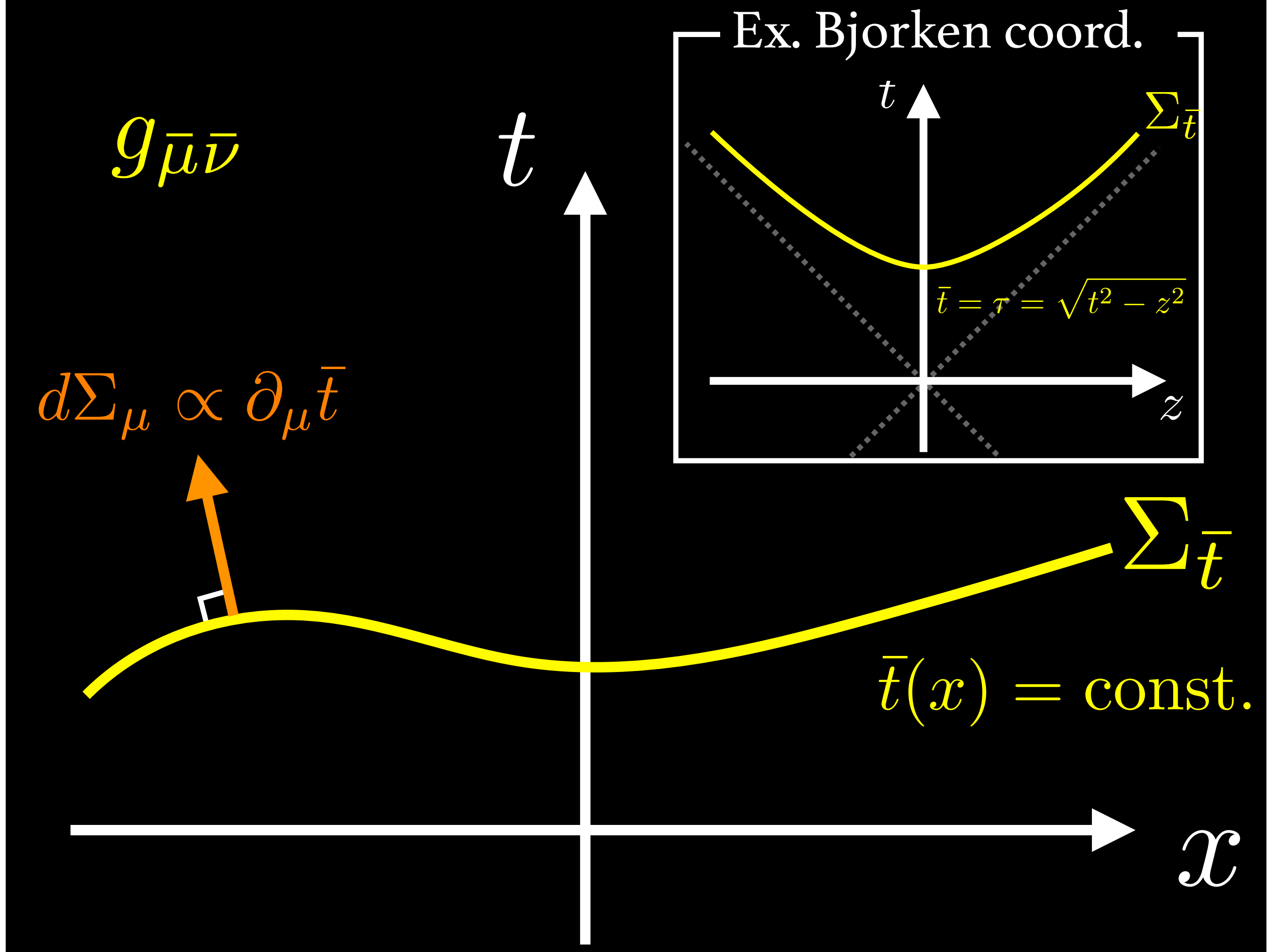

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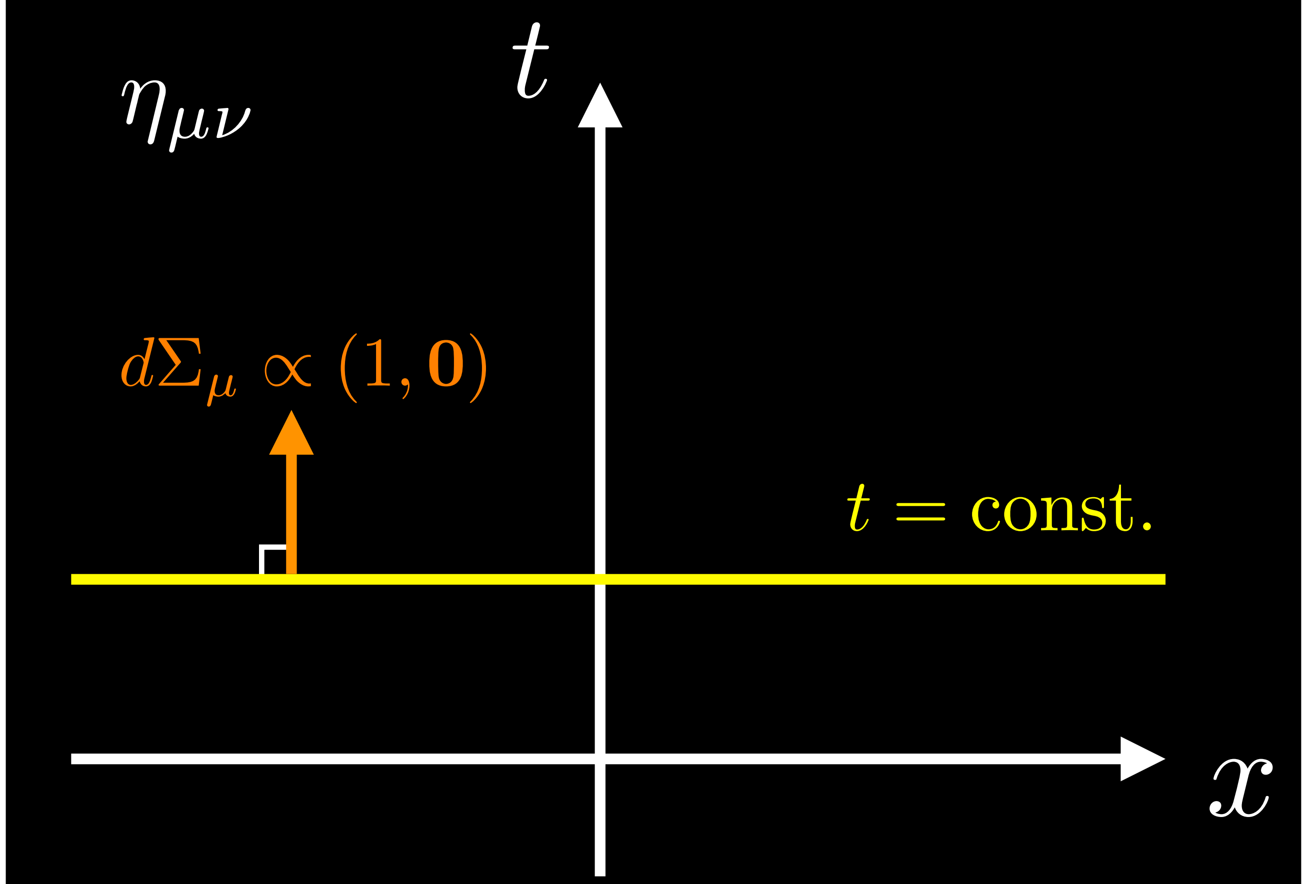
→ ① Formulation becomes manifestly covariant

# Introducing background metric

## Flat spacetime



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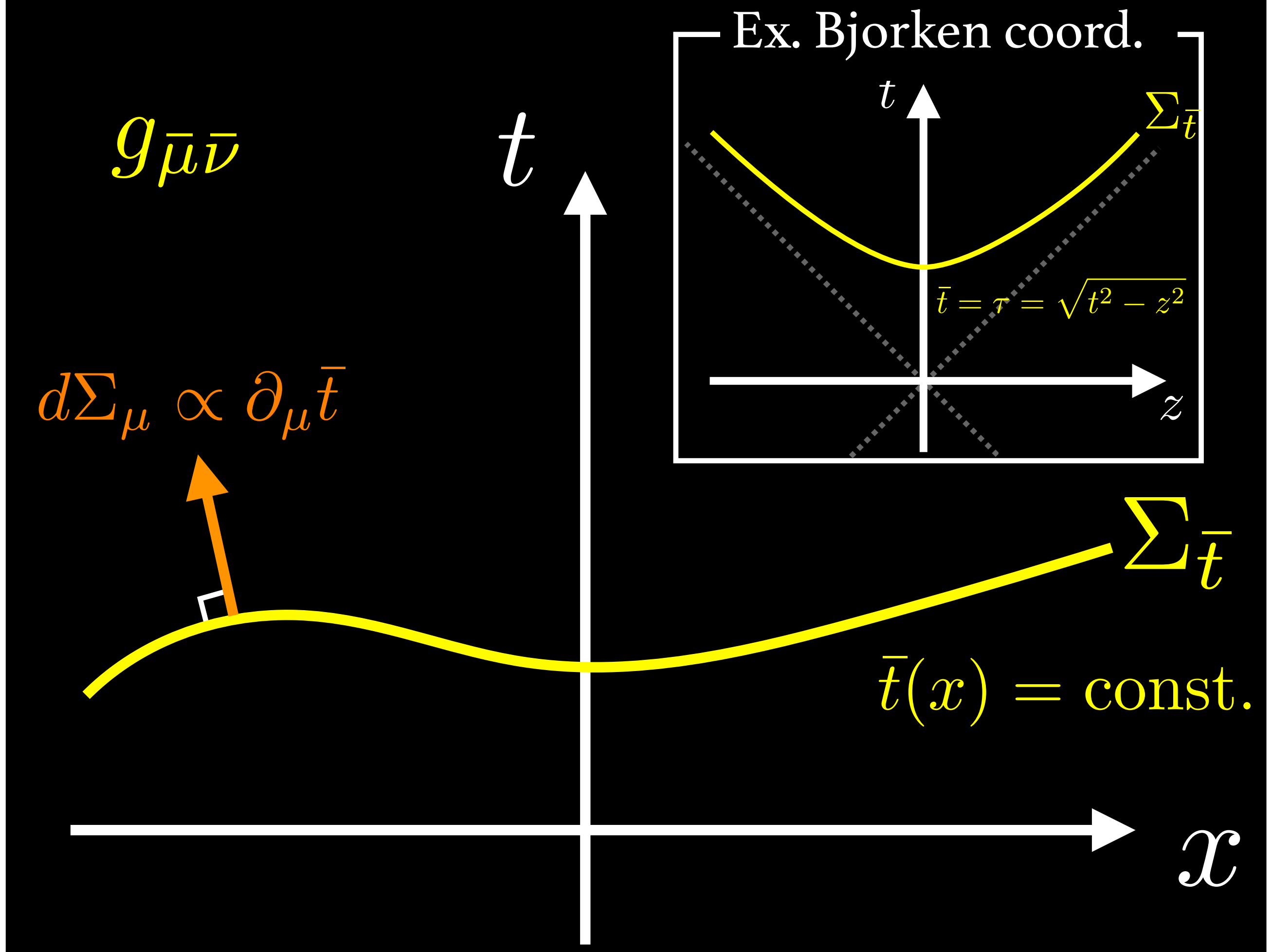

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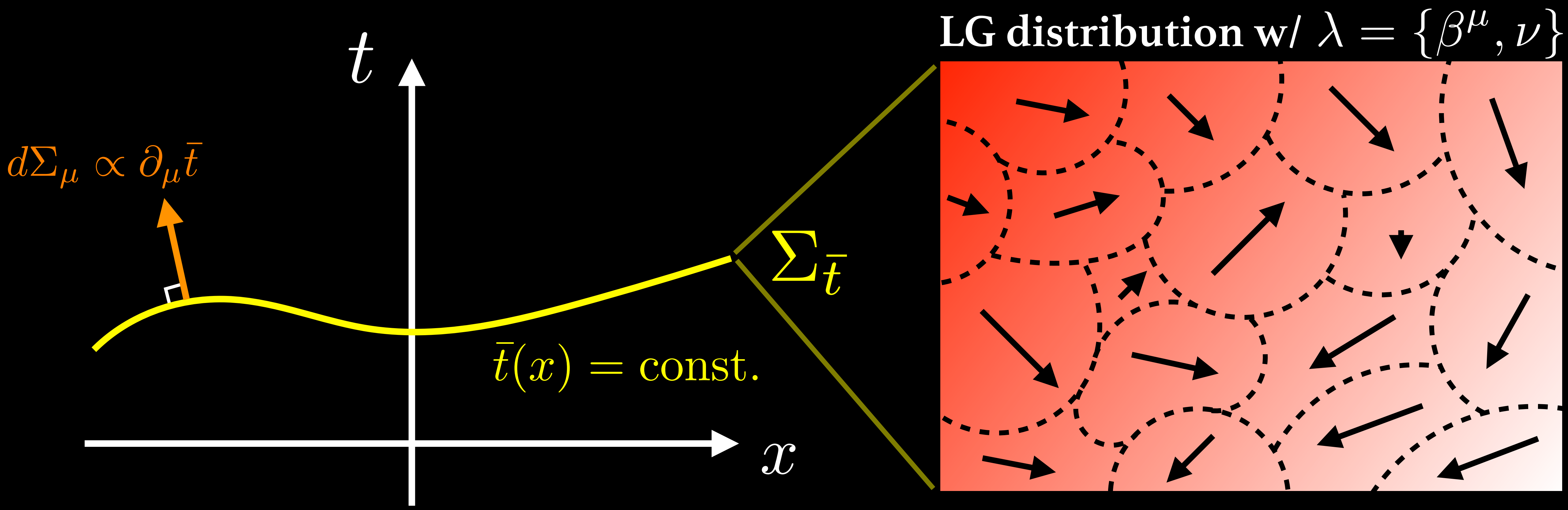
→ ① Formulation becomes manifestly covariant

② Background metric plays a role as external field coupled to $T^{\mu\nu}$

# (Local) Thermodynamic Potential



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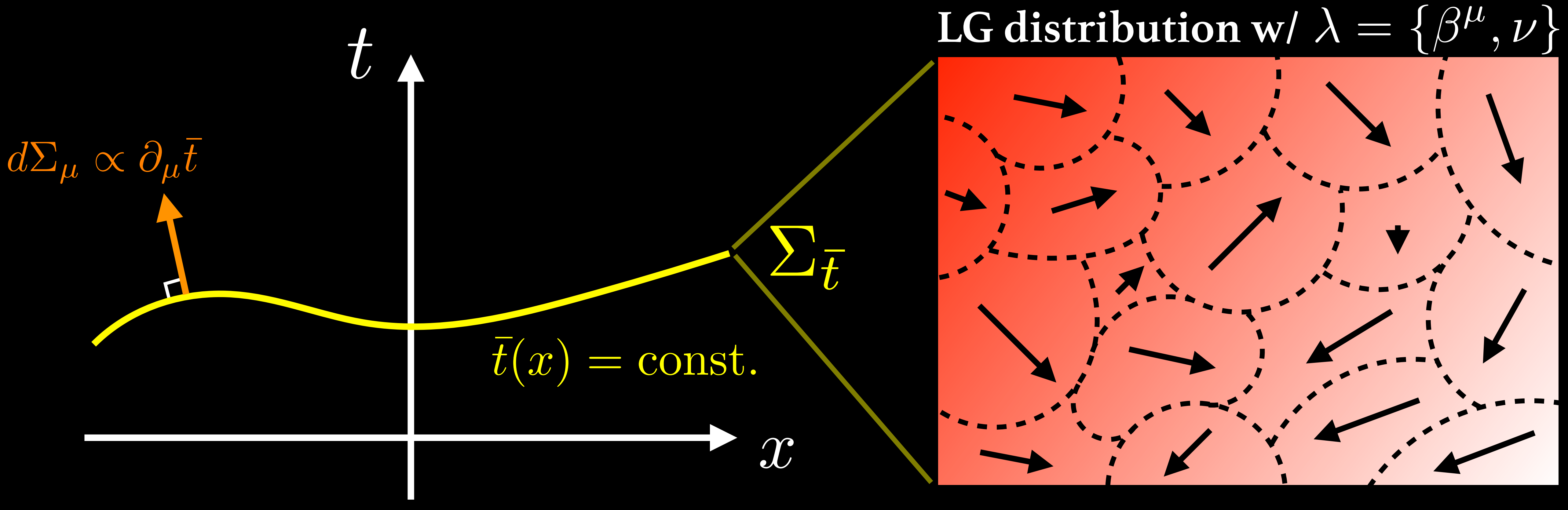

**Masseiu-Planck functional**

$$\Psi[\bar{t};\lambda] \equiv \log \mathrm{Tr} \exp \left[ \int d\Sigma_{\bar{t}\nu} \Big( \beta^\mu(x) \hat{T}^\nu{}_\mu(x) + \nu(x) \hat{J}^\nu(x) \Big) \right]$$

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# Variation formula for local equil.

[ Banerjee et al.(2012), Jensen et al.(2012) , Haehl et al. (2015), MH(2017)]

**Variation formula in "hydrostatic gauge"**

$$\langle \hat{T}^{\mu\nu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \Psi[\bar{t};\lambda], \;\; \langle \hat{J}^{\mu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta A_{\mu}(x)} \Psi[\bar{t};\lambda]$$

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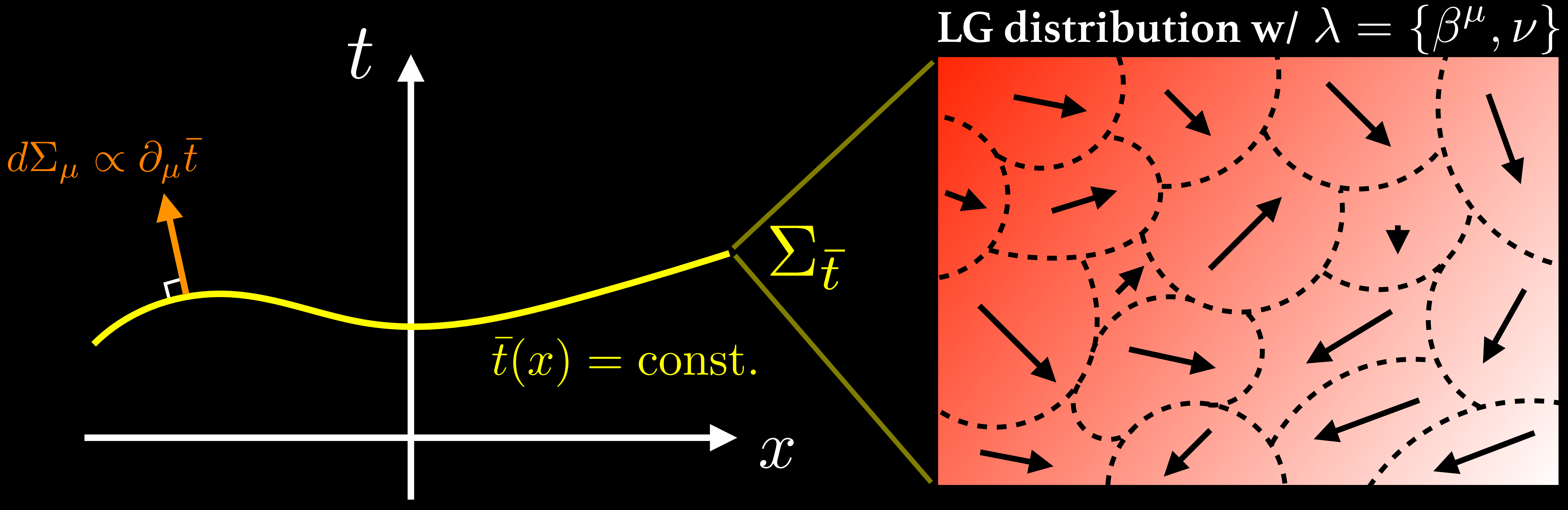

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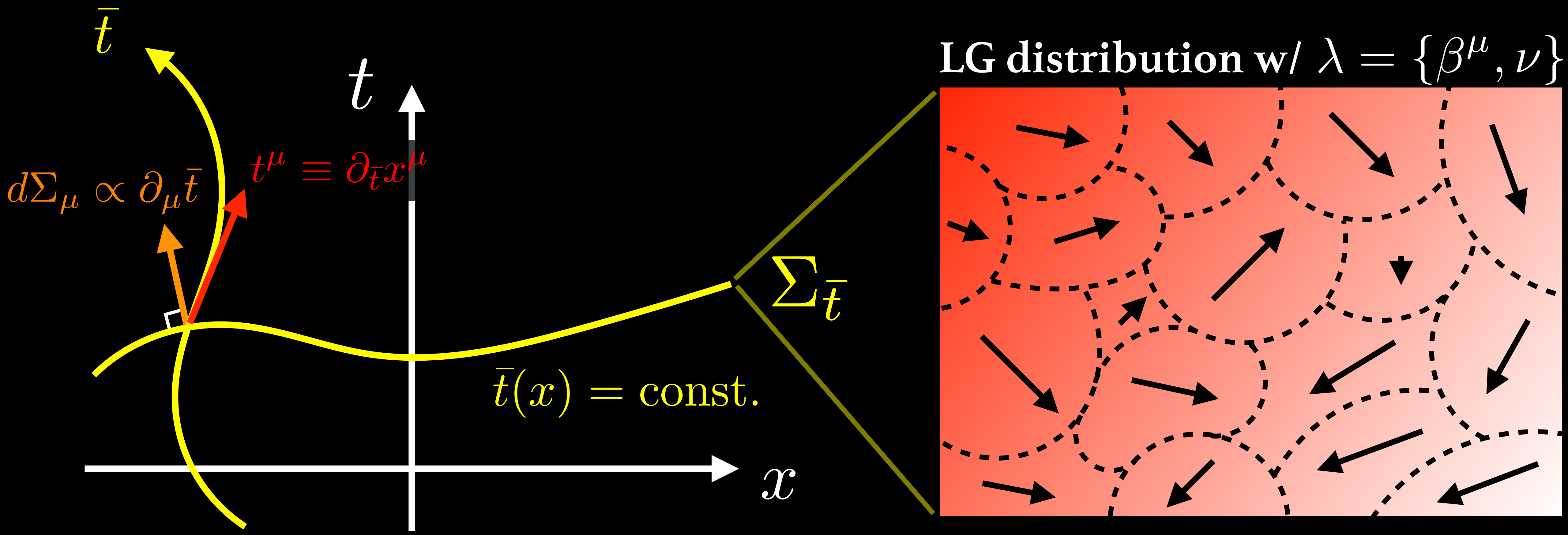

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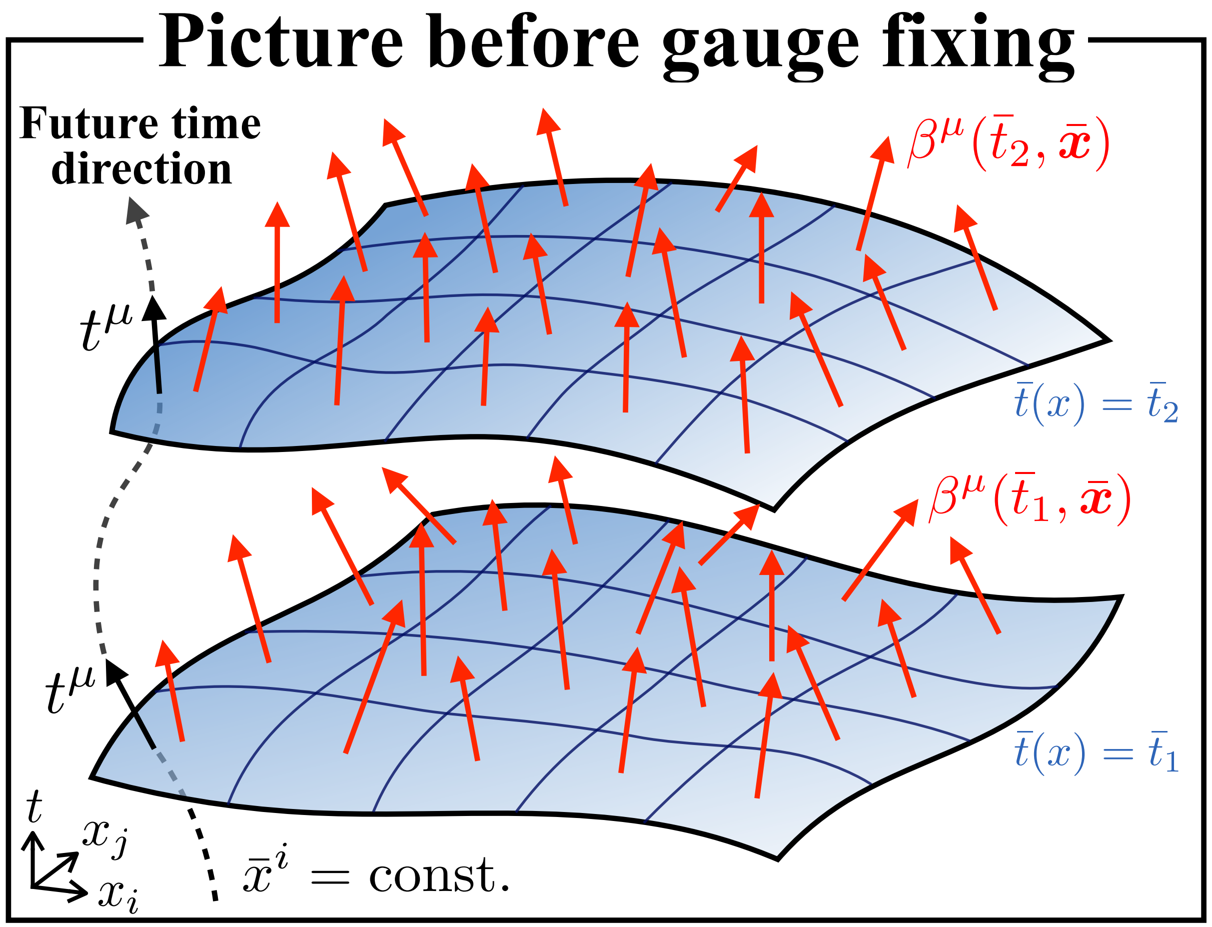
# Hydrostatic gauge fixing



We can choose the time direction vector $t^\mu(x) \equiv \partial_{\bar{t}} x^\mu$

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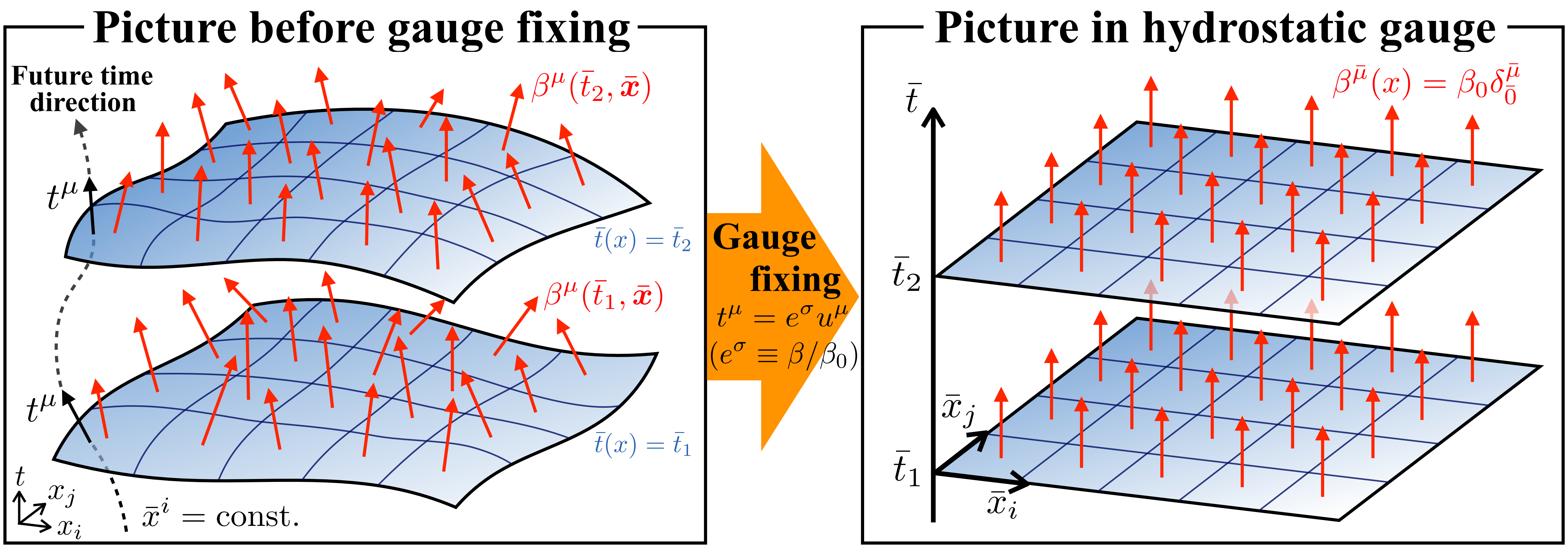

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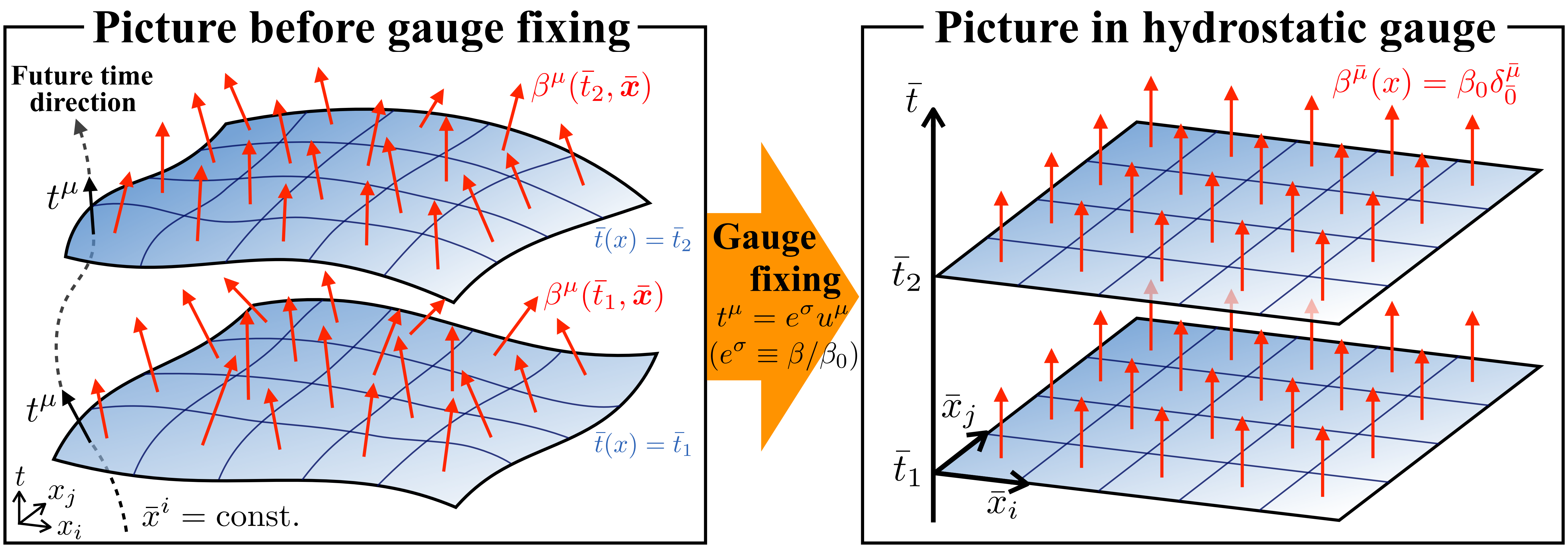

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## Hydrostatic gauge fixing

**Let us choose** $t^\mu(x) = \beta^\mu(x)/\beta_0, \; A_{\bar 0}(x) = \nu(x)$

# Variation formula for local equil.

[ Banerjee et al.(2012), Jensen et al.(2012) , Haehl et al. (2015), MH(2017)]

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$$\partial_{\bar{t}} \Psi[\bar{t};\lambda] = \int d^{d-1}\bar{x} \sqrt{-g} \left( \nabla_\mu \beta_\nu \langle \hat{T}^{\mu\nu} \rangle_{\bar{t}}^{\mathrm{LG}} + (\nabla_\mu \nu + F_{\nu\mu}\beta^\nu) \langle \hat{J}^\mu \rangle_{\bar{t}} \right)$$

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Matching them gives the above variation formula! □

**Q. How can we calculate $\Psi \equiv \log Z$ ?**

# Thermal QFT in a Nutshell

**Global equil.** $\beta_0$

$$T = \text{const.}$$

---

**Gibbs dist.:** $$\hat{\rho}_G = \frac{e^{-\beta(\hat{H}-\mu\hat{N})}}{Z} = e^{-\beta(\hat{H}-\mu\hat{N})-\Psi[\beta,\nu]}$$

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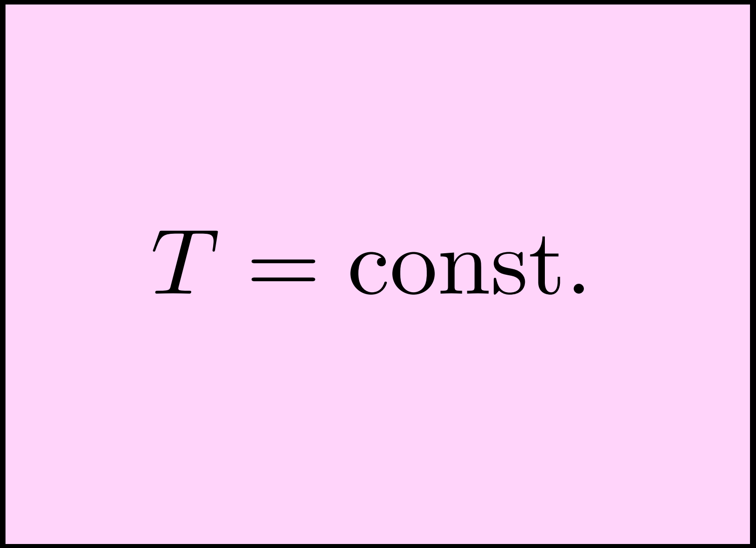

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**Thermodynamic potential with Euclidean action**

$$\Psi[\beta,\nu] = \log \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \log \int d\varphi \langle \pm\varphi | e^{-\beta(\hat{H}-\mu\hat{N})} | \varphi \rangle$$

$$= \log \int_{\varphi(\beta)=\pm\varphi(0)} \mathcal{D}\varphi\, e^{+S_E[\varphi]}, \quad S_E[\varphi] = \int_0^\beta d\tau \int d^3x\, \mathcal{L}_E(\varphi, \partial_\mu\varphi)$$

# Thermal QFT in a Nutshell

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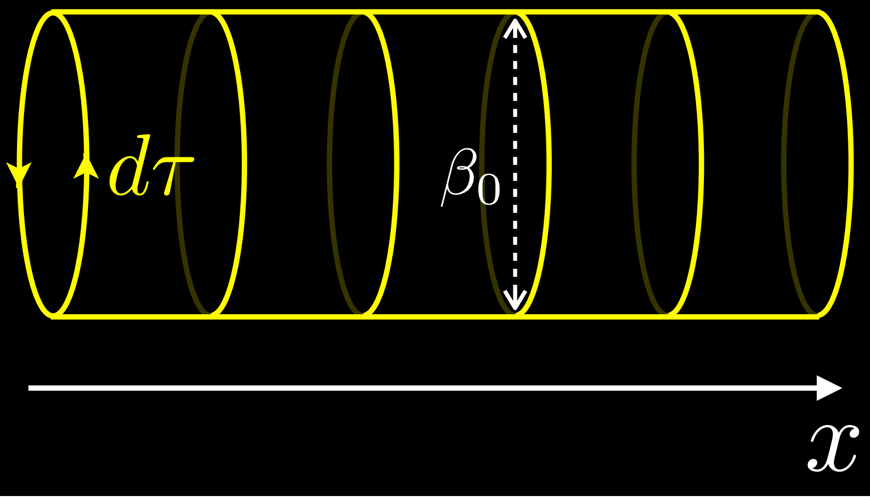
**Path int.**

## Thermal QFT (Matsubara formalism)

[ Matsubara, 1955 ]



**QFT in the flat spacetime with size** $\beta_0$

---

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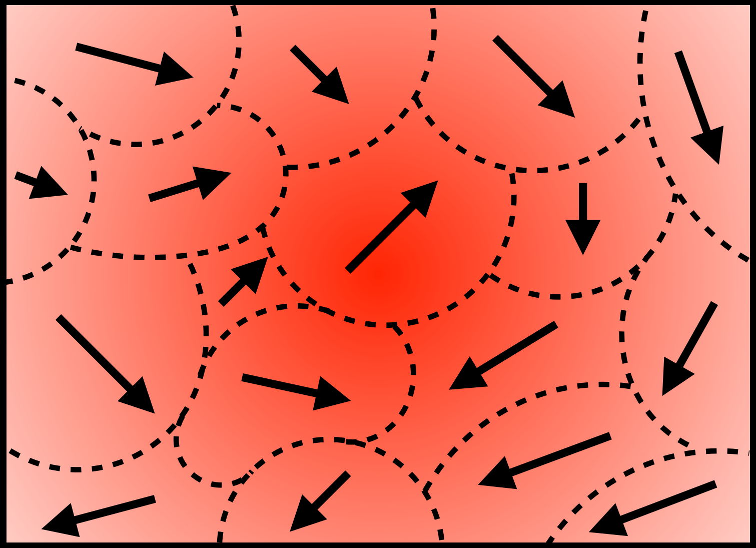
# QFT for local thermal equilibrium?

**Local equil.** $\{\beta(x),\ \vec{v}(x)\}$



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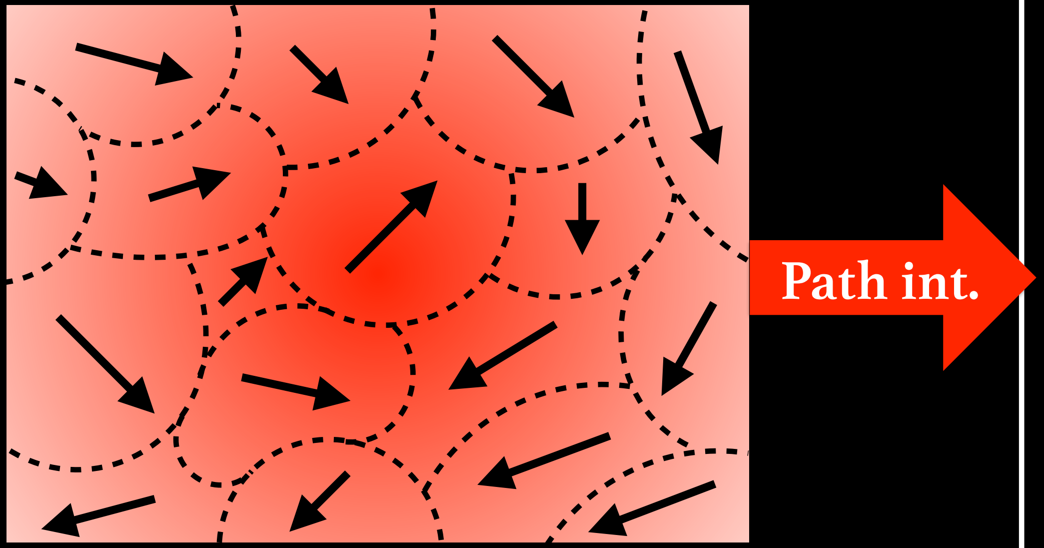

**Local Thermal QFT**

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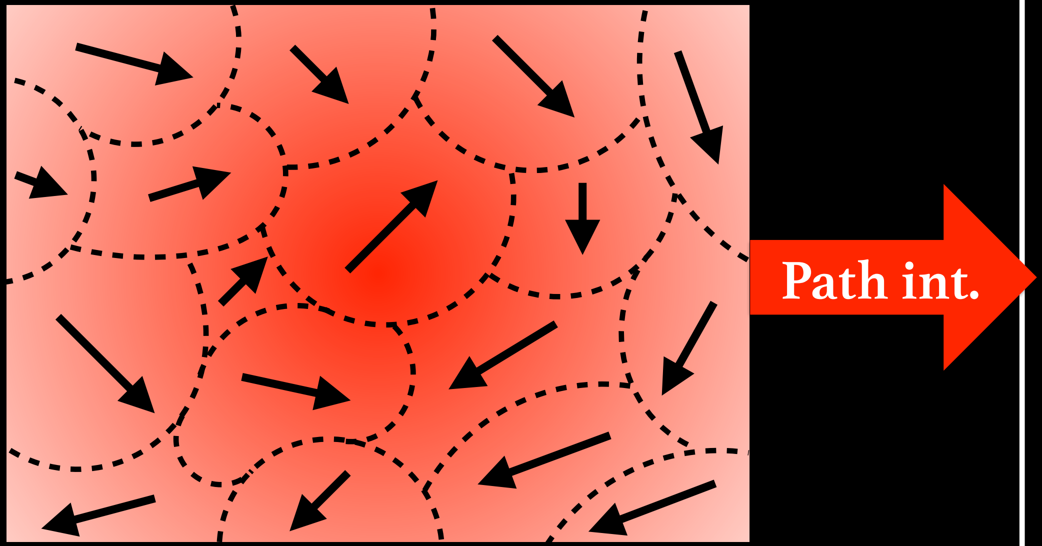
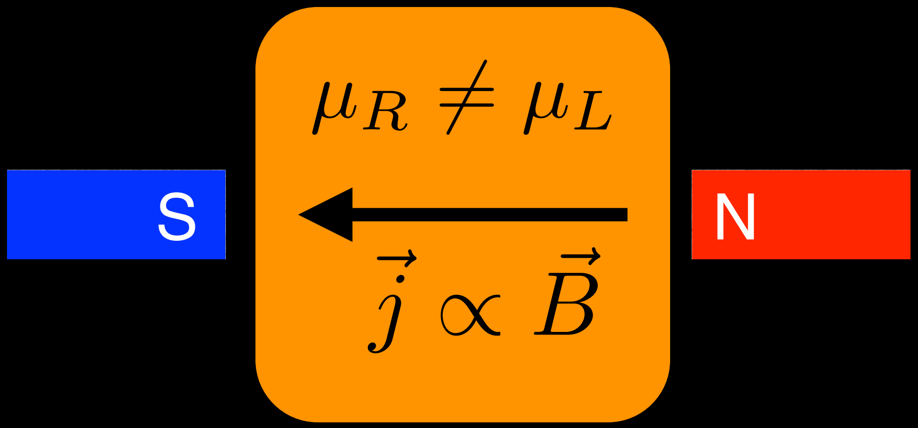
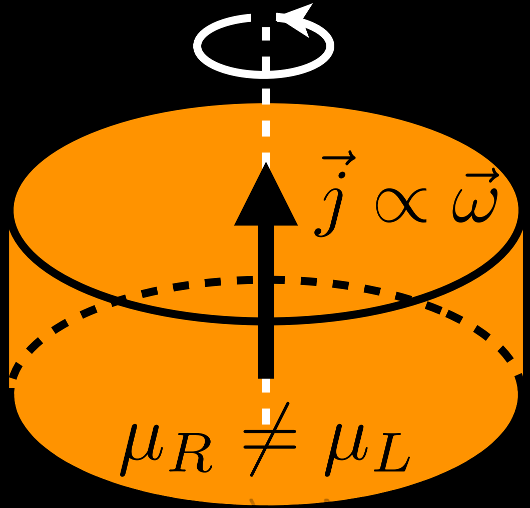


**Local Thermal QFT**

Local thermal QFT can describe anomaly-induced transport



Chiral Magnetic Effect



Chiral Vortical Effect

# Case study 1: Scalar field

$$\mathcal{L} = -\frac{g^{\bar{\mu}\bar{\nu}}}{2}\partial_{\bar{\mu}}\phi\partial_{\bar{\nu}}\phi - V(\phi)$$

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$$S[\phi,\beta^{\mu}] = \int_0^{\beta_0} d\tau \int d^3\bar{x}\sqrt{-g}e^{\sigma}u^{\bar{0}}\left[-\frac{e^{-2\sigma}}{2u^{\bar{0}}u_{\bar{0}}}(i\dot{\phi})^2 - \frac{-e^{-\sigma}u^{\bar{i}}}{u^{\bar{0}}u_{\bar{0}}}(i\dot{\phi})\partial_{\bar{i}}\phi - \frac{1}{2}\left(\gamma^{\bar{i}\bar{j}} + \frac{u^{\bar{i}}u^{\bar{j}}}{u^{\bar{0}}u_{\bar{0}}}\right)\partial_{\bar{i}}\phi\partial_{\bar{j}}\phi - V(\phi)\right]$$

$$(e^{\sigma(\bar{x})} \equiv \beta(\bar{x})/\beta_0)$$

# Case study I: Scalar field

$$\mathcal{L} = -\frac{g^{\bar{\mu}\bar{\nu}}}{2}\partial_{\bar{\mu}}\phi\partial_{\bar{\nu}}\phi - V(\phi)$$

➡ $$\hat{T}^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = \partial^{\mu}\hat{\phi}\partial^{\nu}\hat{\phi} + g^{\mu\nu}\mathcal{L}(\hat{\phi}, \partial_{\rho}\hat{\phi})$$

$$\boxed{\begin{aligned}\Psi[\bar{t};\lambda] &= \log \mathrm{Tr}\exp\left[-\int d^{d-1}\bar{x}\sqrt{-g}\beta^{\mu}(x)\hat{T}^{\bar{0}}{}_{\mu}(x)\right] \\ &= \log\int\mathcal{D}\phi\exp\left(S_E[\phi,\beta^{\mu}]\right) = \log\int\mathcal{D}\phi\exp\left(S_E[\phi,\tilde{g}]\right)\end{aligned}}$$

$$S[\phi,\beta^{\mu}] = \int_0^{\beta_0} d\tau\int d^3\bar{x}\sqrt{-g}e^{\sigma}u^{\bar{0}}\left[-\frac{e^{-2\sigma}}{2u^{\bar{0}}u_{\bar{0}}}(i\dot{\phi})^2 - \frac{-e^{-\sigma}u^{\bar{i}}}{u^{\bar{0}}u_{\bar{0}}}(i\dot{\phi})\partial_{\bar{i}}\phi - \frac{1}{2}\left(\gamma^{\bar{i}\bar{j}} + \frac{u^{\bar{i}}u^{\bar{j}}}{u^{\bar{0}}u_{\bar{0}}}\right)\partial_{\bar{i}}\phi\partial_{\bar{j}}\phi - V(\phi)\right]$$

$$= \int_0^{\beta_0} d\tau\int d^3\bar{x}\sqrt{-\tilde{g}}\left[-\frac{\tilde{g}^{\bar{\mu}\bar{\nu}}}{2}\partial_{\bar{\mu}}\phi\partial_{\bar{\nu}}\phi - V(\phi)\right] \qquad (e^{\sigma(\bar{x})} \equiv \beta(\bar{x})/\beta_0)$$

# ψ in terms of thermal metric

$$\Psi[\bar{t}; \lambda] = \log \int \mathcal{D}\phi \exp\left(S_E[\phi, ; \tilde{g}]\right)$$

### Thermal metric

$$\tilde{g}_{\bar{\mu}\bar{\nu}} = \begin{pmatrix} -e^{2\sigma} & e^{\sigma} u_{\bar{j}} \\ e^{\sigma} u_{\bar{i}} & \gamma_{\bar{i}\bar{j}} \end{pmatrix}$$

$$(e^{\sigma(\bar{x})} \equiv \beta(\bar{x})/\beta_0)$$

### Inverse thermal metric

$$\tilde{g}^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} \dfrac{e^{-2\sigma}}{u^{\bar{0}} u_{\bar{0}}} & -\dfrac{e^{-\sigma} u^{\bar{j}}}{u^{\bar{0}} u_{\bar{0}}} \\ -\dfrac{e^{-\sigma} u^{\bar{i}}}{u^{\bar{0}} u_{\bar{0}}} & \gamma^{\bar{i}\bar{j}} + \dfrac{u^{\bar{i}} u^{\bar{j}}}{u^{\bar{0}} u_{\bar{0}}} \end{pmatrix}$$

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### ◆ Interpretation of above result

$\Psi[\bar{t};\lambda]$ **is described by QFT in ”curved spacetime” s. t.**

$$d\tilde{s}^2 = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}}dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}}dx^{\bar{i}}dx^{\bar{j}}$$

$$(a_{\bar{i}} \equiv e^{-\sigma}u_{\bar{i}}, \quad \gamma'_{\bar{i}\bar{j}} \equiv \gamma_{\bar{i}\bar{j}} + u_{\bar{i}}u_{\bar{j}}, \quad d\tilde{t} = -id\tau)$$

# Case study 2: Dirac field

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}\left(\gamma^a e_a{}^{\bar{\mu}} \overrightarrow{D}_{\bar{\mu}} - \overleftarrow{D}_{\bar{\mu}} \gamma^a e_a{}^{\bar{\mu}}\right)\psi - m\bar{\psi}\psi$$

# Case study 2: Dirac field

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}\left(\gamma^a e_a{}^{\bar{\mu}}\overrightarrow{D}_{\bar{\mu}} - \overleftarrow{D}_{\bar{\mu}}\gamma^a e_a{}^{\bar{\mu}}\right)\psi - m\bar{\psi}\psi$$

**Symmetric energy-momentum tensor**

$$T^{\bar{\mu}}{}_{\bar{\nu}} = -\delta^{\bar{\mu}}_{\bar{\nu}}\mathcal{L} - \frac{1}{4}\bar{\psi}(\gamma^{\bar{\mu}}\overrightarrow{D}_{\bar{\nu}} + \gamma_{\bar{\nu}}\overrightarrow{D}^{\bar{\mu}} - \overleftarrow{D}_{\bar{\nu}}\gamma^{\bar{\mu}} - \overleftarrow{D}^{\bar{\mu}}\gamma_{\bar{\nu}})\psi$$

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◆ **Result of path integral**

$$\Psi[\bar{t};\lambda] \equiv \log \mathrm{Tr}\exp\left[\int d\Sigma_{\bar{t}\nu}\left(\beta^{\mu}(x)\hat{T}^{\nu}{}_{\mu}(x) + \nu(x)\hat{J}^{\nu}(x)\right)\right]$$

$$= \log\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left(S_E[\psi,\bar{\psi};\tilde{e}]\right)$$

# ψ in terms of thermal vielbein

$$\Psi[\bar{t};\lambda] = \log \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(S_E[\psi,\bar{\psi};\tilde{e}]\right)$$

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- **Euclidean action with thermal vielbein**

$$S_E[\psi,\bar{\psi};\tilde{e}] = \int_0^{\beta_0} d\tau \int d^3\bar{x}\,\tilde{e}\left[-\frac{1}{2}\bar{\psi}\left(\gamma^a \tilde{e}_a{}^{\bar{\mu}}\overrightarrow{D}_{\bar{\mu}} - \overleftarrow{D}_{\bar{\mu}}\gamma^a \tilde{e}_a{}^{\bar{\mu}}\right)\psi - m\bar{\psi}\psi\right]$$

**Thermal vielbein :** $\tilde{e}_{\bar{0}}{}^{a} = e^{\sigma}u^{a},\ \tilde{e}_{\bar{i}}{}^{a} = e_{\bar{i}}{}^{a} \quad (e^{\sigma} \equiv \beta(x)/\beta_0)$

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**Thermal vielbein :** $\tilde{e}_{\bar{0}}{}^a = e^\sigma u^a, \ \tilde{e}_{\bar{i}}{}^a = e_{\bar{i}}{}^a \quad (e^\sigma \equiv \beta(x)/\beta_0)$

## ◆ Interpretation of above result

$\Psi[\bar{t};\lambda]$ **is described by QFT in "curved spacetime" s. t.**

$$d\tilde{s}^2 = \tilde{e}_{\bar{\mu}}{}^a\tilde{e}_{\bar{\nu}}{}^b\eta_{ab}dx^{\bar{\mu}}dx^{\bar{\nu}} = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}}dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}}dx^{\bar{i}}dx^{\bar{j}}$$

$$(a_{\bar{i}} \equiv e^{-\sigma}u_{\bar{i}}, \quad \gamma'_{\bar{i}\bar{j}} \equiv \gamma_{\bar{i}\bar{j}} + u_{\bar{i}}u_{\bar{j}}, \quad d\tilde{t} = -id\tau)$$

# Local Thermal QFT

**Global equil.** $\beta_0$

$$T = \text{const.}$$

# Local Thermal QFT

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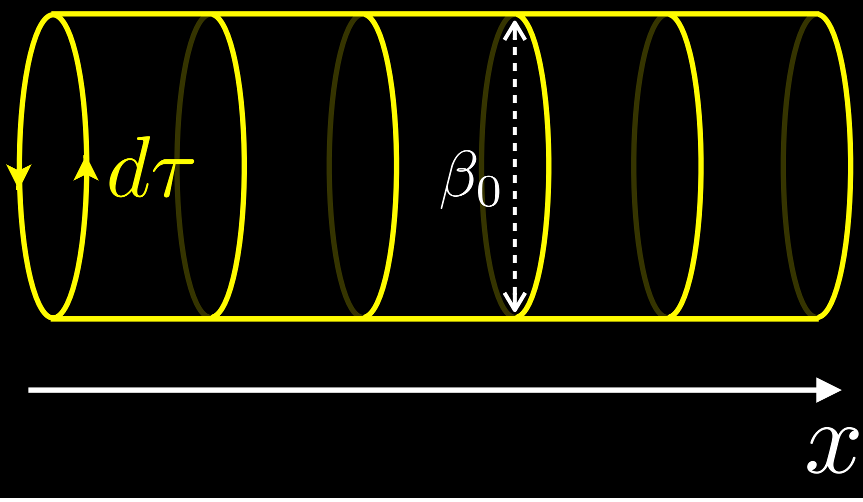
**Path int.**

## Thermal QFT (Matsubara formalism)

[ Matsubara, 1955 ]



**QFT in the flat spacetime with size** $\beta_0$

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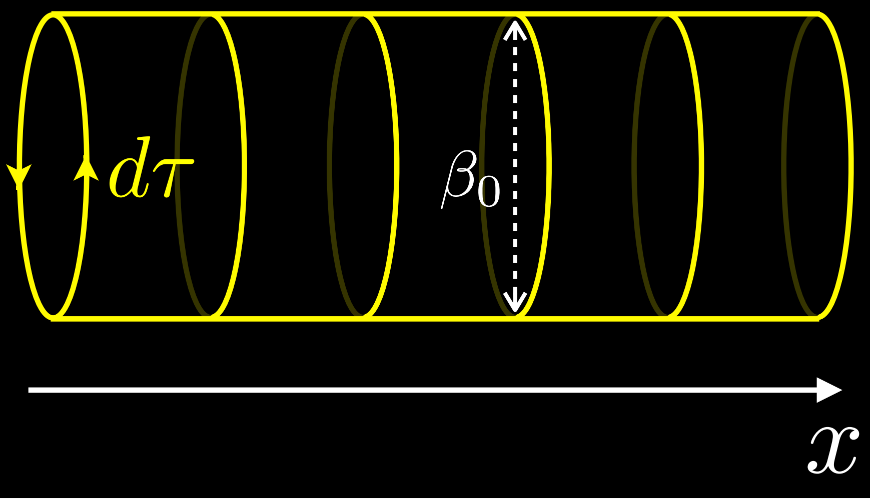

**QFT in the flat spacetime with size** $\beta_0$

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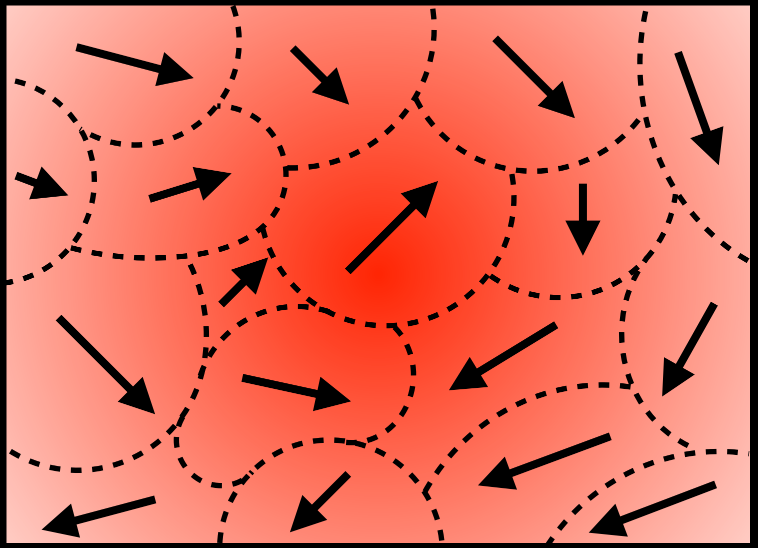
**Local equil.** $\{\beta(x),\ \vec{v}(x)\}$



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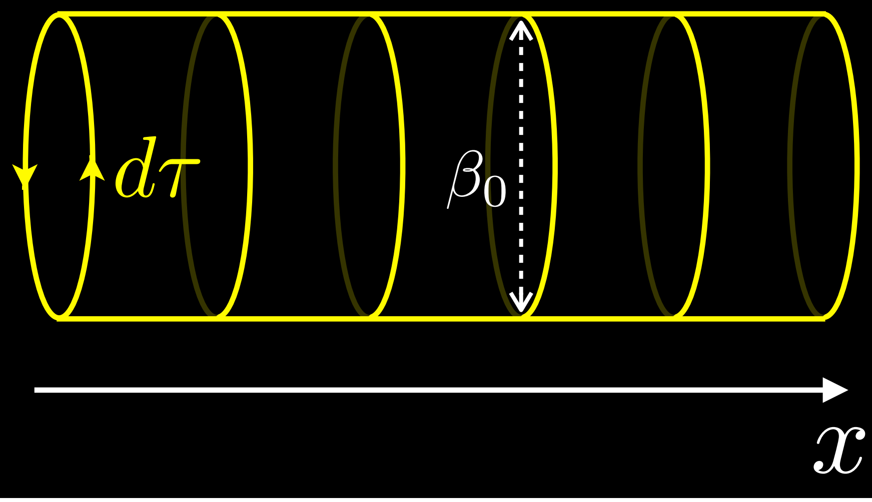

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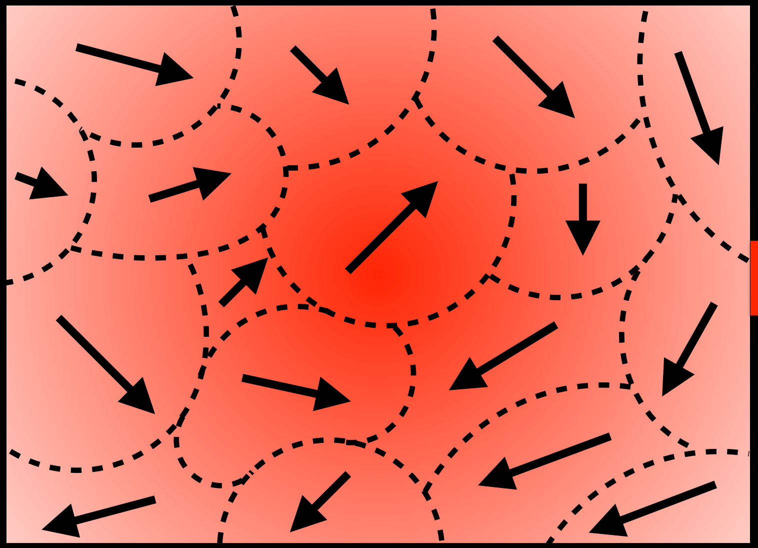
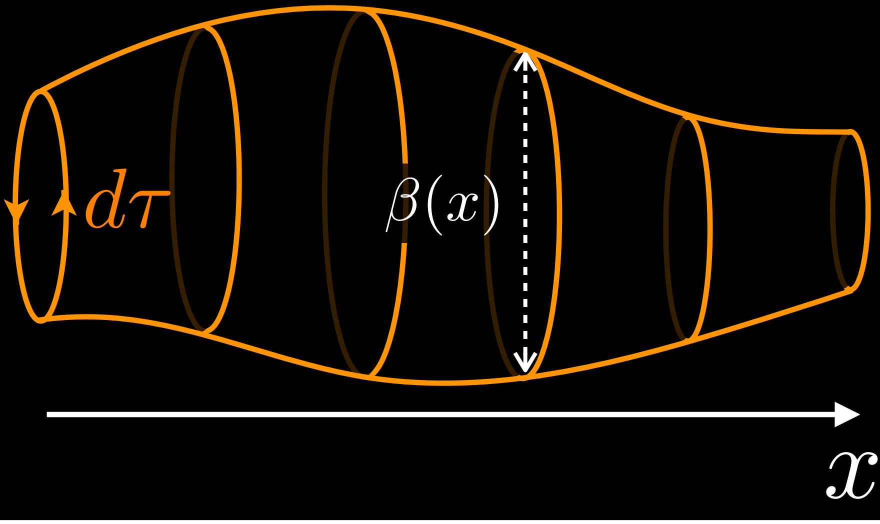


**Path int.**

## Local Thermal QFT

[ Hayata-Hidaka-MH-Noumi PRD(2015) ]
[ MH (2017) ]



**QFT in the "curved spacetime" with "line element"**

$$d\tilde{s}^2 = d\tilde{s}^2(\beta, \vec{v})$$

# Symmetry of Local Thermal QFT

$$d\tilde{s}^2 = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}} dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}} dx^{\bar{i}} dx^{\bar{j}}$$

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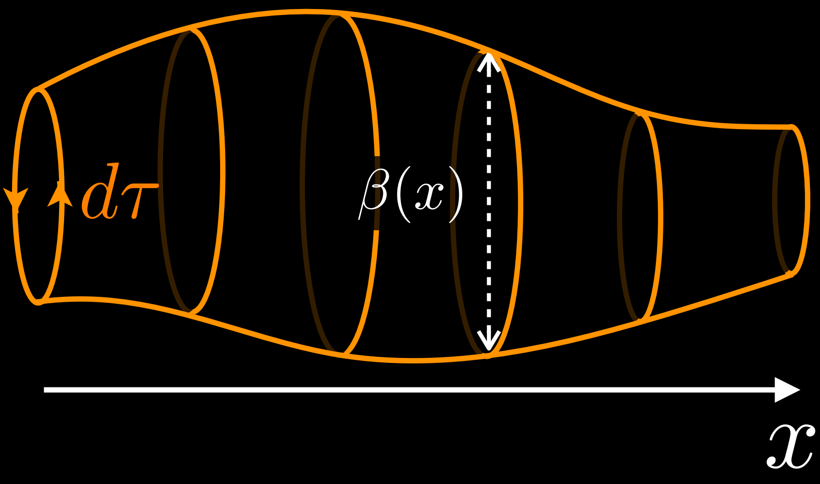

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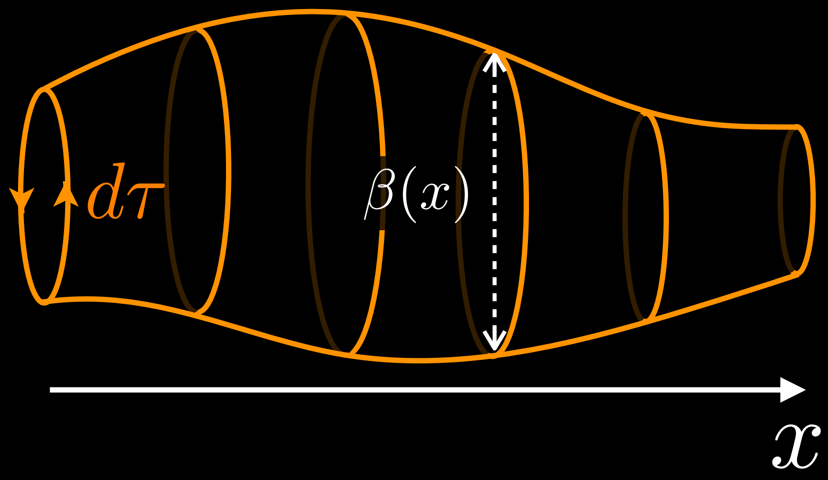

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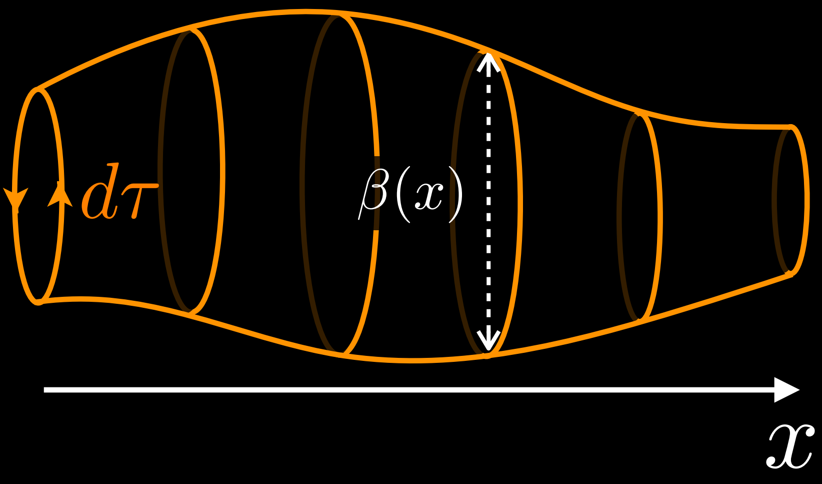

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**(1) Spatial diffeomorphism** :

Physics does not depend on the choice of Spatial coordinate system!!

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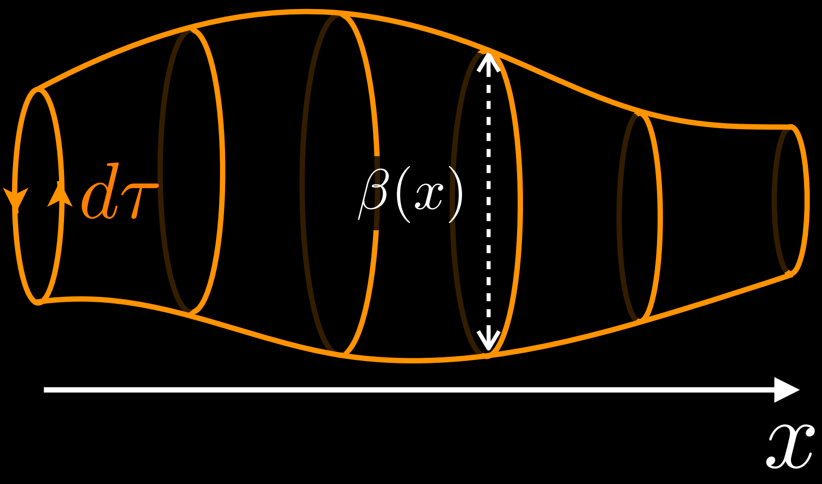

## Symmetry of "curved spacetime"

**( 1 ) Spatial diffeomorphism** :

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**( 2 ) Imaginary time tran. & Kaluza-Klein gauge** :

Parameters $\lambda$ (e.g. T) does not depend on imaginary time!!

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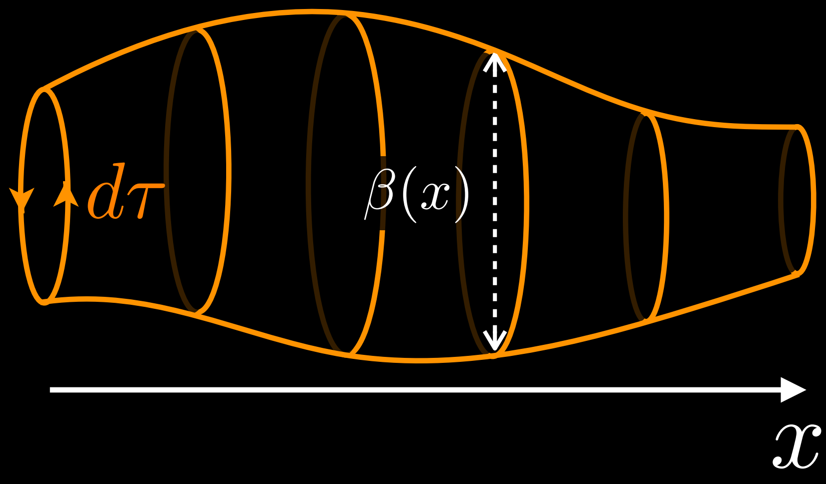

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$\Psi[\lambda] = \log \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{S[\psi,\bar{\psi},\tilde{e}]}$ **should respect the above symmetries!!**

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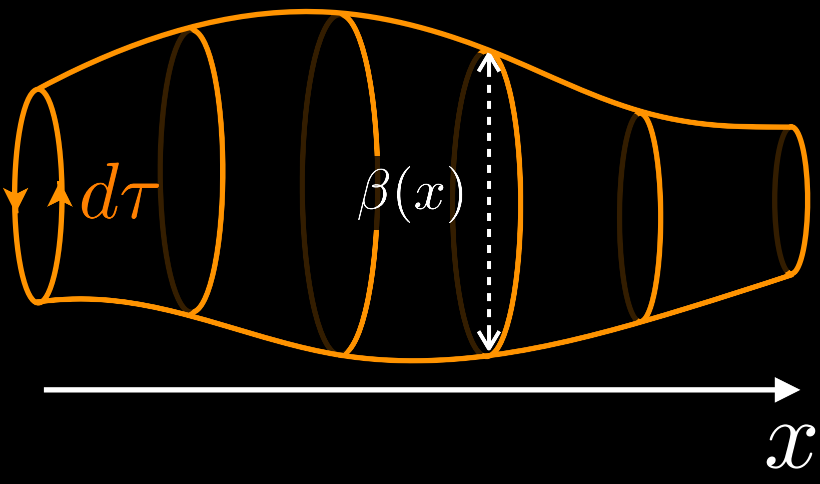

## Symmetry of "curved spacetime"

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**cf. Hydrostatic partition function method**

Banerjee et al.(2012), Jensen et al.(2012)

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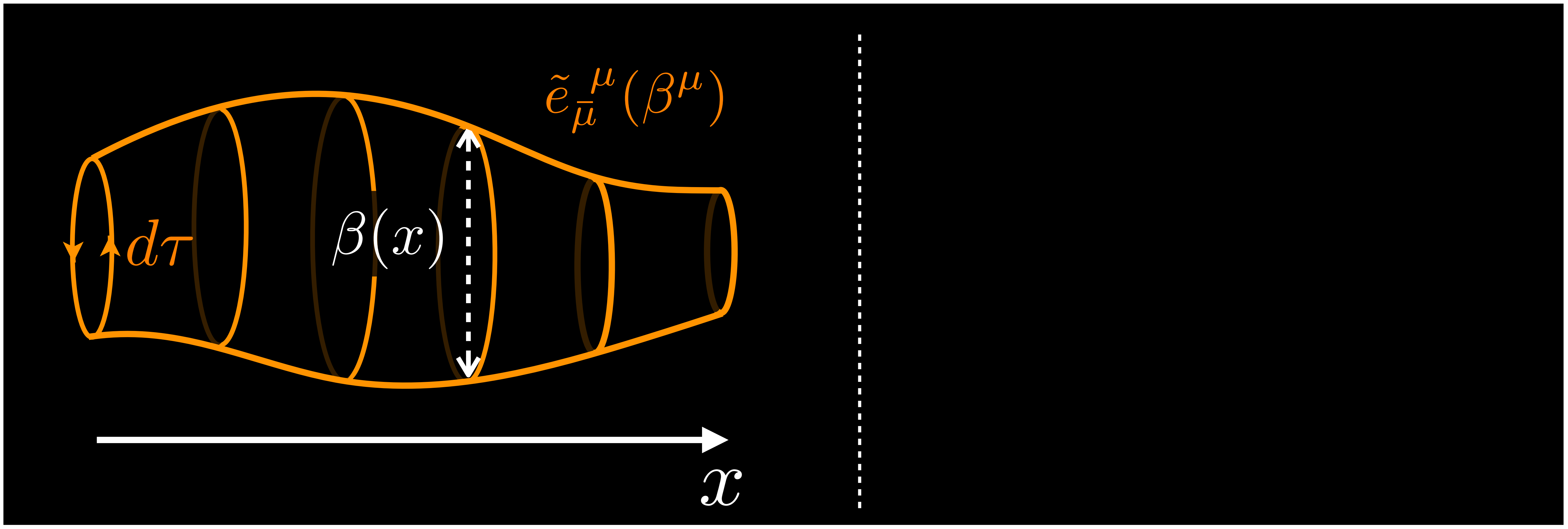
$$d\tilde{s}^2 = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}}dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}}dx^{\bar{i}}dx^{\bar{j}} \quad (d\tilde{t} = -id\tau)$$



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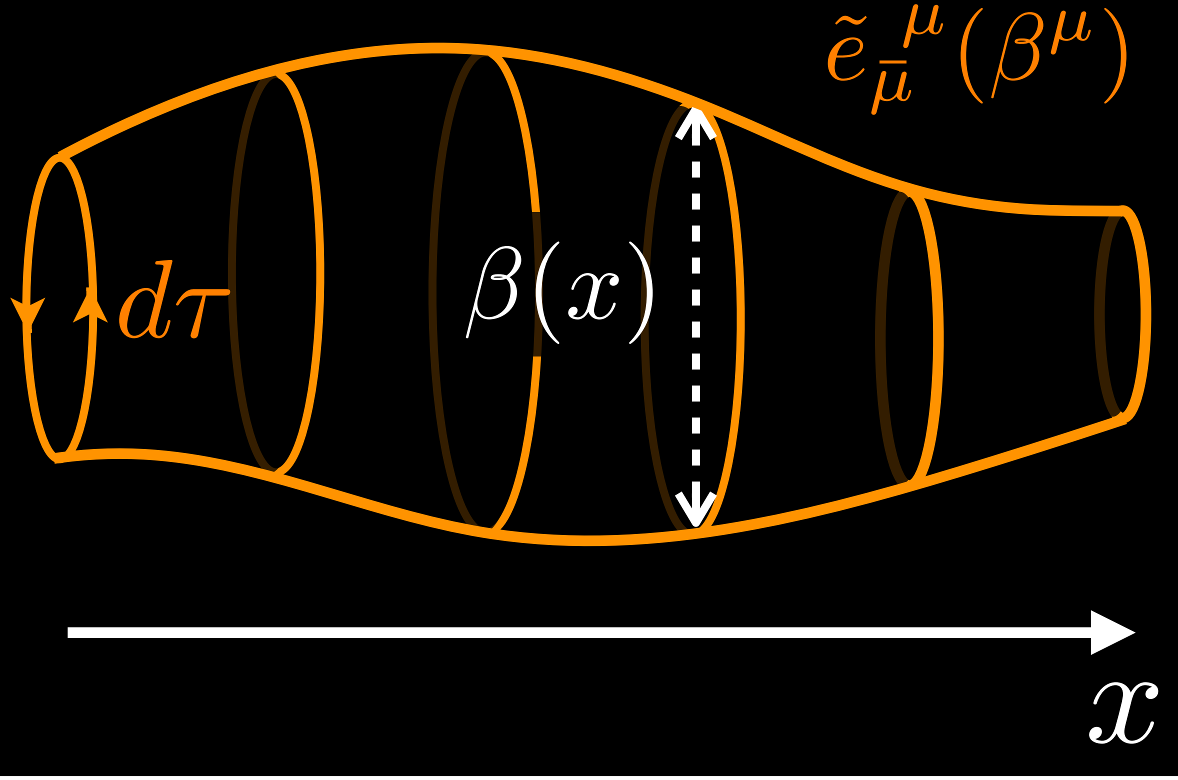

**Parameters $\lambda$ don't depend on imaginary time $\tau$.**

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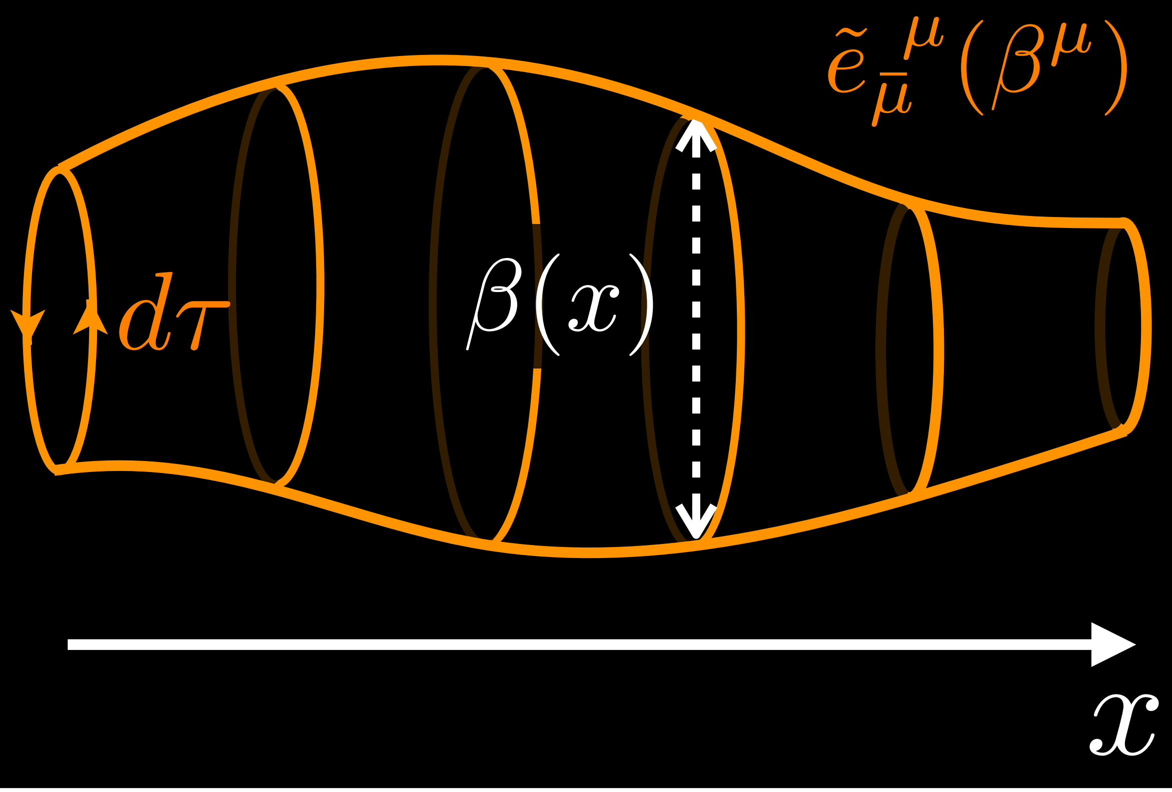

**Parameters $\lambda$ don't depend on imaginary time $\tau$.**

**"Kaluza-Klein" gauge tr.**

$$\begin{cases} \tilde{t} \to \tilde{t} + \chi(\bar{\boldsymbol{x}}) \\ \bar{\boldsymbol{x}} \to \bar{\boldsymbol{x}} \\ a_{\bar{i}}(\bar{\boldsymbol{x}}) \to a_{\bar{i}}(\bar{\boldsymbol{x}}) - \partial_{\bar{i}}\chi(\bar{\boldsymbol{x}}) \end{cases}$$

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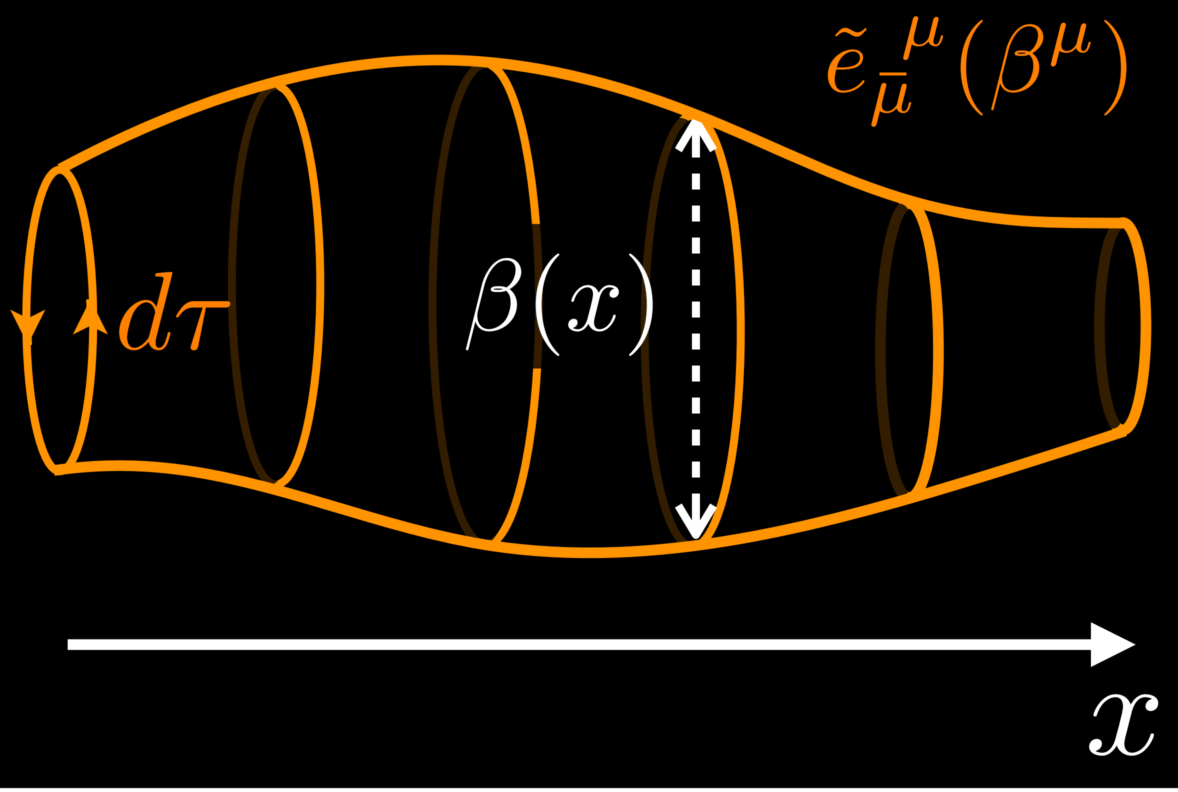

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$$\Psi[\lambda] = \log \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{S[\psi,\bar{\psi},\tilde{e}]} \ni$$

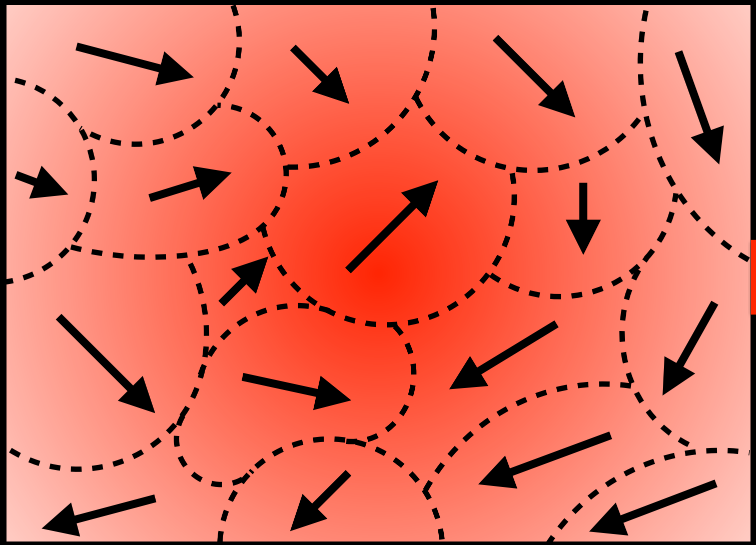
👍 $f^{\bar{i}\bar{j}} f_{\bar{i}\bar{j}}, \cdots$

👎 $a_{\bar{i}},\ a_{\bar{i}}a^{\bar{i}}, \cdots$

$$(f_{\bar{i}\bar{j}} \equiv \partial_{\bar{i}}a_{\bar{j}} - \partial_{\bar{j}}a_{\bar{i}})$$

# Short Summary: Local Thermal QFT

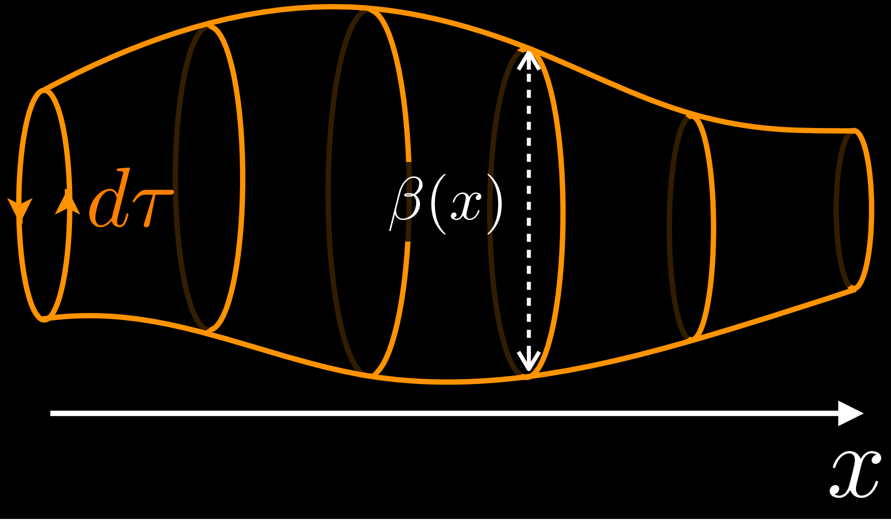
Local equil. $\{\beta(x),\ \vec{v}(x)\}$



Path int.

## Local Thermal QFT

[ Hayata-Hidaka-MH-Noumi PRD(2015) ]
[ MH (2017) ]



QFT in the **"curved spacetime"**
with **"line element"**

$$d\tilde{s}^2 = d\tilde{s}^2(\beta, \vec{v})$$

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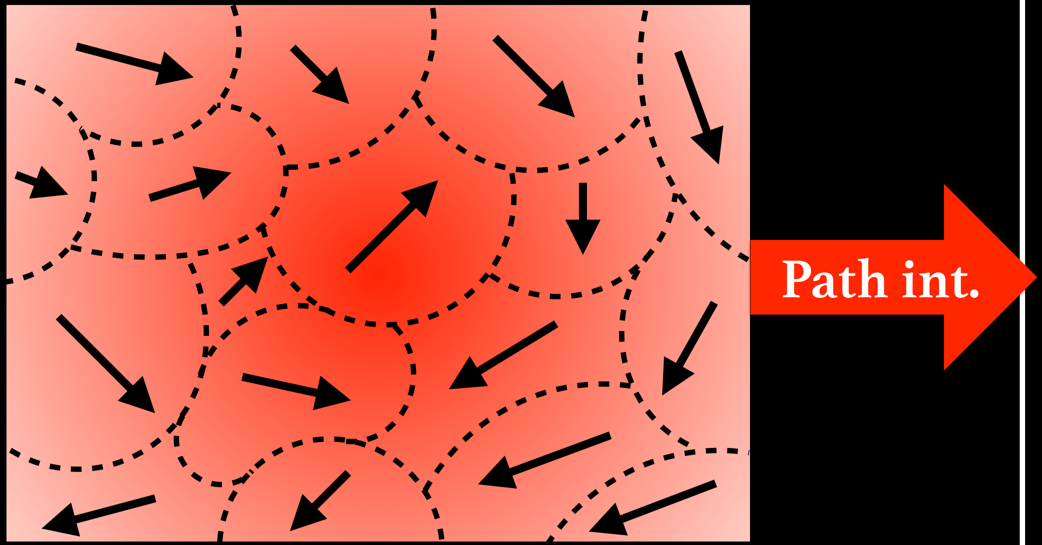

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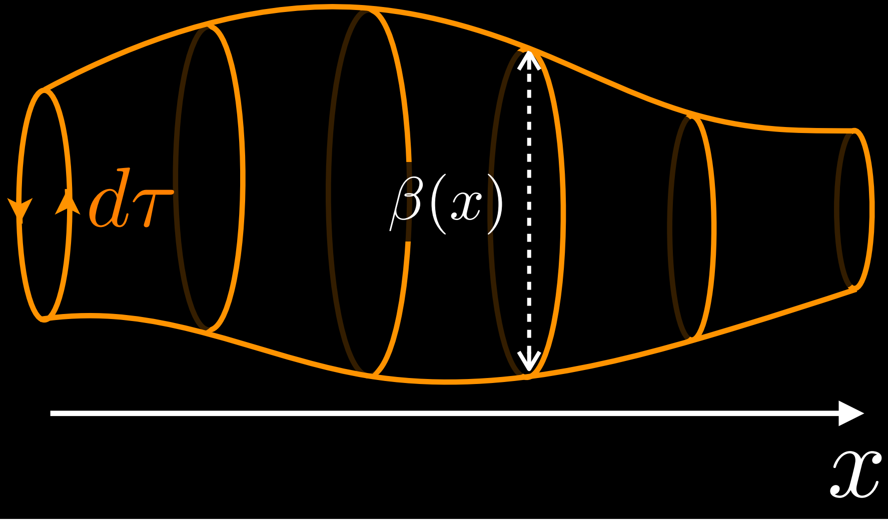
[ Hayata-Hidaka-MH-Noumi PRD(2015) ]
[ MH (2017) ]



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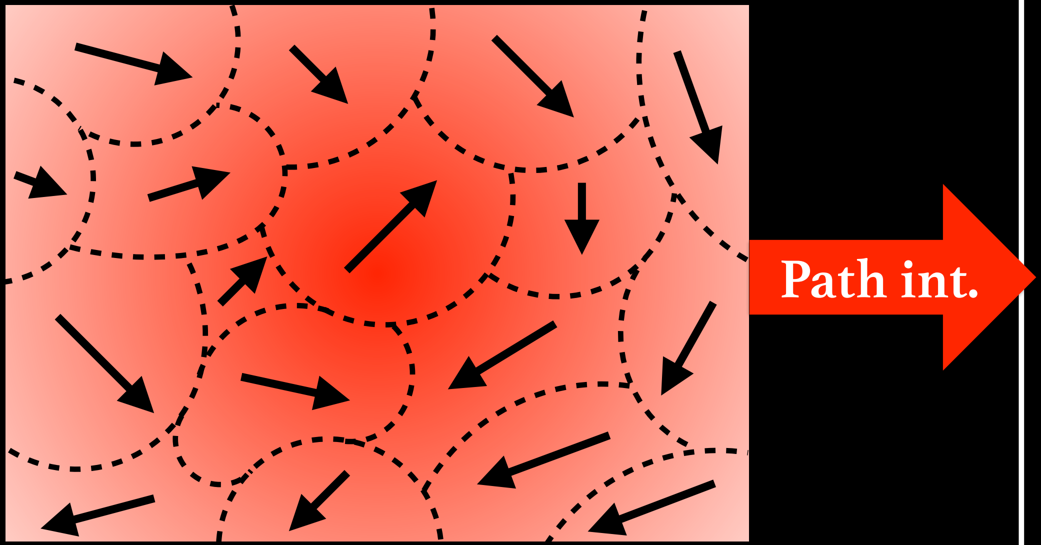

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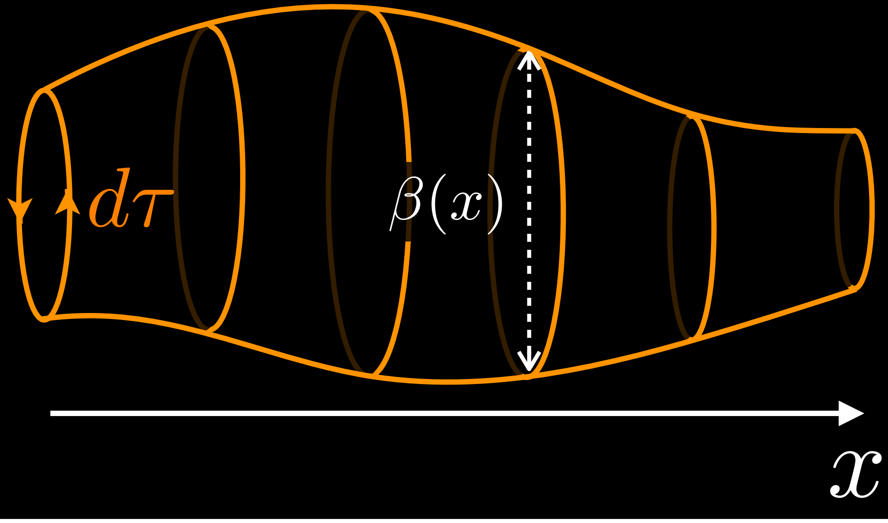

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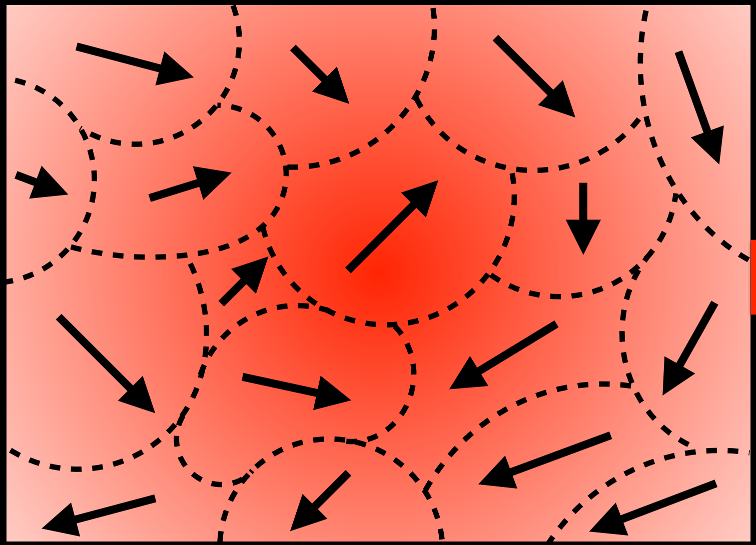
① $\Psi[\lambda]$ plays a role as the generating functional: $\langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}} = \dfrac{2}{\sqrt{-g}}\dfrac{\delta}{\delta g_{\mu\nu}(x)}\Psi[\lambda]$

# Short Summary: Local Thermal QFT

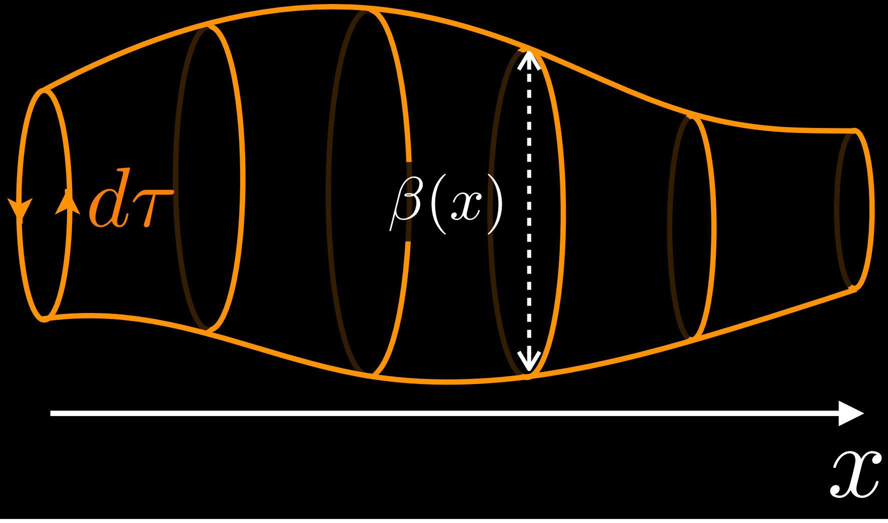
**Local equil.** $\{\beta(x),\ \vec{v}(x)\}$



**Path int.**

## Local Thermal QFT

[ Hayata-Hidaka-MH-Noumi PRD(2015) ]
[ MH (2017) ]



QFT in the
“curved spacetime”
with “line element”

$$d\tilde{s}^2 = d\tilde{s}^2(\beta, \vec{v})$$

$$\Psi[\bar{t};\lambda] \equiv \log \mathrm{Tr} \exp\left[\int d\Sigma_{\bar{t}\nu}\left(\beta^{\mu}(x)\hat{T}^{\nu}{}_{\mu}(x)+\nu(x)\hat{J}^{\nu}(x)\right)\right]$$

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$$d\tilde{s}^2 = -e^{2\sigma}(d\tilde{t}+a_{\bar{i}}dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}}dx^{\bar{i}}dx^{\bar{j}}$$

# Short Summary: Local Thermal QFT

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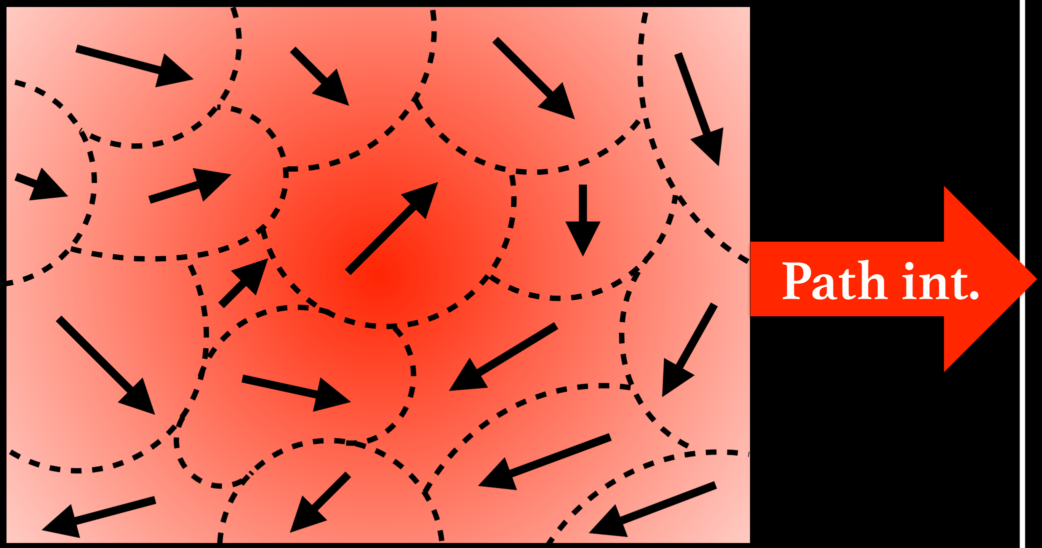

## Local Thermal QFT

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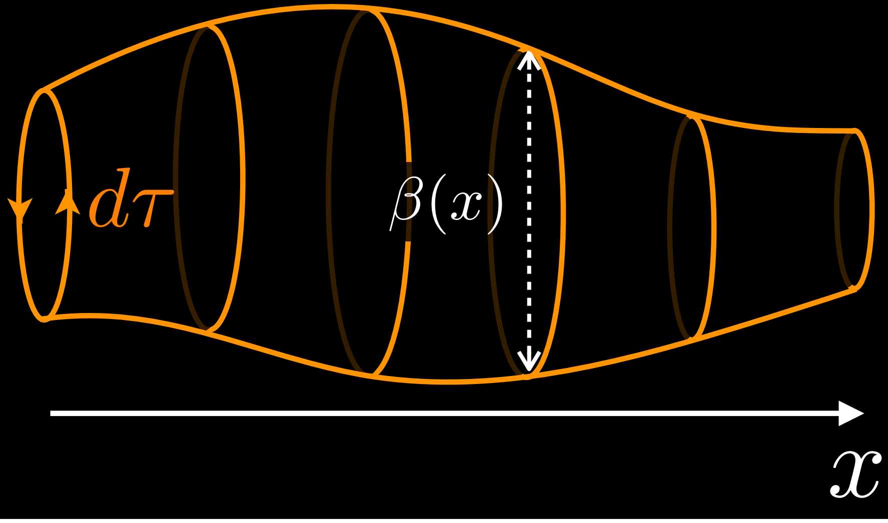

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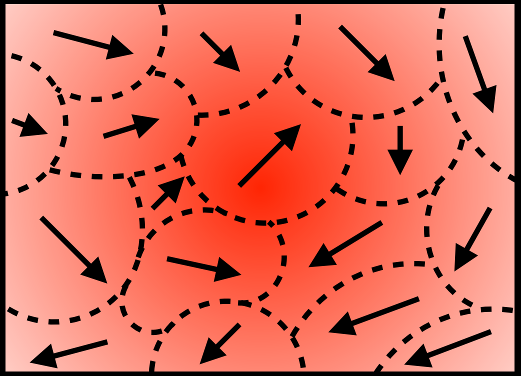
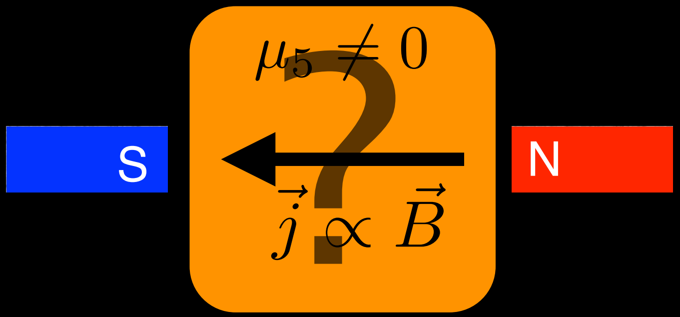
**Symmetry = Spatial diffeomorphism + Kaluza-Klein gauge**

# Outline

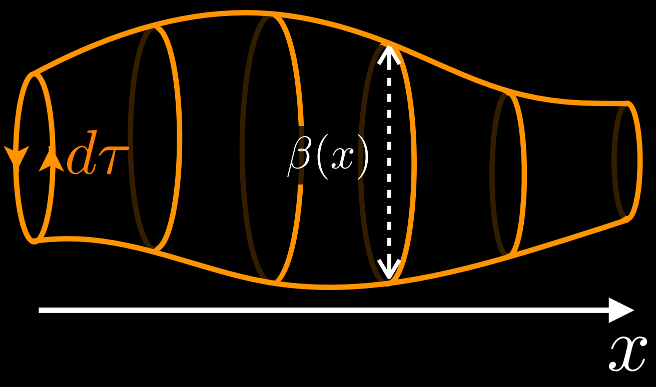
## MOTIVATION:

Quantum field theory under
local thermal equilibrium?





## APPROACH:



QFT for Local Gibbs distribution

① Variation formula: $\langle \hat{T}^{\mu\nu}(x)\rangle^{\mathrm{LG}} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\Psi[\lambda]$

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## APPLICATION:

Derivation of
Anomalous hydrodynamics

# Outline

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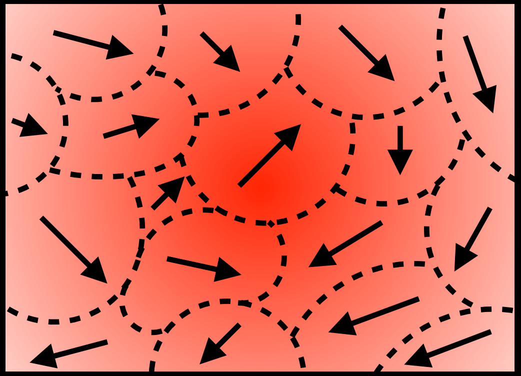

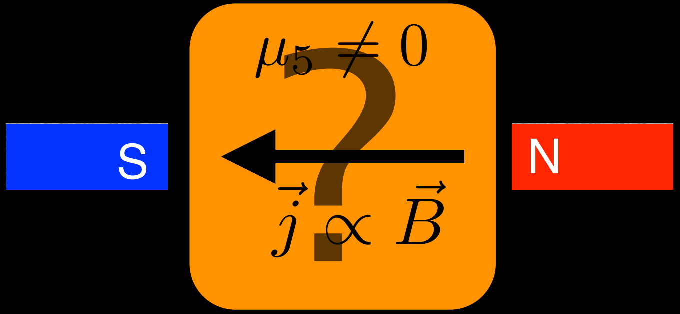

## APPROACH:

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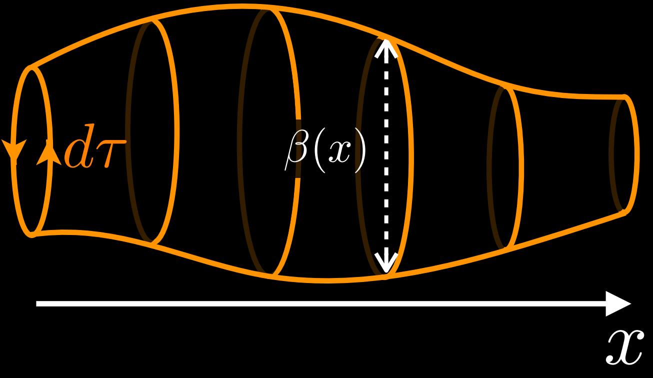

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## APPLICATION:

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# Parity-even case

$$\mu_R = \mu_L$$

# Derivative expansion of ψ

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**Non-dissipative constitutive relation**

$$\langle \hat{T}^{\mu\nu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\Psi[\bar{t};\lambda] = T^{\mu\nu}_{(0)}[\lambda(x)] + T^{\mu\nu}_{(1)}[\lambda(x),\nabla\lambda(x)] + \cdots$$

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Symmetry property

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[ Banerjee et al.(2012), Jensen et al.(2012) ]

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$$\Psi^{(0)}[\lambda] = \int_0^{\beta_0} d\tau \int d^3\bar{x} \sqrt{\gamma'} e^\sigma p(\beta, \mu)$$

# ψ<sup></sup>

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**Perfect fluid**

$$\langle \hat{T}^{\mu\nu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = (e+p)u^\mu u^\nu + p\eta^{\mu\nu}$$

$$\langle \hat{J}^{\mu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = nu^\mu$$

# Parity-odd case

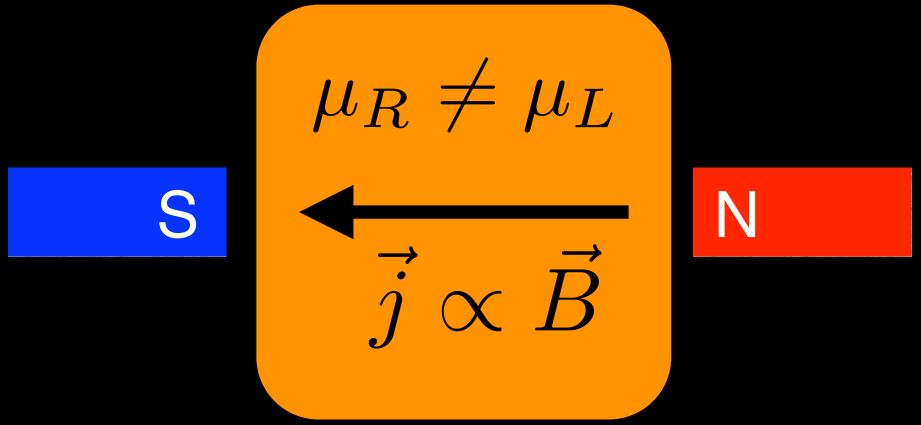
$$\mu_R \neq \mu_L$$

# Anomaly-induced transport

- **Chiral Magnetic Effect (CME)**

[ Fukushima et al.2008, Vilenkin 1980 ]

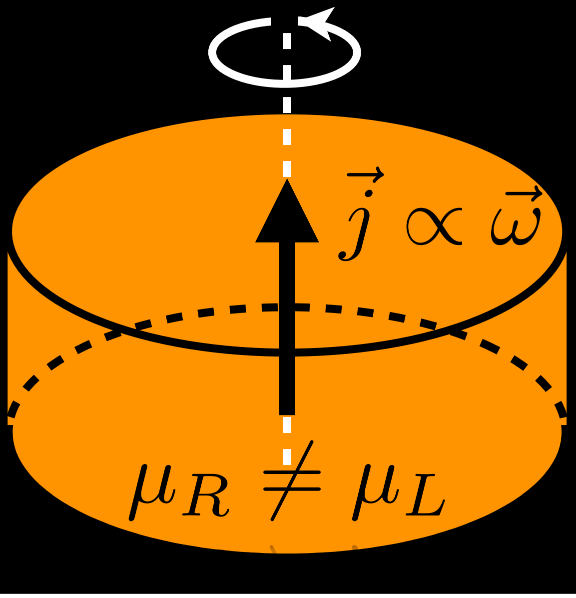
$$\vec{j} = \frac{e\mu_5}{2\pi^2}\vec{B}$$



- **Chiral Vortical Effect (CVE)**

[ Erdmenger et al. 2008, Son-Surowka 2009 ]

$$\vec{j} = \frac{\mu\mu_5}{2\pi^2}\vec{\omega}$$



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$\simeq \beta p$ $= 0$ **Parity-even system**

Symmetry property

## Non-dissipative constitutive relation

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$= 0 \qquad \neq 0$

# Recipe for Masseiu-Planck fcn.

**Weyl fermion**：$\mathcal{L} = \frac{i}{2}\xi^\dagger\left(e_m{}^\mu\sigma^m\overrightarrow{D}_\mu - \overleftarrow{D}_\mu\sigma^m e_m{}^\mu\right)\xi$

$$\Psi[\lambda] = \log\int\mathcal{D}\xi^\dagger\mathcal{D}\xi e^{S[\xi,\xi^\dagger,A,\tilde{e}]} = \underset{\mathcal{O}(p^0)}{\Psi^{(0)}[\lambda]} + \underset{\mathcal{O}(p^1)}{\Psi^{(1)}[\lambda,\partial]} + \mathcal{O}(\partial^2)$$

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**Weyl fermion** : $\mathcal{L} = \frac{i}{2}\xi^\dagger\left(e_m^{\ \mu}\sigma^m \overrightarrow{D}_\mu - \overleftarrow{D}_\mu\sigma^m e_m^{\ \mu}\right)\xi$

$$\Psi[\lambda] = \log\int \mathcal{D}\xi^\dagger\mathcal{D}\xi e^{S[\xi,\xi^\dagger,A,\tilde{e}]} = \underset{\mathcal{O}(p^0)}{\Psi^{(0)}[\lambda]} + \underset{\mathcal{O}(p^1)}{\Psi^{(1)}[\lambda,\partial]} + \mathcal{O}(\partial^2)$$

- **Building blocks** : $\lambda = \{e^\sigma,\ a_{\bar{i}},\ \mu_R,\ \bar{A}_{\bar{i}}\}$

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# ψ⁽⁰⁾ : Order $\mathcal{O}(p^0)$

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$$\Psi[\lambda] = \log\int\mathcal{D}\xi^\dagger\mathcal{D}\xi\, e^{S[\xi,\xi^\dagger,A,\tilde{e}]} = \Psi^{(0)}[\lambda] + \Psi^{(1)}[\lambda,\partial] + \mathcal{O}(\partial^2)$$

$$\mathcal{O}(p^0) \qquad \mathcal{O}(p^1)$$

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$$\Psi^{(0)}[\lambda] = \int_0^{\beta_0} d\tau \int d^3\bar{x}\sqrt{\gamma'}e^\sigma p(\beta,\mu_R)$$

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→

**Perfect fluid**

$$\langle\hat{T}^{\mu\nu}(x)\rangle_{\bar{t}}^{\mathrm{LG}} = (e+p)u^\mu u^\nu + p\eta^{\mu\nu}$$

$$\langle\hat{J}_R^\mu(x)\rangle_{\bar{t}}^{\mathrm{LG}} = n_R u^\mu$$

# ψ⁽¹⁾ : Order $\mathcal{O}(p)$

**Weyl fermion** : $\mathcal{L} = \frac{i}{2}\xi^\dagger \left( e_m^{\ \mu} \sigma^m \overrightarrow{D}_\mu - \overleftarrow{D}_\mu \sigma^m e_m^{\ \mu} \right) \xi$

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# $\psi^{(1)}$ : Order $\mathcal{O}(p)$

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$$\Psi[\lambda] = \log\int\mathcal{D}\xi^\dagger\mathcal{D}\xi e^{S[\xi,\xi^\dagger,A,\tilde{e}]} = \underset{\mathcal{O}(p^0)}{\Psi^{(0)}[\lambda]} + \underset{\mathcal{O}(p^1)}{\Psi^{(1)}[\lambda,\partial]} + \mathcal{O}(\partial^2)$$

- **Building blocks** : $\lambda = \{e^\sigma,\ a_{\bar{i}},\ \mu_R,\ \bar{A}_{\bar{i}}\}$

$$\int d^3\bar{x}\sqrt{\gamma'}C_1(\beta,\mu_R)\epsilon^{\bar{i}\bar{j}\bar{k}}\bar{A}_{\bar{i}}\partial_{\bar{j}}\bar{A}_{\bar{k}}$$

$$\int d^3\bar{x}\sqrt{\gamma'}C_2(\beta,\mu_R)\epsilon^{\bar{i}\bar{j}\bar{k}}\bar{A}_{\bar{i}}\partial_{\bar{j}}a_{\bar{k}}$$

# $\psi^{(\mathrm{I})}$ : Order $\mathcal{O}(p)$

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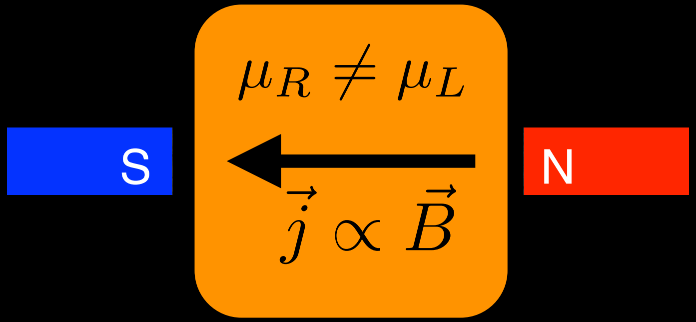

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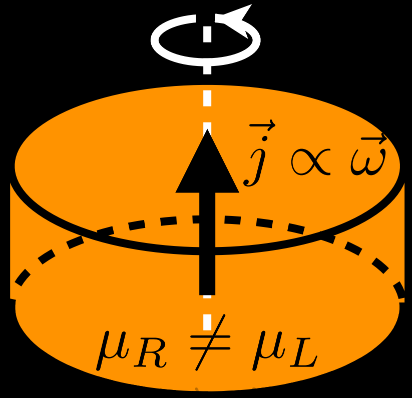

# Anomalous transport coefficients

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① **Non-perturbative way (WZ consistency condition …)**

[Banerjee et al.(2012), Jensen et al.(2012) , Haehl et al. (2015)]

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[Recall Prokhorov's talk]

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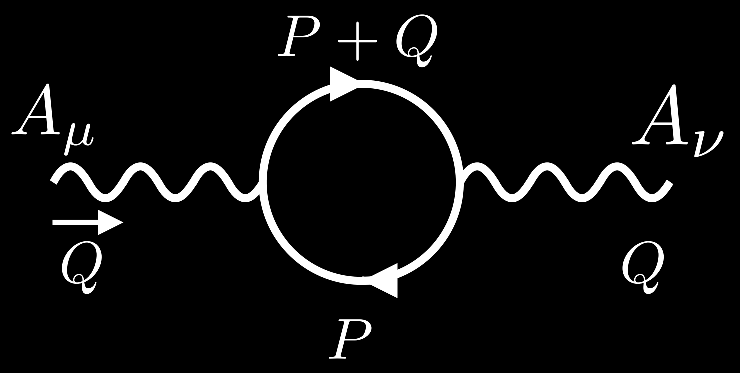
## ② Perturbative evaluation of ψ in external field

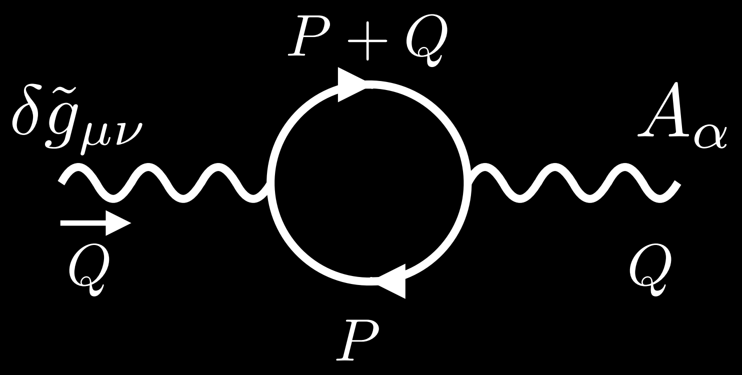
[Recall Prokhorov's talk]

$$\frac{\delta^2\Psi}{\delta A_\mu \delta A_\nu} =$$



$$\simeq -i\varepsilon^{0\mu\rho\nu}\tilde{Q}_\rho \frac{\mu_R}{4\pi^2}$$

$$\frac{\delta^2\Psi}{\delta \tilde{g}_{\mu\nu} \delta A_\alpha} =$$



$$\simeq i\tilde{Q}_\rho \underbrace{C}_{=\frac{\mu_R^2}{8\pi^2}+\frac{T^2}{24}}(\eta^{\nu 0}\varepsilon^{\rho\mu 0\alpha} + \delta_{ij}\eta^{\nu i}\epsilon^{\rho\mu j\alpha})$$

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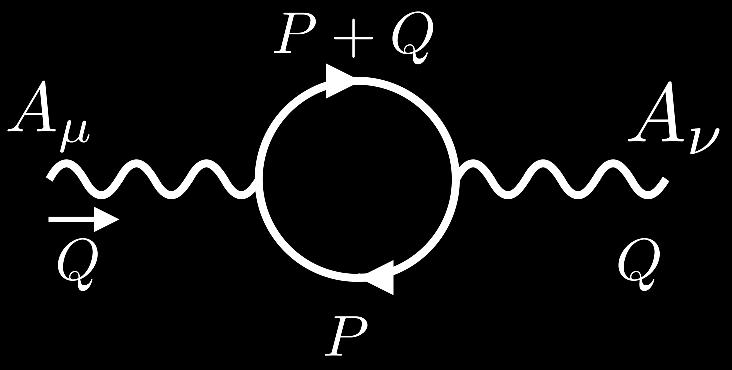

$$\simeq -i\varepsilon^{0\mu\rho\nu}\tilde{Q}_\rho \frac{\mu_R}{4\pi^2}$$

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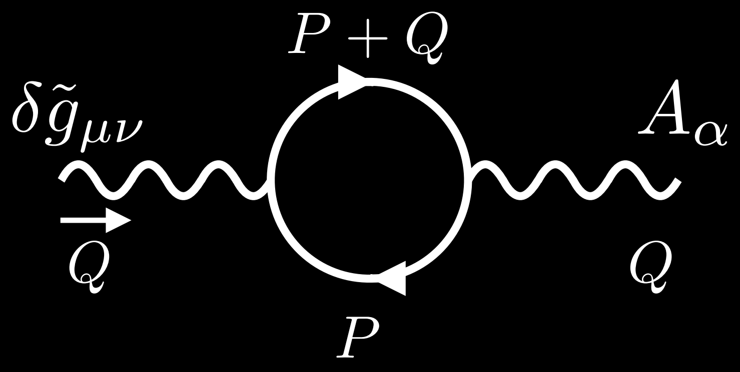

$$\simeq i\tilde{Q}_\rho \underbrace{C}_{=\frac{\mu_R^2}{8\pi^2}+\frac{T^2}{24}}(\eta^{\nu 0}\varepsilon^{\rho\mu 0\alpha} + \delta_{ij}\eta^{\nu i}\epsilon^{\rho\mu j\alpha})$$

$$\Psi^{(1)}[\lambda] = \int d^3x \varepsilon^{0ijk}\left[\frac{\nu_R}{8\pi^2}A_i\partial_j A_k + \left(\frac{\nu_R\mu_R}{8\pi^2}+\frac{T}{24}\right)A_i\partial_j\tilde{g}_{0k}\right]$$

# Derivation of CME/CVE

$$\Psi^{(1)}[\lambda] = \int d^3x \varepsilon^{0ijk} \left[ \frac{\nu_R}{8\pi^2} A_i \partial_j A_k + \left( \frac{\nu_R \mu_R}{8\pi^2} + \frac{T}{24} \right) A_i \partial_j \tilde{g}_{0k} \right]$$

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$$\Rightarrow \quad \langle \hat{J}^i_R(x) \rangle^{\mathrm{LG}}_{(0,1)} = \frac{1}{\sqrt{-g}} \frac{\delta \Psi^{(1)}}{\delta A_i(x)} = \frac{\mu_R}{4\pi^2} B^i + \left( \frac{\mu_R^2}{8\pi^2} + \frac{T^2}{24} \right) \omega^i$$

# Derivation of CME/CVE

$$\Psi^{(1)}[\lambda] = \int d^3x\varepsilon^{0ijk}\left[\frac{\nu_R}{8\pi^2}A_i\partial_jA_k + \left(\frac{\nu_R\mu_R}{8\pi^2}+\frac{T}{24}\right)A_i\partial_j\tilde{g}_{0k}\right]$$

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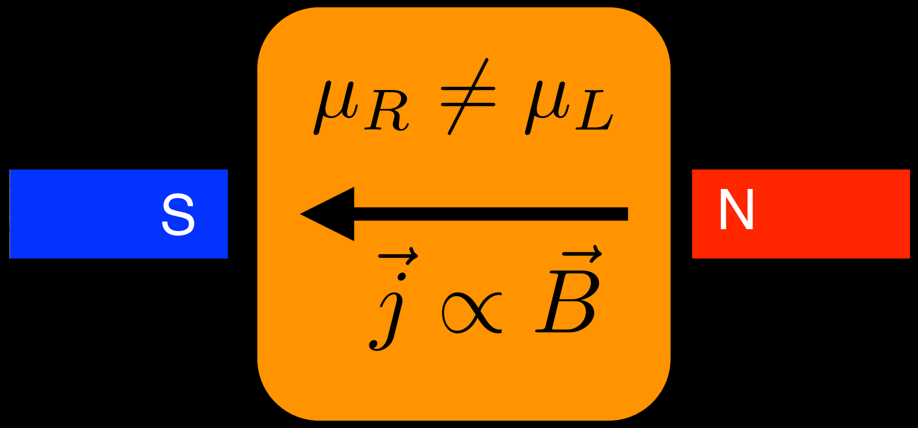
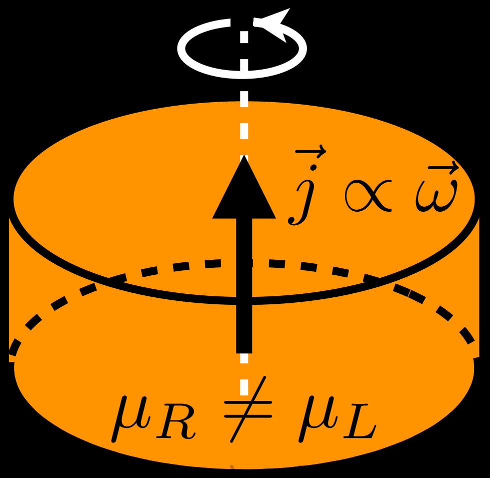
$$\boxed{\begin{aligned}\langle\hat{J}^i_V(x)\rangle^{\mathrm{LG}}_{(0,1)} &= \frac{\mu_5}{2\pi^2}B^i + \frac{\mu\mu_5}{2\pi^2}\omega^i\\ \langle\hat{J}^i_A(x)\rangle^{\mathrm{LG}}_{(0,1)} &= \frac{\mu}{2\pi^2}B^i + \left(\frac{\mu^2+\mu_5^2}{4\pi^2}+\frac{T^2}{12}\right)\omega^i\end{aligned}}$$

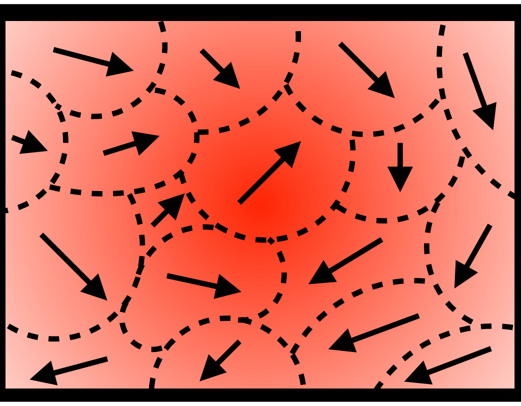


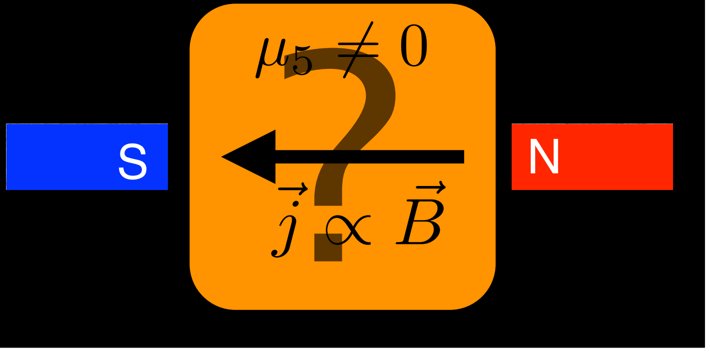


# Summary

**MOTIVATION:**

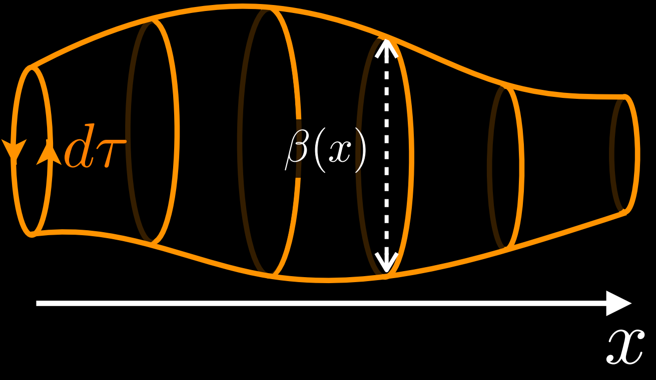
Quantum field theory under
local thermal equilibrium?





**APPROACH:**

QFT for Local Gibbs distribution



① Variation formula: $\langle \hat{T}^{\mu\nu}(x) \rangle^{\mathrm{LG}} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \Psi[\lambda]$

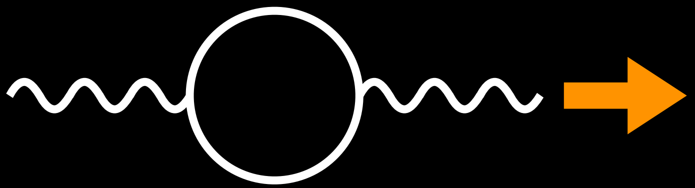
② $\Psi[\lambda]$ is written in terms of QFT in “curved spacetime”

$$d\tilde{s}^2 = -e^{2\sigma}(d\tilde{t} + a_{\bar{i}} dx^{\bar{i}})^2 + \gamma'_{\bar{i}\bar{j}} dx^{\bar{i}} dx^{\bar{j}}$$

Symmetry = Spatial diffeomorphism + Kaluza-Klein gauge

**APPLICATION:**

Derivation of
Anomalous hydrodynamics



$\Psi^{(1)} \to \vec{j} = \frac{e\mu_5}{2\pi^2} \vec{B}$

# Outlook

## DISSIPATION AND FLUCTUATION:

How to implement dissipation and fluctuation based on QFT?

- Zubarev et al. (1979)
- Becattini et al. (2015)
- Hayata, Hidaka, **MH,** Noumi (2015)

- Haehl, Loganayagam, Rangamani (2015-)
- Harder, Kovtun, Ritz (2015)
- Crossley, Giorioso, Liu (2015-)
- Jensen et al. (2017-)

## NON-DISSIPATIVE TRANSPORT:

Evaluation of Masseiu-Planck fcn. in several situations

s.t. in the presence of magnetic field/vorticity …

**- Hattori, Yin(2016)**

**- Becattinil et al. (2015)**

## SUPERFLUID / MAGNETO-HYDRODYNAMICS:

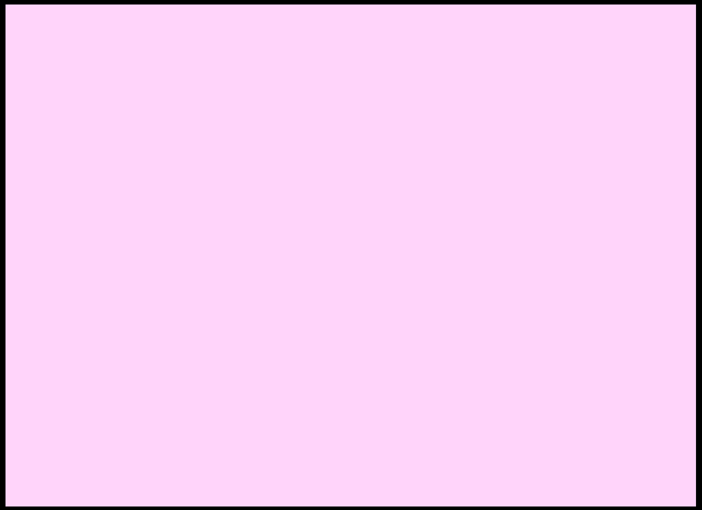
Extension to cases with other zero modes

s.t. Nambu-Goldstone-mode, Photon, Topological defect

# Backup

# What is Local Gibbs distribution?

## Gibbs distribution



What is the state with maximizing information entropy: $S(\hat{\rho}) = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho}$

under constraints:

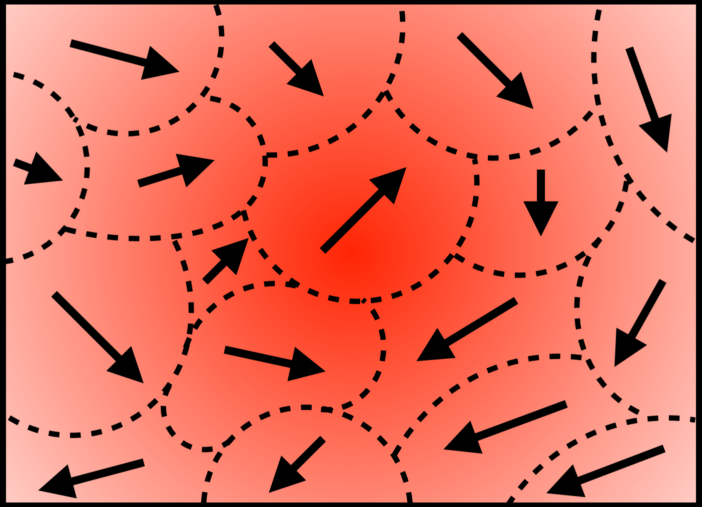
$$\langle \hat{H} \rangle = E = \text{const.},\ \ \langle \hat{N} \rangle = N = \text{const.}$$

**Answer:**

$$\hat{\rho}_{\mathrm{G}} = e^{-\beta\hat{H} - \nu\hat{N} - \Psi[\beta,\nu]}$$

Lagrange multipliers: $\Lambda^a = \{\beta, \nu = \beta\mu\}$

## Local Gibbs distribution



What is the state with maximizing information entropy: $S(\hat{\rho}) = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho}$

under constraints:

$$\langle \hat{T}^0{}_\mu(x) \rangle = p_\mu(x),\ \ \langle \hat{J}^0(x) \rangle = n(x)$$

**Answer:**

$$\hat{\rho}_{\mathrm{LG}} = e^{-\int d^{d-1}x(\beta^\mu \hat{T}^0{}_\mu + \nu\hat{J}^0) - \Psi[\beta^\mu,\nu]}$$

Lagrange multipliers: $\lambda^a(x) = \{\beta^\mu(x), \nu(x)\}$