# Universal Lie Algebra, Vogel Parameters and Color Factors in Nonabelian Gauge Theories

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*Секция Международной конференции «Математика в созвездии наук», приуроченная к юбилею академика В.А. Садовничего*
*Лаборатория информационных технологий им. М.Г. Мещерякова, ОИЯИ,*
*2 April 2024.*

Norwegian mathematician Sophus Lie (17.12.1842 – 18.02.1899) was born in Christiania (Oslo).
German mathematician Felix Klein (25.04.1849 – 22.06.1925)
French mathematician Élie Joseph Cartan (09.04.1869 – 06.05.1951)

German mathematician Hermann Klaus Hugo Weyl (09.11.1885 – 08.12.1955)

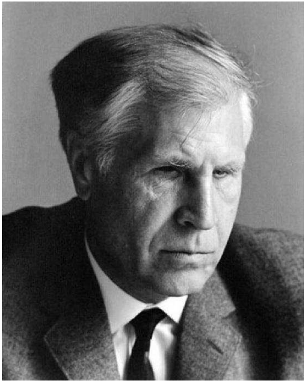
Sophus Lie

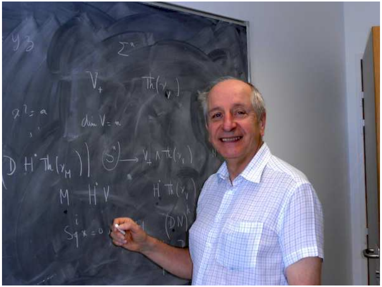
Felix Klein

Hermann Weyl

Élie Cartan

Lev Semenovich Pontryagin (03.09.1908 – 03.05.1988)
"Topological Groups", Princeton University Press (1939)

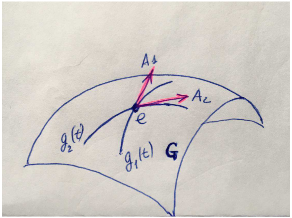




Pierre Vogel (Université Paris VII)
"The universal Lie algebra", Preprint (1999)

A Lie group $G$ is a smooth manifold. Consider the tangent vector space $T_e(G)$ to the Lie group $G$ at the unit element $e \in G$.

**Definition.** *The tangent vector space $T_e(G)$, equipped with the multiplication $[A_1,\ A_2] \in T_e(G)$ $(\forall A_1, A_2 \in T_e(G))$ with axioms:*
*1) Anticommutativity: $[A_1,\ A_2] = -[A_2,\ A_1]$,*
*2) Jacoby identity: $[[A_1,\ A_2],\ A_3] + [[A_3,\ A_1],\ A_2] + [[A_2,\ A_3],\ A_1] = 0$,*
*is called the Lie algebra $\mathfrak{g}$ of the Lie group $G$.*




Let $X_a \,|_{a=1,\dots,\dim \mathfrak{g}} \in T_e(G)$ be basis elements of Lie algebra (LA) $\mathfrak{g}$:

$$[X_a,\ X_b] = C^d_{ab}\, X_d \ , \tag{1}$$

$C^d_{ab}$ – are structure constants. Matrices $\mathrm{ad}(X_a)^d_b = C^d_{ab}$ define **the adjoint representation** of $\mathfrak{g}$. The **invariant Cartan-Killing metric** in $T_e(G)$ is

$$\mathrm{g}_{ab} \equiv \mathrm{Tr}(\mathrm{ad}(X_a) \cdot \mathrm{ad}(X_b)) = C^d_{ac}\, C^c_{bd} \ . \tag{2}$$

The C-K metric $g_{ab}$ defines invariant scalar product for $A, B \in T_e(G)$:

$$(A,\ B) = (A^a X_a,\ B^b X_b) = A^a\, g_{ab}\, B^b\ .$$

For **simple Lie algebras**, the metric $g_{ab}$ is invertible:

$$g_{ab}\, g^{bc} = \delta^c_a\ ,$$

and unique up to a normalization factor: $g_{ab} \to \lambda g_{ab}$.
For compact Lie algebras $\mathfrak{g}$, one can chose the basis: $g_{ab} = -\delta_{ab}$.

The **classification of simple Lie algebras** (E.Cartan-H.Weyl):
4 infinite series (accidental isomorphisms are not taken into account):

$$\mathbf{1}.\, A_n :\ \mathfrak{s}\ell(n+1);\quad \mathbf{2}.\, B_n :\ \mathfrak{so}(2n+1);\quad \mathbf{3}.\, C_n :\ \mathfrak{sp}(2n);\quad \mathbf{4}.\, D_n :\ \mathfrak{so}(2n);$$

$$\dim \mathfrak{s}\ell(N) = N^2 - 1\,,\qquad \dim \mathfrak{so}(N) = \frac{N(N-1)}{2}\,,\qquad \dim \mathfrak{sp}(N) = \frac{N(N+1)}{2}\,,$$

and 5 exceptional LA: $\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$ with dims: 14, 52, 78, 133, 248.

The main object is **split (or polarized) Casimir operator** of LA $\mathfrak{g}$ is

$$\widehat{C} = \mathrm{g}^{ab} X_b \otimes X_a \equiv X^a \otimes X_a \in \mathfrak{g} \otimes \mathfrak{g} \,. \quad (3)$$

The operator $\widehat{C}$ is independent of the choice of the basis $X_a$ in $\mathfrak{g}$ and is related to the standard **quadratic Casimir operator** (central element in the enveloping algebra $\mathcal{U}(\mathfrak{g})$)

$$C^{(2)} = \mathrm{g}^{ab} X_b \cdot X_a \in \mathcal{U}(\mathfrak{g}) \,. \quad (4)$$

Relation is via comultiplication $\Delta(X_a) = (X_a \otimes I + I \otimes X_a)$:

$$\Delta(C^{(2)}) = \Delta(X^a) \cdot \Delta(X_a) = C^{(2)} \otimes I + I \otimes C^{(2)} + 2\,\widehat{C} \,. \quad (5)$$

**Physics analogy** $\mathfrak{su}(2)$: $X_a$ are components of the spin-vector, operator $C^{(2)}$ is the squire of the spin-vector, $\Delta$ is s rule for addition of spins, ad for $\mathfrak{su}(2)$ – representation numerated by spin j=1 .

**Remark**. The split Casimir operator $\widehat{C}$ commutes with the action of $\mathfrak{g}$:

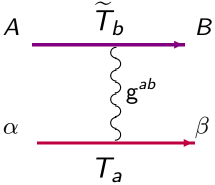
$$[\Delta(A), \widehat{C}] = [(A \otimes I + I \otimes A), \widehat{C}] = 0 \,, \quad \forall A \in \mathfrak{g} \,.$$

Let $T$ and $\widetilde{T}$ be two representations of $\mathfrak{g}$. One can visualize split Casimir operator in the representation $T \otimes \widetilde{T}$:

$$((T \otimes \widetilde{T})\,\widehat{C})^{\alpha A}_{\beta B} \;=\; \mathrm{g}^{ab}\; T^{\alpha}_{\beta}(X_a)\;\widetilde{T}^{A}_{B}(X_b) \;\equiv\; \mathrm{g}^{ab}\,(T_a)^{\alpha}_{\beta}\,(\widetilde{T}_b)^{A}_{B}\,, \qquad (6)$$

where $\alpha, \beta = 1, ..., \dim T$ and $A, B = 1, ..., \dim \widetilde{T}$:

$$(T_a)^{\alpha}_{\beta}\;\; \mathrm{g}^{ab}\;\; (\widetilde{T}_b)^{A}_{B} \qquad =$$




Colour factor for the Feynman diagram describing scattering of two particles in the representations $T$ and $\widetilde{T}$ by gauge field $A \in \mathfrak{g}$.

Split Casimir operator $\widehat{C}$ appears in many applications: in the representation theory, in the theory of integrable systems, as colour factors in the nonabelian gauge theories, ...

## Universal Lie algebra.

**1.** Consider tensor products of $r$ adjoint representations of the simple LA $\mathfrak{g}$ and consider Clebsch-Gordan expansion of these products:

$$\mathrm{ad}^{\otimes r} := \underbrace{\mathrm{ad} \otimes \mathrm{ad} \otimes \cdots \otimes \mathrm{ad}}_{r} = \oplus_\lambda \, n_\lambda \, T_\lambda \, , \qquad (7)$$

where $T_\lambda$ are irreps, $\lambda$ – parameters which numerate irreps (e.g. highest weights) and $n_\lambda \in \mathbb{Z}_{>0}$ are multiplicities.

**2.** The elements of the vector space of rep $\mathrm{ad}^{\otimes r}$ are rank $r$ tensors $t^{a_1 a_2 \dots a_r}$. Invariant subspaces in $V_{\mathrm{ad}}^{\otimes r}$ are spaces of $t^{a_1 a_2 \dots a_r}$ with special symmetrization of indices $(a_1, a_2, \dots a_r)$ (according to Young diagrams $\vdash r$): $t_{\pm}^{a_1 a_2} = \frac{1}{2}(t^{a_1 a_2} \pm t^{a_2 a_1})$.

**3. Amazing fact**: it was noticed [P.Deligne (1996), P.Vogel (1999), J.M.Landsberg and L.Manivel (2002),...] that, for first $r = 2, 3, \dots$, formula (7) is universal for all simple Lie algebras $\mathfrak{g}$. Moreover, there are **remarkable universal formulas** for all $\dim(T_\lambda)$.

**4.** Formulas for $\dim(T_\lambda)$ are written as **rational and homogeneous symmetric functions of 3 parameters** $(\alpha, \beta, \gamma)$, which parameterize all simple Lie algebras $\mathfrak{g}$ and called **Vogel parameters**.

First we give famous P.Deligne formula (appeared in the example $r=2$)

$$\dim \mathfrak{g} \equiv \dim(\mathrm{ad}) = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma} =$$

$$= \frac{(\hat{\alpha}-1)(\hat{\beta}-1)(\hat{\gamma}-1)}{\hat{\alpha}\hat{\beta}\hat{\gamma}}, \quad \hat{\alpha} := \frac{\alpha}{2t},\ \hat{\beta} := \frac{\beta}{2t},\ \hat{\gamma} := \frac{\gamma}{2t}, \qquad \boxed{t := \alpha+\beta+\gamma}.$$

Since all $\dim(T_\lambda)$ are homogeneous symmetric functions of Vogel parameters $(\alpha,\beta,\gamma)$, it is possible to fix one of them, e.g. $\alpha=-2$. For this choice the sum $t := \alpha+\beta+\gamma$ coincides with dual Coxeter number $h^\vee$.

Table 1

| Type | Lie algebra | $\alpha$ | $\beta$ | $\gamma$ | $t = h^\vee = \alpha+\beta+\gamma$ | $\hat{\gamma} = \frac{\gamma}{2t}$ |
|---|---|---|---|---|---|---|
| $A_n$ | $s\ell(n+1)$ | $-2$ | $2$ | $n+1$ | $n+1$ | $1/2$ |
| $B_n$ | $so(2n+1)$ | $-2$ | $4$ | $2n-3$ | $2n-1$ | $\frac{2n-3}{2(2n-1)}$ |
| $C_n$ | $sp(2n)$ | $-2$ | $1$ | $n+2$ | $n+1$ | $\frac{n+2}{2(n+1)}$ |
| $D_n$ | $so(2n)$ | $-2$ | $4$ | $2n-4$ | $2n-2$ | $\frac{n-2}{2(n-1)}$ |
| $G_2$ | $\mathfrak{g}_2$ | $-2$ | $10/3$ | $8/3$ | $4$ | $1/3$ |
| $F_4$ | $\mathfrak{f}_4$ | $-2$ | $5$ | $6$ | $9$ | $1/3$ |
| $E_6$ | $\mathfrak{e}_6$ | $-2$ | $6$ | $8$ | $12$ | $1/3$ |
| $E_7$ | $\mathfrak{e}_7$ | $-2$ | $8$ | $12$ | $18$ | $1/3$ |
| $E_8$ | $\mathfrak{e}_8$ | $-2$ | $12$ | $20$ | $30$ | $1/3$ |

Note that, for all exceptional Lie algebras we have $2t = 3\gamma \ \to\ \hat{\gamma} = 1/3$.

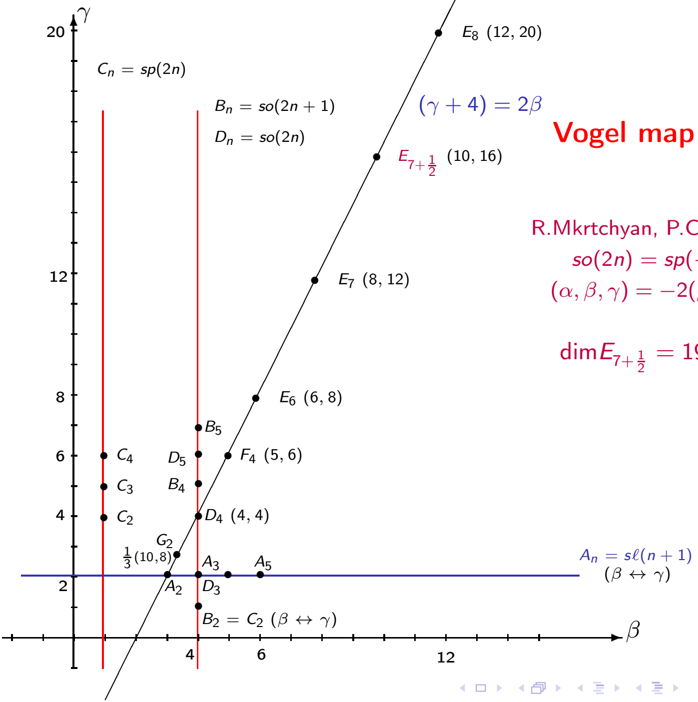
In view that all $\dim(T_\lambda)$ are homogeneous symmetric functions of the Vogel parameters, one can consider all simple Lie algebras as points on the 2d plane $\mathcal{P}_{(\alpha=-2)}$ in 3d space of the Vogel parameters $(\alpha, \beta, \gamma)$. More precisely they are points in $\mathbb{RP}^2/S_3$ (the Vogel map).

Before we represent the Vogel map, we note that condition $2t = 3\gamma$, for exceptional LAs, defines the line $(\gamma + 4) = 2\beta$ on the plane $\mathcal{P}_{(\alpha=-2)} \in \mathbb{R}^3$. Remarkable fact: points of Lie algebras $s\ell(3)$ and $so(8)$ are also on this line.

| Type | Lie algebra | $\alpha$ | $\beta$ | $\gamma$ | $t = h^\vee = \alpha + \beta + \gamma$ | $\hat{\gamma} = \frac{\gamma}{2t}$ |
|---|---|---|---|---|---|---|
| $A_2$ | $s\ell(3)$ | $-2$ | 2 | 3 | 3 | $\frac{\beta}{2t} = 1/3$ |
| $D_4$ | $so(8)$ | $-2$ | 4 | 4 | 6 | $1/3$ |
| $G_2$ | $\mathfrak{g}_2$ | $-2$ | $10/3$ | $8/3$ | 4 | $1/3$ |
| $F_4$ | $\mathfrak{f}_4$ | $-2$ | 5 | 6 | 9 | $1/3$ |
| $E_6$ | $\mathfrak{e}_6$ | $-2$ | 6 | 8 | 12 | $1/3$ |
| $E_7$ | $\mathfrak{e}_7$ | $-2$ | 8 | 12 | 18 | $1/3$ |
| $E_8$ | $\mathfrak{e}_8$ | $-2$ | 12 | 20 | 30 | $1/3$ |

Unified description of all simple LA by means of 3 parameters $(\alpha, \beta, \gamma)$ leads to the conjecture of existing the universal LA.

# Vogel map (1999)




R.Mkrtchyan, P.Cvitanović

$so(2n) = sp(-2n)$

$(\alpha, \beta, \gamma) = -2(\beta, \alpha, \gamma)$

$\dim E_{7+\frac{1}{2}} = 190$

**Example:** $\mathrm{ad}^{\otimes 2}$, $(r=2)$. For all simple LAs (with rank $>1$) we have decomposition

$$\mathrm{ad}^{\otimes 2} = \mathbb{A}(\mathrm{ad}^{\otimes 2}) + \mathbb{S}(\mathrm{ad}^{\otimes 2}) \; = \; (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta) + \mathsf{Y}(\gamma)\big)\;.$$

Dim. formulas for reps are homogeneous rational functions in $(\alpha, \beta, \gamma)$ (symmetry in $(\alpha, \beta, \gamma)$ is permutation of $\mathsf{Y}(\alpha), \mathsf{Y}(\beta), \mathsf{Y}(\gamma)$) [Vogel(1999)]:

$$\dim(\mathrm{ad}) \equiv \dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}\,, \quad \boxed{t := \alpha + \beta + \gamma}\,, \tag{8}$$

$$\dim(\mathsf{X}_2) = \tfrac{1}{2}\dim\mathfrak{g}\,(\dim\mathfrak{g} - 3)\;, \;\; 20|_{sl(3)},\; 350|_{so(8)}\;, \qquad \dim(\mathbf{1}) = 1\,,$$

$$\begin{aligned} \dim \mathsf{Y}(\alpha) &= \tfrac{(2t-3\alpha)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)(\alpha-\gamma)\beta\gamma}\,, \quad 27|_{sl(3)} \sim [4,2],\; 300|_{so(8)} \\ \dim \mathsf{Y}(\beta) &= \dim \mathsf{Y}(\alpha)|_{\alpha\leftrightarrow\beta}\;, \quad 0|_{sl(3)},\; \tfrac{0}{0} = 35, 35', 35''|_{so(8)} \\ \dim \mathsf{Y}(\gamma) &= \dim \mathsf{Y}(\alpha)|_{\alpha\leftrightarrow\gamma}\;, \quad 8|_{sl(3)},\; \tfrac{0}{0} = 0|_{so(8)} \end{aligned} \tag{9}$$

In the rhs we give dims for two "exceptional" algebras $\mathfrak{sl}_3, \mathfrak{so}_8$.

**Remark.** For the exceptional line $2t = 3\gamma$, we have $\dim \mathsf{Y}(\gamma) = 0$. It means that, for axceptional LA, in the decomposition of $\mathrm{ad}^{\otimes 2}$, the representation $\mathsf{Y}(\gamma)$ is missing:

$$\mathrm{ad}^{\otimes 2} = \mathbb{A}(\mathrm{ad}^{\otimes 2}) + \mathbb{S}(\mathrm{ad}^{\otimes 2}) = (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta)\big)\;. \tag{10}$$

Some achievements in the universal description of simple LA & LG.

**1.)** The generating function of universal eigenvalues $C^{(k)}_{\rm ad}$ of the higher Casimir operators in the ad-representation of $\mathfrak{g}$ [R.Mkrtchyan, A.Sergeev and A.Veselov (2012)]

$$\hat{C}(z) \; = \; \sum_{k=0}^{\infty} C^{(k)}_{\rm ad} z^k = \frac{1}{\dim(\mathfrak{g})}\left(\frac{1}{1+z} + \frac{\dim Y(\alpha)}{1+\frac{z\alpha}{2t}} + \frac{\dim Y(\beta)}{1+\frac{z\beta}{2t}} + \frac{\dim Y(\gamma)}{1+\frac{z\gamma}{2t}}\right) + \\ +\frac{1}{2}\dim(\mathfrak{g}) + \frac{1}{1+\frac{z}{2}} - \frac{3}{2}\;.$$

**2.)** Formula for volumes of compact simple Lie groups $G$ [R.Mkrtchyan, A.Veselov]

$$\mathrm{Vol}(G) = (2^{3/2}\pi)^{\dim \mathfrak{g}} \; e^{-\Phi(\alpha,\beta,\gamma)}\;,$$

where $\Phi(\alpha,\beta,\gamma) = \int_0^\infty dz \frac{F(z/t)}{z(e^z-1)}$ and

$$F(z) = \frac{\mathrm{sh}\frac{z}{4}(\alpha-2t)\,\mathrm{sh}\frac{z}{4}(\beta-2t)\,\mathrm{sh}\frac{z}{4}(\gamma-2t)}{\mathrm{sh}\frac{z}{4}\alpha\;\mathrm{sh}\frac{z}{4}\beta\;\mathrm{sh}\frac{z}{4}\gamma} - \dim\mathfrak{g}\;. \qquad (11)$$

Here the first term in rhs is the deformation of the universal formula (8) for $\dim\mathfrak{g} = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}$ (it is clear that $F(z)|_{z=0} = 0$).

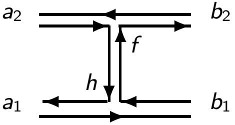
Our method of universal description of the LA is based on the extracting of invariant subspaces in $V_{\rm ad}^{\otimes r} = {\rm ad}^{\otimes r}$ by means of the investigations of the characteristic identities for $r$-split CO $\widehat{C}_{\rm ad}$.

**The split Casimir operators and universality in ad$^{\otimes 2}$:**

$$(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} \equiv ({\rm ad} \otimes {\rm ad})^{a_1 a_2}_{b_1 b_2}(X_h \otimes X^h) = (X_h)^{a_1}_{b_1}\,(X^h)^{a_2}_{b_2} = C^{a_1}_{hb_1}\,C^{a_2}_{fb_2}\,{\rm g}^{hf},$$

acts in the space $V_{\rm ad}^{\otimes 2}$ and $V_{\rm ad} \simeq {\rm ad} \simeq \mathfrak{g}$ is the space of ad-representation. Since ad-representation embedded in $T \otimes (T^{\sf T})^{-1}$, one can consider adj. indices $a, b, c, \ldots$ as pairs of fundamental and antifundamental indices $a = (i, \bar{j}),\; b = (k, \bar{\ell}), \ldots$. In view of this, matrices $(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2}$ can be represented as Feynman "colour" diagrams (oriented and not oriented lines correspond to $\mathfrak{sl}_N$ and $\mathfrak{so}_N$, $\mathfrak{sp}_{2n}$ cases)

$$(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} = C^{a_1}_{hb_1}\,C^{a_2}_{fb_2}\,{\rm g}^{hf} =$$



Let split CO $\widehat{C}_{\rm ad}$ satisfies char. identity $\prod_{i=1}^{k}(\widehat{C}_{\rm ad} - a_i) = 0$. Then we find $k$ projectors in $\mathsf{ad}^{\otimes 2}$ onto invariant subspaces of $\widehat{C}_{\rm ad}$ with eigenvalues $a_j$:

$$\mathrm{P}_{(a_j)} = \prod_{i \neq j} \frac{(\widehat{C}_{\rm ad} - a_i)}{a_j - a_i} \qquad \Rightarrow \qquad \mathrm{ad}^{\otimes 2} = \sum_j \mathrm{P}_{(a_j)} \cdot \left(\mathrm{ad}^{\otimes 2}\right) .$$

The invariant subspaces $\mathrm{P}_{(a_j)} \cdot \left(\mathrm{ad}^{\otimes 2}\right)$ are called Casimir subspaces.

Define invariant operators in $V_{\rm ad}^{\otimes 2}$ by their action on the basis $(X_{a_1} \otimes X_{a_2})$:

$$\mathbf{I}(X_{a_1} \otimes X_{a_2}) = (X_{a_1} \otimes X_{a_2}) \; , \quad \mathbf{P}(X_{a_1} \otimes X_{a_2}) = (X_{a_2} \otimes X_{a_1}) \; ,$$

and construct projectors $\mathbf{P}_{+}^{(ad)} = \frac{1}{2}(\mathbf{I} + \mathbf{P})$ and $\mathbf{P}_{-}^{(ad)} = \frac{1}{2}(\mathbf{I} - \mathbf{P})$.
Introduce symmetrized and antisymmetrized parts of $\widehat{C}_{\rm ad}$

$$\widehat{C}_{\pm} = \mathbf{P}_{\pm}^{(ad)} \; \widehat{C}_{\rm ad} \; , \qquad (\widehat{C}_{\pm})^{a_1 a_2}_{b_1 b_2} = \frac{1}{2}((\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} \pm (\widehat{C}_{\rm ad})^{a_2 a_1}_{b_1 b_2}) \; ,$$

where $\widehat{C}_{+}$ – symmetric and $\widehat{C}_{-}$ – antisymmetric parts of $\widehat{C}_{\rm ad}$ in $(V_{\rm ad})^{\otimes 2} \simeq {\rm ad}^{\otimes 2}$.

**Proposition 1.** *For all simple LA $\mathfrak{g}$ the SCO $\widehat{C}_-$ satisfy char. identity*

$$\boxed{\widehat{C}_-(\widehat{C}_- + \frac{1}{2}) = 0} \Leftrightarrow \boxed{\widehat{C}_-^2 = -\frac{1}{2}\widehat{C}_-} \, , \qquad (12)$$

Since identity (12) is quadratic, we have two projectors $P_{(0)}, P_{(-\frac{1}{2})}$ on two subrepresentations $X_1, X_2$ in $\mathbb{A}(\mathrm{ad} \otimes \mathrm{ad}) \equiv \mathbf{P}_-^{(ad)}(\mathrm{ad} \otimes \mathrm{ad})$

$$\boxed{\mathbb{A}(\mathrm{ad} \otimes \mathrm{ad}) = P_{(0)}(\mathrm{ad} \otimes \mathrm{ad}) + P_{(-\frac{1}{2})}(\mathrm{ad} \otimes \mathrm{ad}) = X_1 + X_2 = \mathrm{ad} + X_2},$$

where $\dim X_1 = \mathbf{Tr}(P_{(0)}) = \dim \mathfrak{g}$,
$\dim X_2 = \mathbf{Tr}(P_{(-\frac{1}{2})}) = \frac{\dim \mathfrak{g}}{2}(\dim \mathfrak{g} - 3)$.

**Proposition 2.** *For all LA of the classical series $A_n = \mathfrak{sl}_{n+1}$, $B_n = \mathfrak{so}_{2n+1}$, $C_n = \mathfrak{sp}_{2n}$, $D_n = \mathfrak{so}_{2n}$ (except $\mathfrak{sl}_3$ and $\mathfrak{so}_8$), in ad-representation, $\widehat{C}_+$ has the universal char identity*

$$\boxed{(\widehat{C}_+ + 1)(\widehat{C}_+^3 + \frac{1}{2}\widehat{C}_+^2 - \mu_1 \widehat{C}_+ - 2\mu_2)\mathbf{P}_+^{(\mathrm{ad})} = 0}\,, \quad \#4\,, \tag{13}$$

*where*

$$\begin{cases} \text{for} \quad \mathfrak{sl}_N : & \mu_1 = \frac{1}{N^2}\,, \qquad \mu_2 = \frac{1}{4N^2}\,; \\ \text{for} \quad \mathfrak{so}_N, \mathfrak{sp}_N : & \mu_1 = \frac{8-\epsilon N}{2(\epsilon N-2)^2}\,, \quad \mu_2 = \frac{\epsilon N-4}{2(\epsilon N-2)^3}\,; \end{cases} \tag{14}$$

$\epsilon = 1, -1$ for $\mathfrak{so}_N, \mathfrak{sp}_N$ and $2(\dim \mathfrak{g} - 1)\mu_2 + \mu_1 = 1/2$. The factorized form of (13) is

$$\boxed{(\widehat{C}_+ + 1)(\widehat{C}_+ + \frac{\alpha}{2t})(\widehat{C}_+ + \frac{\beta}{2t})(\widehat{C}_+ + \frac{\gamma}{2t})\mathbf{P}_+^{(\mathrm{ad})} = 0}\,, \tag{15}$$

where $(\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t}) = 1/2 \quad \Rightarrow \quad (t = \alpha + \beta + \gamma)$,

$$\mu_1 = -\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{4t^2}\,, \quad \mu_2 = -\frac{\alpha\beta\gamma}{16t^3}\,, \tag{16}$$

If we fix $\alpha = -2$, then from (14),(16) we deduce the values of the Vogel parameters $\alpha, \beta, \gamma$ for $s\ell(N), so(N), sp(N)$ in Table 1.

And find the expression for $\dim \mathfrak{g}$ in terms the parameters $\mu_1, \mu_2$ and $\alpha, \beta, \gamma$:

$$\dim \mathfrak{g} = \frac{2\mu_2 - \mu_1 + 1/2}{2\mu_2} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} \, .$$

From char. identity (15), we deduce four universal projectors $\mathrm{P}^{(+)}_{(a_i)}$ on the invariant subspaces $V_{(a_i)}$ (with eigenvalues $a_i$ of $\widehat{C}_+$)

$$\mathbf{P}^{(\mathrm{ad})}_+ (V_{\mathrm{ad}}^{\otimes 2}) = (\mathrm{P}^{(+)}_{(-1)} + \mathrm{P}^{(+)}_{(-\frac{\alpha}{2t})} + \mathrm{P}^{(+)}_{(-\frac{\beta}{2t})} + \mathrm{P}^{(+)}_{(-\frac{\gamma}{2t})}) V_{\mathrm{ad}}^{\otimes 2} = \\ = V_{(-1)} + V_{(-\frac{\alpha}{2t})} + V_{(-\frac{\beta}{2t})} + V_{(-\frac{\gamma}{2t})} \, .$$

$$\mathrm{P}^{(+)}_{(-\frac{\alpha}{2t})} = \mathrm{P}^{(+)}(\alpha|\beta,\gamma) \, , \quad \mathrm{P}^{(+)}_{(-\frac{\beta}{2t})} = \mathrm{P}^{(+)}(\beta|\alpha,\gamma) \, , \quad \mathrm{P}^{(+)}_{(-\frac{\gamma}{2t})} = \mathrm{P}^{(+)}(\gamma|\alpha,\beta) \, .$$

The representations of $\mathfrak{g}$ in the subspaces $V_{(-1)}$, $V_{(-\frac{\alpha}{2t})}$, $V_{(-\frac{\beta}{2t})}$, $V_{(-\frac{\gamma}{2t})}$ were respectively denoted by Vogel as $\mathrm{X}_0 = \mathbf{1}$, $Y_2(\alpha)$, $Y_2(\beta)$, $Y_2(\gamma)$

$$\boxed{\mathbb{S}\,(\mathrm{ad}^{\otimes 2}) = \mathrm{X}_0 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)} \, .$$

Thus, Prop. 1,2 justify the Vogel statement for LA of classical series.
**Theorem.** (P.Vogel, 1999)

$$\mathrm{ad}^{\otimes 2} = \mathbb{A}(\mathrm{ad}^{\otimes 2}) + \mathbb{S}(\mathrm{ad}^{\otimes 2}) \;=\; (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta) + \mathsf{Y}(\gamma)\big).$$

Finally, we calculate (by means of trace formulas) the Vogel universal expressions for the dim of the invariant eigenspaces $V_{(a_i)}$:

$$\begin{aligned}
&\dim V_{(-1)} = \mathbf{Tr}\, \mathsf{P}^{(+)}_{(-1)} = 1\,,\\
&\dim Y_2(\alpha) \equiv \dim V_{(-\frac{\alpha}{2t})} = \mathbf{Tr}\, \mathsf{P}^{(+)}_{(-\frac{\alpha}{2t})} = -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)\beta(\alpha-\gamma)\gamma}\,,\\
&\dim Y_2(\beta) \equiv \dim V_{(-\frac{\beta}{2t})} = \mathbf{Tr}\, \mathsf{P}^{(+)}_{(-\frac{\beta}{2t})} = -\frac{(3\beta-2t)(\alpha-2t)(\gamma-2t)t(\alpha+t)(\gamma+t)}{\beta^2(\beta-\alpha)\alpha(\beta-\gamma)\gamma}\,,\\
&\dim Y_2(\gamma) \equiv \dim V_{(-\frac{\gamma}{2t})} = \mathbf{Tr}\, \mathsf{P}^{(+)}_{(-\frac{\gamma}{2t})} = -\frac{(3\gamma-2t)(\beta-2t)(\alpha-2t)t(\beta+t)(\alpha+t)}{\gamma^2(\gamma-\beta)\beta(\gamma-\alpha)\alpha}\,.
\end{aligned}$$

Remark. The cases of algebras $\mathfrak{sl}_3$ and $\mathfrak{so}_8$ are exceptional.

## Universal char identities for $\widehat{C}$ for exceptional Lie algebras in $\mathrm{ad}^{\otimes 2}$.

The antisymmetric $\widehat{C}_-$ and symmetric $\widehat{C}_+$ parts of the split Casimir operators in the ad-representation for all exceptional Lie algebras $\mathfrak{g}$ obey the universal identities

$$\boxed{\widehat{C}_-\left(\widehat{C}_- + \frac{1}{2}\right) = 0}, \quad \boxed{(\widehat{C}_+ + 1)\big(\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu\big)\ \mathbf{P}_+^{(\mathrm{ad})} = 0}, \quad \#3\,, \qquad (17)$$

where the universal parameter $\mu$ is fixed as follows

$$\mu = \frac{5}{6(2+\dim(\mathfrak{g}))}\,, \qquad (18)$$

and for algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ with dimensions $14, 52, 78, 133, 248$ we have respectively $\mu = \frac{5}{96}, \frac{5}{324}, \frac{1}{96}, \frac{1}{162}, \frac{1}{300}$.

Moreover the char identities for $\widehat{C}_+$ for algebras $\mathfrak{s}\ell_3$ and $\mathfrak{so}_8$ have the same structure (17) with $\mu = \frac{1}{12}$ and $\mu = \frac{1}{36}$.

From (17) we obtain the factorized form of the universal char. identity for $\widehat{C}_+$

$$(\widehat{C}_+ + 1)(\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu)\mathbf{P}_+^{(\mathrm{ad})} \equiv (\widehat{C}_+ + 1)(\widehat{C}_+ + \frac{\alpha}{2t})(\widehat{C}_+ + \frac{\beta}{2t})\mathbf{P}_+^{(\mathrm{ad})} = 0 \, , \quad (19)$$

where we introduced the notation for two roots of eq. $\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu = 0$:

$$\frac{\alpha}{2t} = \frac{1-\mu'}{12} \, , \quad \frac{\beta}{2t} = \frac{1+\mu'}{12} \, , \qquad \mu' := \sqrt{1+288\mu} = \sqrt{\frac{\dim \mathfrak{g} + 242}{\dim \mathfrak{g} + 2}} \, . \quad (20)$$

These roots are related by $3(\alpha+\beta) = t$, and for $\alpha = -2$ this relation defines the line of the exceptional LA on the Vogel $(\beta, \gamma)$ plane (as we discussed above). We note that $\mu'$ is a rational number (since $\frac{\alpha}{2t}$ and $\frac{\beta}{2t}$ are rational) only for certain finite sequence of $\dim \mathfrak{g}$:

$$\dim \mathfrak{g} = 3, 8, 14, 28, 47, 52, 78, 96, 119, 133, 190, 248, 287, 336, \\ 484, 603, 782, 1081, 1680, 3479 \, , \quad (21)$$

which includes the dimensions $14, 52, 78, 133, 248$ of the exceptional Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, and the dimensions $8$ and $28$ of $s\ell(3)$ and $so(8)$, which are sometimes also referred to as exceptional.

**Remark.** The sequence (21) contains $\dim \mathfrak{g}^* = (10m - 122 + 360/m)$, $(m \in \mathbb{N})$ referring to the adjoint representations of the so-called $E_8$ family of algebras $\mathfrak{g}^*$; see the Cvitanović book. For such dimensions we have relation $\mu' = |(m+6)/(m-6)|$. Two numbers $47$ and $119$ from sequence (21) do not belong to the sequence $\dim \mathfrak{g}^*$. Thus, the interpretation of these two numbers as the dimensions of some algebras is missing. Moreover, for values $\dim \mathfrak{g}$ given in (21), using (20), one can calculate dimensions of the corresponding representations $Y(\alpha)$:

$$\dim V_{(-\frac{\alpha}{2t})} = \big\{5, 27, 77, 300, \tfrac{14553}{17}, 1053, 2430, \tfrac{48608}{13}, \tfrac{111078}{19}, 7371, 15504, \\ 27000, \tfrac{841279}{23}, \tfrac{862407}{17}, 107892, \tfrac{2205225}{13}, \tfrac{578151}{2}, 559911, \tfrac{42507504}{31}, \tfrac{363823677}{61}\big\}$$

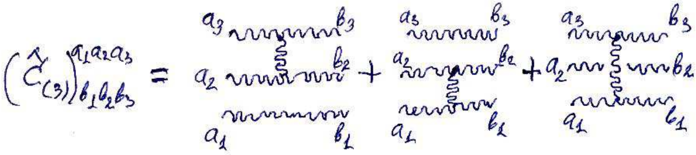
Since $\dim V_{(-\frac{\alpha}{2t})}$ should be integer, we conclude that there not exist Lie algebras with dimensions $47, 96, 119, 287, 336, 603, 782, 1680, 3479$, for which we assume characteristic identity (19) and the trace formulas.

# Universal formulas for 3-split Casimir operator in $\mathrm{ad}^{\otimes 3}$

The matrix $\widehat{C}^{a_1 a_2 a_3}_{b_1 b_2 b_3} := (\widehat{C}_{(3)})^{a_1 a_2 a_3}_{b_1 b_2 b_3}$ of the 3-split Casimir operator is

$$(\widehat{C}_{(3)})^{a_1 a_2 a_3}_{b_1 b_2 b_3} = (\widehat{C}_{12} + \widehat{C}_{13} + \widehat{C}_{23})^{a_1 a_2 a_3}_{b_1 b_2 b_3} \,, \qquad (22)$$

and acts in the space $V_{\mathrm{ad}}^{\otimes 3}$ of the representation $\mathrm{ad}^{\otimes 3}$.



According to

$$\mathrm{ad}^{\otimes 3} = (\mathrm{P}_{[3]} + \mathrm{P}_{[2,1]} + \mathrm{P}_{[1^3]})\,\mathrm{ad}^{\otimes 3} \,,$$

we have decomposition

$$\widehat{C}_{(3)} = (\mathrm{P}_{[3]} + \mathrm{P}_{[2,1]} + \mathrm{P}_{[1^3]})\widehat{C}_{(3)} = \widehat{C}_{[3]} + \widehat{C}_{[2,1]} + \widehat{C}_{[1^3]} \,.$$

A.P. Isaev, S.O. Krivonos, A.A. Provorov,
Int.J.Mod.Phys.A 38 (2023) 06n07, 2350037; e-Print: 2212.14761 [math-ph]
All calculations were done with "Mathematica".

**Proposition 3.** *For 3-split Casimirs* $\widehat{C}_{[1^3]}$, $\widehat{C}_{[3]}$ *and* $\widehat{C}_{[2,1]}$ *we have the universal char. identities*

$$\widehat{C}_{[1^3]}\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[1^3]}+\tfrac{3}{2}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\alpha}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\beta}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\gamma}\right)=0$$

$$\begin{aligned}&\left(\widehat{C}_{[3]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[3]}+1\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\alpha}\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\beta}\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\gamma}\right)\times\\&\left(\widehat{C}_{[3]}+3\hat{\alpha}\right)\left(\widehat{C}_{[3]}+3\hat{\beta}\right)\left(\widehat{C}_{[3]}+3\hat{\gamma}\right)\mathrm{P}_{[3]}=0\,,\end{aligned} \tag{23}$$

$$\begin{aligned}&\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[2,1]}+1\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\gamma}\right)\times\\&\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\gamma}\right)\times\\&\left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\gamma}\right)\mathrm{P}_{[2,1]'}=0.\end{aligned}$$

*where* $\hat{\alpha}=\frac{\alpha}{2t}$, $\hat{\beta}=\frac{\beta}{2t}$, $\hat{\gamma}=\frac{\gamma}{2t}$. All formulas in (23) are homogeneous and symmetric under permutations $(\alpha,\beta,\gamma)$. Our results are in agreement with [P.Vogel (1999), A.M. Cohen and R. de Man (1996)].

The dimensions of irreps corresponding to the eigenvalues of $\widehat{C}_{[2,1]}$
[A.P. Isaev, S.O. Krivonos, A.A. Provorov,
Int.J.Mod.Phys.A 38 (2023) 06n07, 2350037; e-Print: 2212.14761 [math-ph]]

$$
\begin{aligned}
\dim_{-\frac{1}{2}} &= 2X_2 = & 2\times\tfrac{1}{2}\,\dim(g)\,(\dim(g)-3)\,,\\
\dim_{-1} &= 2X_1 = & 2\times\,\dim(g),\\
\dim_{\hat\alpha-\frac{1}{2}} &= B = & \frac{(\hat\alpha-1)(\hat\beta-1)(\hat\gamma-1)(2\,\hat\alpha+\hat\beta)(2\,\hat\alpha+\hat\gamma)(2\,\hat\beta+1)(2\,\hat\gamma+1)(3\,\hat\beta-1)(3\,\hat\gamma-1)}{8\,\hat\alpha^2\,(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)(2\,\hat\beta-\hat\gamma)(2\hat\gamma-\hat\beta)\,\hat\beta^2\,\hat\gamma^2},\\
\dim_{\hat\beta-\frac{1}{2}} &= B' = & \frac{(\hat\beta-1)(\hat\gamma-1)(\hat\alpha-1)(2\,\hat\beta+\hat\gamma)(2\,\hat\beta+\hat\alpha)(2\,\hat\gamma+1)(2\,\hat\alpha+1)(3\,\hat\gamma-1)(3\,\hat\alpha-1)}{8\,\hat\beta^2\,(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)(2\,\hat\gamma-\hat\alpha)(2\hat\alpha-\hat\gamma)\,\hat\gamma^2\,\hat\alpha^2},\\
\dim_{\hat\gamma-\frac{1}{2}} &= B'' = & \frac{(\hat\gamma-1)(\hat\alpha-1)(\hat\beta-1)(2\,\hat\gamma+\hat\alpha)(2\,\hat\gamma+\hat\beta)(2\,\hat\alpha+1)(2\,\hat\beta+1)(3\,\hat\alpha-1)(3\,\hat\beta-1)}{8\,\hat\gamma^2\,(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)(2\,\hat\alpha-\hat\beta)(2\hat\beta-\hat\alpha)\,\hat\alpha^2\,\hat\beta^2},\\
\dim_{-\hat\alpha-\frac{1}{2}} &= Y_2 = & -\frac{(3\hat\alpha-1)(\hat\beta-1)(\hat\gamma-1)(2\hat\beta+1)(2\hat\gamma+1)}{8\hat\alpha^2(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)\hat\beta\hat\gamma},\\
\dim_{-\hat\beta-\frac{1}{2}} &= Y_2' = & -\frac{(3\hat\beta-1)(\hat\gamma-1)(\hat\alpha-1)(2\hat\gamma+1)(2\hat\alpha+1)}{8\hat\beta^2(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)\hat\gamma\hat\alpha},\\
\dim_{-\hat\gamma-\frac{1}{2}} &= Y_2'' = & -\frac{(3\hat\gamma-1)(\hat\alpha-1)(\hat\beta-1)(2\hat\alpha+1)(2\hat\beta+1)}{8\hat\gamma^2(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)\hat\alpha\hat\beta},\\
\dim_{-\frac{3}{2}\hat\alpha} &= C = & -\frac{2}{3}\,\frac{(1+2\hat\alpha)(1+2\hat\beta)(1+2\hat\gamma)(1-\hat\beta)(1-\hat\gamma)(\hat\beta+\hat\gamma)(2\hat\beta+\hat\gamma)(2\hat\gamma+\hat\beta)}{\hat\alpha^3\hat\beta\hat\gamma(\hat\alpha-2\hat\beta)(\hat\alpha-2\hat\gamma)(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)}\\
\dim_{-\frac{3}{2}\hat\beta} &= C' = & -\frac{2}{3}\,\frac{(1+2\hat\beta)(1+2\hat\gamma)(1+2\hat\alpha)(1-\hat\gamma)(1-\hat\alpha)(\hat\gamma+\hat\alpha)(2\hat\gamma+\hat\alpha)(2\hat\alpha+\hat\gamma)}{\hat\beta^3\hat\gamma\hat\alpha(\hat\beta-2\hat\gamma)(\hat\beta-2\hat\alpha)(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)}\\
\dim_{-\frac{3}{2}\hat\gamma} &= C'' = & -\frac{2}{3}\,\frac{(1+2\hat\gamma)(1+2\hat\alpha)(1+2\hat\beta)(1-\hat\alpha)(1-\hat\beta)(\hat\alpha+\hat\beta)(2\hat\alpha+\hat\beta)(2\hat\beta+\hat\alpha)}{\hat\gamma^3\hat\alpha\hat\beta(\hat\gamma-2\hat\alpha)(\hat\gamma-2\hat\beta)(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)}.
\end{aligned}
$$

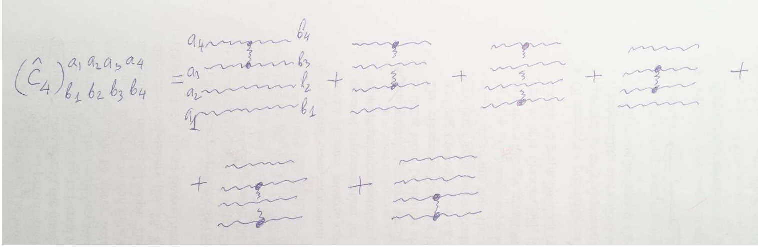
All calculations were done with "Mathematica".

# Universal formulas for 4-split Casimir operator in $\mathrm{ad}^{\otimes 4}$

The matrix of the 4-split Casimir operator is

$$(\widehat{C}_{(4)})^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} = (\widehat{C}_{12} + \widehat{C}_{13} + \widehat{C}_{14} + \widehat{C}_{23} + \widehat{C}_{24} + \widehat{C}_{34})^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} \, , \qquad (24)$$

and acts in the space $V_{\mathrm{ad}}^{\otimes 3}$ of the representation $\mathrm{ad}^{\otimes 3}$.




According to

$$\mathrm{ad}^{\otimes 4} = (\mathrm{P}_{[4]} + \mathrm{P}_{[3,1]} + \mathrm{P}_{[2^2]} + \mathrm{P}_{[2,1^2]} + \mathrm{P}_{[1^4]})\, \mathrm{ad}^{\otimes 4} \, ,$$

we have decomposition

$$\widehat{C}_{(4)} = \widehat{C}_{[4]} + \widehat{C}_{[3,1]} + \widehat{C}_{[2^2]} + \widehat{C}_{[2,1^2]} + \widehat{C}_{[1^4]} \, .$$

**Symmetric module** $P_{[4]}(\mathrm{ad}^{\otimes 4})$ includes the following representations:

[M.Avetisyan, A.P.I., S.Krivonos, R.Mkrtchyan, The uniform structure of $\mathfrak{g}^{\otimes 4}$, ArXive:2311.05358]

$$P_{[4]}(\mathrm{ad}^{\otimes 4}) = 2 \oplus J \oplus J' \oplus J'' \oplus X_2 \oplus \mathbb{Z}_3 \oplus 3Y_2 \oplus 3Y_2' \oplus 3Y_2'' \oplus C \oplus C' \oplus C'' \oplus Y_4 \oplus Y_4' \oplus Y_4'' \oplus D \oplus D' \oplus D'' \oplus D''' \oplus D'''' \oplus D'''''.$$

The universal dimension formulae of some of these representations are:

$$\begin{aligned}
dimJ &= \frac{(\hat\alpha+\hat\beta)(\hat\alpha+\hat\gamma)(2\hat\alpha+\hat\beta-\hat\gamma)(2\hat\alpha+2\hat\beta-\hat\gamma)(2\hat\alpha-\hat\beta+\hat\gamma)(\hat\alpha+2\hat\beta+\hat\gamma)}{4\,\hat\alpha^2\hat\beta^2\hat\gamma^2(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)(\hat\beta-2\hat\gamma)(\hat\beta-\hat\gamma)^2(2\hat\beta-\hat\gamma)(\hat\alpha-\hat\beta-\hat\gamma)} \times \\
&\quad (2\hat\alpha+2\hat\beta+\hat\gamma)(2\hat\alpha-\hat\beta+2\hat\gamma)(\hat\alpha+\hat\beta+2\hat\gamma)(2\hat\alpha+\hat\beta+2\hat\gamma)(\hat\alpha+2\hat\beta+2\hat\gamma), \\
dimJ' &= dimJ_{\hat\alpha\leftrightarrow\hat\beta}, \quad dimJ'' = dimJ_{\hat\alpha\leftrightarrow\hat\gamma}, \\
dim\,\mathbb{Z}_3 &= 2dim\widehat{X}_3 = \frac{2}{9}(N^2-1)^2(N^2-9) \quad \text{for } sl(N), \\
&= dim\,\mathbb{X}_3 = \frac{1}{6}dim\mathfrak{g}(dim\mathfrak{g}-1)(dim\mathfrak{g}-8), \quad \text{for } so(N) \text{ and exceptional algebras}, \\
dimY_4 &= -\frac{(\hat\alpha-1)(2\hat\alpha-1)(7\hat\alpha-1)(\hat\beta-1)(\hat\alpha+\hat\beta-1)(2\hat\alpha+\hat\beta-1)(3\hat\alpha+\hat\beta-1)(\hat\gamma-1)}{24\hat\alpha^4(\hat\alpha-\hat\beta)(2\hat\alpha-\hat\beta)(3\hat\alpha-\hat\beta)\hat\beta(\hat\alpha-\hat\gamma)(2\hat\alpha-\hat\gamma)(3\hat\alpha-\hat\gamma)\hat\gamma} \times \\
&\quad (\hat\alpha+\hat\gamma-1)(2\hat\alpha+\hat\gamma-1)(3\hat\alpha+\hat\gamma-1), \\
dimY_4' &= (dimY_4)_{\hat\alpha\leftrightarrow\hat\beta}, \quad dimY_4'' = (dimY_4)_{\hat\alpha\leftrightarrow\hat\gamma}, \\
dimD &= \frac{(3\hat\alpha-2\hat\beta-2\hat\gamma)(\hat\alpha-\hat\beta-2\hat\gamma)(\hat\beta+\hat\gamma)(\hat\alpha+\hat\beta+\hat\gamma)(2\hat\alpha+\hat\beta+\hat\gamma)(2\hat\beta+\hat\gamma)(\hat\alpha+2\hat\beta+\hat\gamma)}{\hat\alpha^3(\hat\alpha-\hat\beta)^2(3\hat\alpha-\hat\beta)\hat\beta^2(\hat\alpha-2\hat\gamma)(\hat\alpha-\hat\gamma)(2\hat\alpha-\hat\gamma)(\hat\beta-\hat\gamma)\hat\gamma} \times \\
&\quad (2\hat\alpha+2\hat\beta+\hat\gamma)(\hat\alpha+2\hat\gamma)(2\hat\alpha-\hat\beta+2\hat\gamma)(\hat\alpha+\hat\beta+2\hat\gamma)(2\hat\alpha+\hat\beta+2\hat\gamma)(\hat\alpha+2\hat\beta+2\hat\gamma), \qquad (25) \\
dimD' &= (dimD)_{\hat\alpha\leftrightarrow\hat\beta}, \quad dimD'' = (dimD)_{\hat\alpha\leftrightarrow\hat\gamma}, \quad dimD''' = (dimD)_{\hat\beta\leftrightarrow\hat\gamma}, \\
dimD'''' &= (dimD)_{\hat\alpha\to\hat\beta\to\hat\gamma\to\hat\alpha}, \quad dimD''''' = (dimD)_{\hat\alpha\to\hat\gamma\to\hat\beta\to\hat\alpha}. \qquad (26)
\end{aligned}$$

**Conclusion.**

1.) Char. identities for the split COs allow us to calculate colour factors for the amplitudes in "glue-dynamics" and write them in the universal form via Vogel parameters.

2.) The universal description of the subrepresentations in $\mathrm{ad}^{\otimes n}$ for $n \geq 5$ is open problem. We have difficulties with analytical calculations on "Mathematica".

3.) $n$-Split Casimir operators are Hamiltonians for Heisenberg-type spin-chains with interactions between all nodes (not just the closest ones).

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Prof. A.V. Borisov and Prof. V.Ch. Zhukovsky (Theoretical Physics Chair),
Scientific Counsel of Physics Faculty (2023).

The head of the Theoretical Physics Chair from 1953 to 1966 was N.N. Bogolubov.