Oscillator problem on three dimensional two-sheeted hyperboloid

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Geometry of space H_3^1

Three dimensional hyperboloid $H_3^1 \subset \mathbf{R}_{3,1}$

$$x \cdot x = x^2 = x_0^2 - (x_1^2 + x_2^2 + x_3^2) = R^2$$

The group of isometry for hyperboloid is $SO(3,1)$ and corresponding Lie algebra is six dimensional and consists of generators of rotation

$$L_1 = -i\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right), \quad L_2 = -i\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right), \quad L_3 = -i\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$$

and operators of Lorentz transformation

$$K_1 = -i\left(x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0}\right), \quad K_2 = -i\left(x_0 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_0}\right), \quad K_3 = -i\left(x_0 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_0}\right)$$

with commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k, \quad [L_i, K_j] = -i\epsilon_{ijk} K_k, \quad [K_i, K_j] = i\epsilon_{ijk} K_k$$

Hamiltonian on H_3^1 is written as

$$H = -\frac{1}{2R^2}\Delta_{LB} + V(x) = \frac{1}{2R^2}\left(\mathbf{K}^2 - \mathbf{L}^2\right) + V(x)$$

$$\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2 \quad \mathbf{K}^2 = K_1^2 + K_2^2 + K_3^2$$

where Δ_{LB} is Laplace-Beltrami operator

$$\Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k} \quad ds^2 = g_{ik}dx^i dx^k$$

$$g^{ik} = (g_{ik})^{(-1)} \quad g = det(d_{ik}) \quad g_{ik} = G_{\nu\mu}\frac{\partial x_i}{\partial \xi_\mu}\frac{\partial x_k}{\partial \xi_\nu} \quad (i,k = 1,2,3)$$

On H_3^1 hyperboloid the equivalent of oscillator potential is

$$V(x) = \frac{\omega^2 R^2}{2}\frac{\mathbf{x}^2}{x_0^2}$$

and as for integral of motion for the system, it is Demkov tensor

$$D_{ik} = \frac{1}{2R^2}(K_i K_k + K_k K_i) + \omega^2 R^2 \frac{x_i x_k}{x_0^2}$$

For operators L_i and D_{ik} comutation relation are

$$[D_{ij}, L_k] = i\left(\epsilon_{ikl} D_{jl} + \epsilon_{jkm} D_{im}\right)$$

$$[D_{ik}, D_{jl}] = i\left(\omega^2 - \frac{1}{4R^4}\right)\left(\delta_{il} L_{kj} + \delta_{kl} L_{ij} + \delta_{ij} L_{kl} + \delta_{jk} L_{il}\right) - \frac{i}{2R^2}\left(\{L_{ij} D_{lk}\}\right.$$

$$\left. + \{L_{il} D_{kj}\} + \{L_{kj} D_{il}\} + \{L_{kl} D_{lj}\}\right) \qquad L_{ik} = \epsilon_{ikj} L_j$$

$$\sum_i D_{ik} L_i = \sum_i L_i D_{ik} = \frac{1}{2R^2} L_k$$

We will consider our system in two coordinate systems spherical:

$$x_0 = R\cosh\tau, \quad x_1 = R\sinh\tau\sin\theta\cos\varphi, \quad x_2 = R\sinh\tau\sin\theta\sin\varphi, \quad x_3 = R\sinh\tau\cos\theta$$

$$\tau \in (0, \infty), \theta \in [0, \pi], \varphi \in [0, 2\pi)$$

and cylindrical:

$$x_0 = R\cosh\tau_1\cosh\tau_2, \quad x_1 = R\sinh\tau_1\cos\varphi, \quad x_2 = R\sinh\tau_1\sin\varphi, \quad x_3 = R\cosh\tau_1\sinh\tau_2$$

$$\tau_1 \in [0, \infty), \tau_2 \in (-\infty, \infty), \varphi \in [0, 2\pi)$$

$$\Delta_{LB} = \frac{1}{R^2}\left\{\frac{1}{\sinh^2\tau}\frac{\partial}{\partial\tau}\sinh^2\tau\frac{\partial}{\partial\tau} - \frac{1}{\sinh^2\tau}\left(\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right\}$$

Potential

$$V_{osc} = \frac{\omega^2 R^2}{2}\frac{x_1^2 + x_2^2 + x_3^2}{x_0^2} = \frac{\omega^2 R^2}{2}\tanh^2\tau$$

We achieve the separation of variables by substitution

$$\Psi(\tau, \theta, \varphi; R) = \frac{1}{\sqrt{R^3}}(\sinh\tau)^{-1} f(\tau) Y_\ell^m(\theta, \varphi)$$

The ℓ and m are the eigenvalues of full momenta $\mathbf{L}$ and L_3. For bound states $E < 0$ energy spectra is described by

$$E_N(\nu, R) = -\frac{(N+1)(N+3)}{2R^2} + \frac{\nu + 1/2}{R^2}\left(N + \frac{3}{2}\right)$$

where $N = \ell + 2n_\tau \leq \nu - 3/2$, and we will denote wave function by $\Psi_{N\ell m}(\tau, \theta, \varphi; R)$. As for unbound states $E > 0$:

$$E = (\omega^2 + 1 + p^2/4)/2R^2, p \in \mathbf{R}$$

with wave function $\Psi_{p\ell m}(\tau, \theta, \varphi; R)$.

Cylindrical

$$\Delta_{LB} = \frac{1}{R^2}\left\{\frac{1}{\cosh\tau_1\sinh\tau_1}\frac{\partial}{\partial\tau_1}\cosh\tau_1\sinh\tau_1\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\partial^2}{\partial\tau_2^2} + \frac{1}{\sinh^2\tau_1}\frac{\partial^2}{\partial\varphi^2}\right\}$$

Potential

$$V(\tau_1\tau_2) = \frac{\omega^2 R^2}{2}\left(1 - \frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\right)$$

We achieve the separation of variables by substitution

$$\Psi(\tau_1,\tau_2,\varphi;R) = \frac{1}{\sqrt{R^3}}(\cosh\tau_1)^{-\frac{1}{2}}(\sinh\tau_1)^{-\frac{1}{2}}f(\tau_1)S(\tau_2)\frac{e^{im\varphi}}{\sqrt{2\pi}}$$

For bound states $E < 0$ energy spectra is the same as above

$$E_N(\nu,R) = -\frac{(N+1)(N+3)}{2R^2} + \frac{\nu+1/2}{R^2}\left(N+\frac{3}{2}\right)$$

with $N = m + 2n_1 + n_2$, and we will denote wave function by $\Psi_{Nn_2m}(\tau,\theta,\varphi;R)$. Needs mentioning that n_2 and m are connected to eigen values of D_{33} and L_3. As for unbound states $E > 0$ we have the same expected result:

$$E = (\omega^2 + 1 + p^2/4)/2R^2, p \in \mathbf{R}$$

with wave function $\Psi_{pn_2m}(\tau,\theta,\varphi;R)$ and $\Psi_{p\chi m}(\tau,\theta,\varphi;R)$

Expansion for bound states $E < 0$

$$\Psi_{Nn_2m}(\tau_1, \tau_2, \varphi; R) = \sum_{\ell=m,m+1}^{N} W^{\ell,m,n_\tau}_{n_1 n_2 m}(\nu)\Psi_{N\ell m}(\tau, \theta, \varphi; R)$$

$$\Psi_{N\ell m}(\tau, \theta, \varphi; R) = \sum_{n_2=0,1}^{N-m} \left(W^{\ell,m,n_\tau}_{n_1 n_2 m}(\nu)\right)^* \Psi_{Nn_2m}(\tau_1, \tau_2, \varphi; R)$$

$$N = m + 2n_1 + n_2 = \ell + 2n_\tau$$

For convenience we will fixate variable $\tau \to \infty$ which also leads to $\tau_1 \to \infty$

$$W^{\ell,m,n_\tau}_{n_1 n_2 m}(\nu) = const \int_0^\pi (\sin\theta)^{N-n_2+1} \times$$
$$\times\, {}_2F_1\left(-n_2, 2\nu - n_2, \nu + \frac{1}{2} - n_2, \frac{1-\cos\theta}{2}\right) P_\ell^m(\cos\theta) d\theta$$

The answer is described through Racah polynomials

$$W^{\ell,m,n_\tau}_{n_1 n_2 m}(\nu) = const \quad R_n(\lambda(x),\alpha,\beta,\gamma,\delta) \sim {}_4F_3(1)$$

$$x^{(\pm)} = \frac{n_2}{2} - \frac{1}{4} \pm \frac{1}{4} \qquad \beta^{(\pm)} = \frac{N+m}{2} + \frac{3}{4} \mp \frac{1}{4} \qquad \alpha^{(\pm)} = -\frac{N-m}{2} - \frac{3}{4} \mp \frac{1}{4}$$

$$n^{(\pm)} = \frac{\ell-m}{2} - \frac{1}{4} \pm \frac{1}{4} \qquad \gamma^{(\pm)} = \mp\frac{1}{2} \qquad \delta^{(\pm)} = -\nu$$

$$\sum_{\frac{n_2}{2}=0}^{\frac{N-m}{2}} W^{\ell,m,n_\tau(+)}_{n_1,n_2,m} \left(W^{\ell',m,n_\tau(+)}_{n_1,n_2,m}\right)^* = \delta_{\ell\ell'} \qquad \sum_{\frac{n_2}{2}=1}^{\frac{N-m}{2}-\frac{1}{2}} W^{\ell,m,n_\tau(-)}_{n_1,n_2,m} \left(W^{\ell',m,n_\tau(-)}_{n_1,n_2,m}\right)^* = \delta_{\ell\ell'}$$

Expansion for unbound states $E > 0$

As we saw for positive energies we had $\Psi_{p\ell m}(\tau,\theta,\varphi)$ in spherical coordinates and $\Psi_{pn_2 m}(\tau_1,\tau_2,\varphi)$ as well as $\Psi_{p\chi m}(\tau_1,\tau_2,\varphi)$

$$\Psi_{p\ell m}(\tau,\theta,\varphi) = \sum_{n_2} A_{n_2}\Psi_{pn_2 m}(\tau_1,\tau_2,\varphi) + \int B(\chi)\Psi_{p\chi m}(\tau_1,\tau_2,\varphi)d\chi$$

For simlicity let's discuss cylindrical function expansion by sphericalones

$$\Psi_{pn_2m}(\tau_1,\tau_2,\varphi;R)=\sum_{\ell=m,m+1} W_{pn_2m}^{p\ell m}\Psi_{p\ell m}(\tau,\theta,\varphi;R)$$

Coefficients are described by Wilson polynomials

$$W_{pn_2m}^{p\ell m(\pm)}=const\ \mathbb{W}_{n^\pm}(-(a^\pm+k^\pm)^2,a^\pm,b^\pm,c^\pm,d^\pm)$$

$$n^\pm=\frac{\ell}{2}-\frac{m}{2}-\frac{1}{4}\pm\frac{1}{4}\qquad k^\pm=\frac{n_2}{2}-\frac{1}{4}\pm\frac{1}{4}\qquad c^\pm=\frac{m}{2}+\frac{1}{2}+\frac{ip}{2}$$

$$a^\pm=-\frac{\nu}{2}+\frac{1}{2}\mp\frac{1}{4}\qquad b^\pm=\frac{\nu}{2}+\frac{1}{2}\mp\frac{1}{4}\qquad d^\pm=\frac{m}{2}+\frac{1}{2}-\frac{ip}{2}$$

$$\Psi^\nu_{p\chi m}(\tau_1,\tau_2,\varphi;R)=\sum_{\ell=m,m+1} W_{p\chi m}^{p\ell m}\Psi^\nu_{p\ell m}(\tau,\theta,\varphi;R)$$

$$W_{p\chi m}^{p\ell m(\pm)}=const\ \mathbb{W}_{n^\pm}(x^2,a^\pm,b^\pm,c^\pm,d^\pm)$$

$$n^\pm=\frac{\ell}{2}-\frac{m}{2}-\frac{1}{4}\pm\frac{1}{4}\qquad x=\frac{\chi}{2}\qquad c^\pm=\frac{m}{2}+\frac{1}{2}+\frac{ip}{2}$$

$$a^\pm=-\frac{\nu}{2}+\frac{1}{2}\mp\frac{1}{4}\qquad b^\pm=\frac{\nu}{2}+\frac{1}{2}\mp\frac{1}{4}\qquad d^\pm=\frac{m}{2}+\frac{1}{2}-\frac{ip}{2}$$

Orthogonality relation for wilson polynomials writes as

$$\int_0^\infty A(x)\mathbb{W}_n(x^2,a,b,c,d)\mathbb{W}_m(x^2,a,b,c,d)dx + \\ + \sum_k^{a+k<0} B(k)\mathbb{W}_n(-(a+k)^2,a,b,c,d)\mathbb{W}_m(-(a+k)^2,a,b,c,d) = C\,\delta_{nm}$$

which for our coefficients transforms into

$$\int_{-\infty}^{\infty} W_{p\chi m}^{p\ell m(\pm)}\left(W_{p\chi m}^{p\ell' m(\pm)}\right)^* d\chi + \sum_{n_2}^{n_2<\nu-\frac{1}{2}} W_{pn_2 m}^{p\ell m(\pm)}\left(W_{pn_2 m}^{p\ell' m(\pm)}\right)^* = \delta_{\ell\ell'}$$

This allows us to reverse the expansion, and break down spherican functions through cylindricals.

$$\Psi_{p\ell m}^{\nu}(\tau,\theta,\varphi;R) = \int_{-\infty}^{\infty}\left(W_{p\chi m}^{p\ell' m(\pm)}\right)^* \Psi_{p\chi m}^{\nu}(\tau_1,\tau_2,\varphi;R)d\chi + \\ + \sum_{n_2}^{n_2<\nu-\frac{1}{2}}\left(W_{pn_2 m}^{p\ell' m(\pm)}\right)^* \Psi_{pn_2 m}^{\nu}(\tau_1,\tau_2,\varphi;R)$$

Conclusions

- Harmonic oscillator problem on two-sheeted hyperboloid was defined and solved
- Energy spectrum was obtained for bound and unbound states of the system
- Interbasis expansion was performed fully between spherical and cylindrical coordinate systems, with coefficients being described through Racah and Wilson Polynomials

Thank you for your attention!