On the negativity probability of the Wigner function

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Overview

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The main objective

We are interested in phase space representations for finite dimensional quantum systems

THE GOAL:

- **Is it possible to describe the whole family of of Wigner functions $W_\varrho(\Omega|\nu)$ over the classical phase space (Ω) for a generic N-level quantum system with density matrix ϱ? $(+)$**
- **Whats the nature of negative values of quasiprobability distribution functions?(?)**
- **Is there strict correlation between entanglement and presence of negative values of WF?(?)**

The standard form of the Wigner function

For a given state, describing by the density operator ϱ, the Wigner function $W(\boldsymbol{q}, \boldsymbol{p})$ defined on a classical $2n$-dimensional phase space spanned by the canonical coordinates $\boldsymbol{q}$ and momentum $\boldsymbol{p}$ reads

$$W(\boldsymbol{q}, \boldsymbol{p}) := \int \mathrm{d}^n \boldsymbol{z} e^{\frac{\imath}{\hbar} \boldsymbol{z}\boldsymbol{p}} \langle q + \frac{z}{2} \,|\varrho|\, q - \frac{z}{2} \rangle \,.$$

QPD's

Quasi-Probabilty Distributions – "quantum analogue" of the statistical distribution on the phase space of a classical system

Representation for W via displacement operator D

$$W = \operatorname{Tr}\left[\varrho D \Pi D^{\dagger}\right]$$

where D and Π are the displacement and parity operators respectively.

Symbols and Quasi probability distributions

- For any operator A on the Hilbert space $\mathcal{H}$ of quantum system one can define a family of functions $F_A(\Omega;\nu)$ onto the phase space Ω. Here, ν labels the parameters fixing the function.
- When the operator A represents the density matrix, $A = \varrho$, the corresponding phase-space functions $F_\varrho(\Omega;\nu) := P(\Omega;\nu)$ are named as **Quasiprobability Distributions**.

The Stratonovich-Weyl correspondence

The physically motivated properties of $P(\Omega;\nu)$ were formulated by R.L.Stratonovich more than sixty years ago (1955) and are usually referred to as the Stratonovich-Weyl correspondence

The Stratonovich-Weyl Correspondence

Clauses of SW correspondence:

- **Mapping** • For a density matrix ϱ the Wigner function W_ρ on the classical phase-space (Ω) is given by the map:

$$W_\rho(\Omega) = \operatorname{tr}\left(\rho\Delta(\Omega)\right)$$

defined by the Hermitian kernel $\Delta(\Omega) = \Delta(\Omega)^\dagger$, with a unit norm

$$\int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) = 1$$

- **Reconstruction** • The state ρ can be reconstructed as

$$\rho = \int_\Omega \mathrm{d}\Omega\, \Delta(\Omega) W_\rho(\Omega)\,.$$

- **Covariance** • The unitary symmetry $\rho' = U(\alpha)\rho U^\dagger(\alpha)$ induces the kernel transformation: $$\Delta(\Omega') = U(\alpha)^\dagger \Delta(\Omega) U(\alpha)$$

The Wigner distribution kernel

The Wigner distribution $W_\varrho(\Omega)$ over a phase space parametrized by the set Ω is determined by the kernel $\Delta(\Omega|\nu)$:

$$W_\varrho^{(\boldsymbol{\nu})}(\vartheta_1, \vartheta_2, \dots, \vartheta_{d_{\mathbb{F}}}) = \operatorname{tr}[\varrho\, \Delta(\Omega|\nu)] = \operatorname{tr}\left[\varrho\, X P^{(N)}(\boldsymbol{\nu}) X^\dagger\right],$$

Here $P(\nu) = \operatorname{diag}||\pi_1, \pi_2, \dots, \pi_N||$.

In accordance with the SU(n)-covariance of kernel we identify:

d_F - parameters of unitary matrix $U(\theta) \in SU(N)$ with the coordinates of classical phase-space, $\Omega = (\theta_1, \dots \theta_{d_F})$.

Deriving the Master equations for $\Delta(\Omega)$

- **Step 1** • The SU(N) symmetry allows to define the "reconstruction" integral for ϱ over the SU(N) group with the Haar measure:

$$\varrho = Z_N^{-1} \int_{SU(N)} \mathrm{d}\mu_{SU(N)}\, \Delta(\Omega_N)\, \mathrm{tr}\left[\varrho \Delta(\Omega_N)\right] .$$

- **Step 2** • Substitute decomposition $\Delta = U(\theta) P U^\dagger(\theta)$ into the identity, after fixing

$$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_N .$$

and evaluate the integral using the Weingarten formula:

$$\int d\mu U_{i_1 j_1} U_{i_2 j_2} \bar{U}_{k_1 l_1} \bar{U}_{k_2 l_2} = \frac{1}{N^2-1} \left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_1}\delta_{j_2 l_2} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_2}\delta_{j_2 l_1}\right)$$

$$- \frac{1}{N(N^2-1)} \left(\delta_{i_1 k_1}\delta_{i_2 k_2}\delta_{j_1 l_2}\delta_{j_2 l_1} + \delta_{i_1 k_2}\delta_{i_2 k_1}\delta_{j_1 l_1}\delta_{j_2 l_2}\right) .$$

Deriving the Master equations for $\Delta(\Omega)$

$$(\mathrm{tr}[P])^2 = Z_N N\,, \quad \mathrm{tr}[P^2] = Z_N N^2\,.$$

- **Step 3** • Fixing Z_N : Standardization

$$Z_N^{-1} \int \mathrm{d}\mu_{SU(N)} W_A^{(\nu)}(\Omega_N) = \mathrm{tr}[A]\,,$$

is satisfied iff $tr\,[P] = Z_N\, N$, resulting in $Z_N = \frac{1}{N}$ and

master equations

$$\mathrm{tr}\,[\Delta(\Omega_N)] = 1\,, \qquad \mathrm{tr}[\Delta(\Omega_N)^2] = N\,.$$

In $\mu_1, \mu_2, \dots \mu_{N^2-1}$ orthonormal basis of $\mathfrak{su}(N)$

$$\Delta(\Omega_N \,|\, \boldsymbol{\nu}) = \frac{1}{N} U(\Omega_N) \left[I + \sqrt{\frac{N(N^2-1)}{2}} \sum_{\lambda \in H} \mu_s(\boldsymbol{\nu}) \lambda_s \right] U^\dagger(\Omega_N)\,,$$

with coefficients $\mu_s(\nu)$ defined on an unit sphere $\mathbb{S}_{N-2}(1)$.

The Wigner function

for N dimensional quantum system with $N-1$ dimensional Bloch vector $\boldsymbol{\xi}$

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\theta_1, \theta_2, \dots, \theta_d) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}\,(\boldsymbol{n}, \boldsymbol{\xi})\right]\,,$$

$$\boldsymbol{n} = \mu_1 \boldsymbol{n}^{(1)} + \mu_2 \boldsymbol{n}^{(2)} + \dots + \mu_{N-1} \boldsymbol{n}^{(N-1)}\,,$$

$$\boldsymbol{n}^{(s)}_\mu = \frac{1}{2}\,\mathrm{tr}\left(U \lambda_s U^\dagger \lambda_\mu\right)\,, \qquad \mu = 1, 2, \dots, N^2-1\,.$$

Qubit kernel and the Wigner function

For a 2-level system the uniquely defined kernel is
$P^{(2)} = \frac{1}{2} diag||1+\sqrt{3}, 1-\sqrt{3}||$.

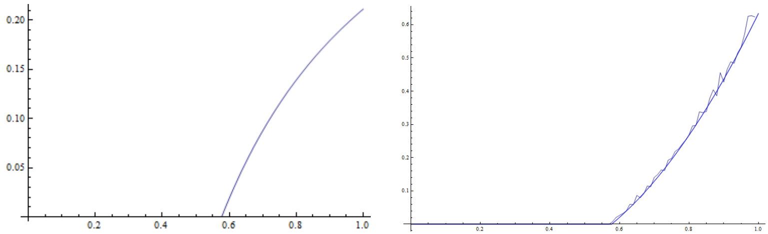
Taking into account that $X = \exp\left(i\, \frac{\alpha}{2}\, \sigma_3\right)\, \exp\left(i\, \frac{\beta}{2}\, \sigma_2\right)\, \exp\left(-i\, \frac{\alpha}{2}\, \sigma_3\right)$, for a qubit parametrized in a standard way by a Bloch vector
$\boldsymbol{r}(\psi, \phi) = (r \sin\psi \cos\phi, r \sin\phi \sin\phi, r \cos\psi)$ as

$$\varrho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma})\,,$$

$$W_{\boldsymbol{r}}(\alpha, \beta) = tr\left[\varrho\, X\, P^{(2)}\, X^{\dagger}\right] = \frac{1}{2} - \frac{\sqrt{3}}{2}\, (\boldsymbol{r}(-\psi, -\phi), \boldsymbol{n})\ .$$

Wigner function negativity probability

a b



a) $Prob\left([W<0]\big|[at\ r]\right) = \frac{1}{2}\cdot H_\theta\left(r-\frac{1}{\sqrt{3}}\right)\cdot\left(1-\frac{1}{\sqrt{3}r}\right)$

b) $PDF\left([W<0]\wedge[at\ r]\right) = \frac{1}{2}H_\theta\left(r-\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{3}r}\right)3r^2$, of generating a density matrix both at distance r and with negative Wigner function.

$$P(W<0) = \frac{1}{2} - \frac{2}{3\sqrt{3}} = 0.1151.$$

Kernel and it's fundamental region

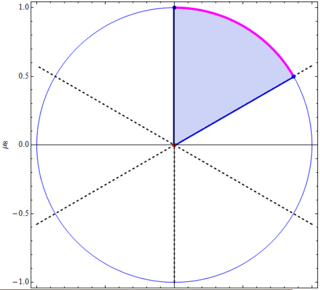
The generic degenerate kernels

$$P^{(3)}(\nu) = \frac{1}{2} diag\left|\left|1-\nu+\sqrt{(1+\nu)(5-3\nu)},\, 1-\nu-\sqrt{(1+\nu)(5-3\nu)},\, 2\nu\right|\right|,$$

$\nu \in (-1, -\frac{1}{3})$.

$P^{(3)}(-1) = diag||1, 1, -1||$ and $P^{(3)}\left(-\frac{1}{3}\right) = diag\left|\left|\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3}\right|\right|$.

The fundamental region



Choosing a convenient parametrization $\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta)$, $\zeta \in [0, \frac{\pi}{3}]$, the fundamental region for the kernel (moduli space of kernels accounting for permutations) becomes the magenta arc.

Qutrit:The Wigner Function

The Euler parametrization for $SU(3)$

$$U(\Omega_3) = V(\alpha,\beta,\gamma)\exp\left(i\theta\lambda_5\right)\ V(a,b,c)\exp\left(i\phi\lambda_8\right)\ ,$$

if being careful with the choice of the embedding one may construct the Wigner function with the suitable set of phase space angles

$$W^{(\nu)}_{\xi}(\alpha,\beta,\gamma,a,b,c,\theta,\phi) = \frac{1}{3} + \frac{2}{3\sqrt{3}}\left[\mu_3\left(\boldsymbol{n}^{(3)},\boldsymbol{\xi}\right) + \mu_8\left(\boldsymbol{n}^{(8)},\boldsymbol{\xi}\right)\right],$$

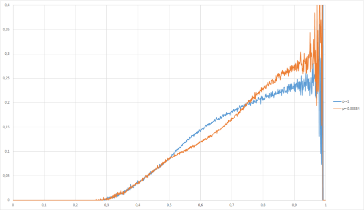
$$W^{(-1)}_{\boldsymbol{\xi}}(\alpha,\beta,\gamma,\theta) = \frac{1}{3} + \frac{4}{3}\left(\boldsymbol{n}^{(8)},\boldsymbol{\xi}\right),$$

$$W^{(-1/3)}_{\boldsymbol{\xi}}(\alpha,\beta,\gamma,\theta,a,b) = \frac{1}{3} + \frac{2}{\sqrt{3}}\left(\boldsymbol{n}^{(3)} + \frac{1}{\sqrt{3}}\boldsymbol{n}^{(8)},\boldsymbol{\xi}\right),$$

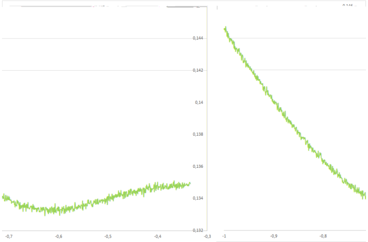
$$W^{(-1/3)}_{\boldsymbol{\xi}}(\alpha',\beta',\gamma',\theta') = \frac{1}{3} + \frac{4}{3}\left(\boldsymbol{n}',\boldsymbol{\xi}\right),$$

Qutrit: Wigner function negativity probability

a



b



$$\text{a)} Prob\left([W < 0] \Big| [at\ r]\right) \text{ for } p = -1 \text{ and } p = -\frac{1}{3}$$
$$\text{b)} Prob\left(W < 0\right)$$

Thank you!