Pseudo-Riemannian Spectral Triples for the Standard Model

Arkadiusz Bochniak[1], Andrzej Sitarz[1,2]

[1] Institute of Physics of the Jagiellonian University
[2] Institute of Mathematics of the Polish Academy of Sciences

Dubna, 26.04.2018

Reference

Bochniak A., Sitarz A., *Finite Pseudo-Riemannian spectral triples and The Standard Model*, arXiv:1804.09482 [hep-th]

Authors acknowledge support by NCN grant OPUS 2016/21/B/ST1/02438

What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(M,S)$ of square-integrable spinors and the Dirac operator $\not{D}_M = i\gamma^\mu(\partial_\mu + \omega_\mu)$ (in local coordinates) together with the usual γ_5 grading and the charge conjugation operator.

What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(M,S)$ of square-integrable spinors and the Dirac operator $\not{D}_M = i\gamma^\mu(\partial_\mu + \omega_\mu)$ (in local coordinates) together with the usual γ_5 grading and the charge conjugation operator.

Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$

$\mathcal{A}$ is a $*$-algebra represented on Hilbert space $\mathcal{H}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$, J is an antilinear isometry s.th. $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$. $\mathcal{D}$ is essentially self-adjoint operator with compact resolvent and s.th. $[\mathcal{D}, a]$ is bounded for all $a \in \mathrm{Dom}(\mathcal{D})$ and $\mathcal{D}\gamma = -\gamma\mathcal{D}$.

What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(M,S)$ of square-integrable spinors and the Dirac operator $\not{D}_M = i\gamma^\mu\left(\partial_\mu + \omega_\mu\right)$ (in local coordinates) together with the usual γ_5 grading and the charge conjugation operator.

Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$

$\mathcal{A}$ is a $*$-algebra represented on Hilbert space $\mathcal{H}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$, J is an antilinear isometry s.th. $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$. $\mathcal{D}$ is essentially self-adjoint operator with compact resolvent and s.th. $[\mathcal{D}, a]$ is bounded for all $a \in \mathrm{Dom}(\mathcal{D})$ and $\mathcal{D}\gamma = -\gamma\mathcal{D}$. Moreover $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$ defining KO-dimension.

What is a geometry ?

Connes' reconstruction theorem

The whole metric and spin structure of a compact, orientable, Riemannian, spin^c manifold can be encoded in the $*$-algebra $C^\infty(M)$ of smooth functions, Hilbert space $L^2(M,S)$ of square-integrable spinors and the Dirac operator $\not{D}_M = i\gamma^\mu\left(\partial_\mu + \omega_\mu\right)$ (in local coordinates) together with the usual γ_5 grading and the charge conjugation operator.

Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$

$\mathcal{A}$ is a $*$-algebra represented on Hilbert space $\mathcal{H}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$, J is an antilinear isometry s.th. $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$. $\mathcal{D}$ is essentially self-adjoint operator with compact resolvent and s.th. $[\mathcal{D}, a]$ is bounded for all $a \in \mathrm{Dom}(\mathcal{D})$ and $\mathcal{D}\gamma = -\gamma\mathcal{D}$. Moreover $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon''\gamma J$ with $\epsilon, \epsilon', \epsilon'' = \pm 1$ defining KO-dimension.There are additional compatibility conditions for $\mathcal{D}$ and for γ (related with the orientability axiom).

Almost-commutative geometry for the Standard Model

$$\left(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})), L^2(M,S) \otimes H_f, \not{D}_M \otimes 1 + \gamma_5 \otimes D_f, \gamma_5 \otimes \gamma_f, J_M \otimes J_f\right)$$

$$H_f = H_L \oplus H_R \oplus H_L^c \oplus H_R^c$$

$$D_f \in M_{96}(\mathbb{C})$$

γ_f - chirality operator
J_f - exchange particle with antiparticle and complex conjugates

Almost-commutative geometry for the Standard Model

$$\left(C^\infty(M)\otimes(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})), L^2(M,S)\otimes H_f, \not{D}_M\otimes 1+\gamma_5\otimes D_f, \gamma_5\otimes\gamma_f, J_M\otimes J_f\right)$$

$$H_f = H_L\oplus H_R\oplus H_L^c\oplus H_R^c$$

$$D_f\in M_{96}(\mathbb{C})$$

γ_f - chirality operator
J_f - exchange particle with antiparticle and complex conjugates

Expansion of the Euclidean spectral action reproduces the effective action for the SM and allows for the expression of bosonic parameters by fermionic one.

Questions

Questions

1. How to include the Lorentzian structure ?

Questions

1. How to include the Lorentzian structure ?
2. Is the finite part also Lorentzian ?

Questions

1. How to include the Lorentzian structure ?
2. Is the finite part also Lorentzian ?
3. What does it imply for the SM ?

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$$

1. $\mathcal{A}$ is a $*$-algebra represented on an Hilbert space $\mathcal{H}$
2. For $p+q$ even $\gamma^* = \gamma,\ \gamma^2 = 1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading commuting with $\mathcal{A}$
3. J is antilinear isometry with $[Ja^*J^{-1}, b] = 0$
4. $\beta = \beta^\dagger, \beta^2 = 1$ commuting with $\mathcal{A}$
5. $\mathcal{D}^\dagger = (-1)^p \beta \mathcal{D} \beta$
6. $[\mathcal{D}, a]$ is bounded
7. $\mathcal{D}\gamma = -\gamma\mathcal{D}$
8. $\mathcal{D}J = \epsilon J\mathcal{D},\ J^2 = \epsilon' \mathrm{id},\ J\gamma = \epsilon'' \gamma J$

$p-q$ mod 8	0	1	2	3	4	5	6	7
ϵ	+	−	+	+	+	−	+	+
ϵ'	+	+	−	−	−	−	+	+
ϵ''	+		−		+		−	

9. $\beta\gamma = (-1)^p \gamma\beta, \ \beta J = (-1)^{\frac{p(p-1)}{2}} \epsilon^p J\beta$
10. $\left[JaJ^{-1}, [\mathcal{D}, b]\right] = 0$
11. orientability : there exists $a^{\circ}_{i_0} \otimes a_{i_0} \otimes a_{i_1} \otimes \ldots \otimes a_{i_n}$ s.th.

$$Ja^{\circ}_{i_0} J^{-1} a_{i_0} [\mathcal{D}, a_{i_1}] \ldots [\mathcal{D}, a_{i_n}] = \begin{cases} \gamma, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

12. time-orientation :

$$\beta = \sum_i Ja^0_i J^{-1} a_i [\mathcal{D}, b^1_i] \ldots [\mathcal{D}, b^p_i]$$

13. $\langle \mathcal{D} \rangle = \sqrt{\frac{1}{2}(\mathcal{D}\mathcal{D}^\dagger + \mathcal{D}^\dagger \mathcal{D})}$ has compact resolvent
14. $[\langle \mathcal{D} \rangle, [\mathcal{D}, a]]$ is bounded

Motivation

Clifford algebra : $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}1$

- $\gamma = i^{\frac{p-q}{2}}\gamma_1 \ldots \gamma_{p+q}$
- there exists unitary B s.th. $B\gamma_i = \epsilon\gamma_i^* B$ and $BB^* = \epsilon'$. Define $J\psi := B\psi^*$.
- $\mathcal{D} = -\sum_j \eta_{jj}\gamma_j\partial_j$
- $B\gamma = \epsilon''\gamma B$
- $\beta = i^{\frac{1}{2}p(p-1)}\gamma_1 \ldots \gamma_p$
- $\beta\mathcal{D}\beta = (-1)^p\mathcal{D}^\dagger$

Finite triples

$$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \qquad \mathcal{H} = \bigoplus_{i,j} \mathcal{H}_{ij}$$

$$\mathcal{H}_{ij} = P_i \mathcal{H} P_j \cong \mathbb{C}^{n_i} \otimes \mathbb{C}^{r_{ij}} \otimes \mathbb{C}^{n_j}$$

- $\gamma_{ij} = \gamma|_{\mathcal{H}_{ij}} = 1_{n_i} \otimes \Gamma_{ij} \otimes 1_{n_j}$
- $q_{ij} := r_{ij}\gamma_{ij}$ is symmetric for KO-dimension 0 and 4 and antisymmetric for KO-dimension 2 and 6
- $\mathcal{D}_{ij,kl} := P_i J P_j J^{-1} \mathcal{D} P_k J P_l J^{-1}$
- there exists $\xi = \sum_{i \neq j} P_i d P_j$ s.th. $\mathcal{D} = \xi + J\xi J^{-1} + \delta$
- $\mathcal{D}_{ji,lk} = \epsilon J D_{ji,lk} J^{-1}$
- for odd p and some $\gamma_{ij} = \pm 1,\ r_{ij} > 0$ there is no pseudo-Riemannian structure

Riemannian from pseudo-Riemannian

$$\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger), \quad \mathcal{D}_- = \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger)$$

We get two Riemannian spectral triples $(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_\pm, J, \gamma)$, that differ by KO-dimensions, with additional selfadjoint grading β s.th.

$$\beta D_\pm = \pm(-1)^p D_\pm \beta,$$

$$\beta\gamma = (-1)^p \gamma\beta, \quad \beta J = (-1)^{\frac{1}{2}p(p-1)} \epsilon^p J\beta.$$

$$\mathcal{D}_+ = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger), \quad \mathcal{D}_- = \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger)$$

We get two Riemannian spectral triples $(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_\pm, J, \gamma)$, that differ by KO-dimensions, with additional selfadjoint grading β s.th.

$$\beta D_\pm = \pm(-1)^p D_\pm \beta,$$

$$\beta\gamma = (-1)^p \gamma\beta, \quad \beta J = (-1)^{\frac{1}{2}p(p-1)} \epsilon^p J\beta.$$

$$\mathcal{D}_E = D_+ + D_-$$

$$J_E = J\beta, \quad J'_E = J_E\gamma$$

$(\mathcal{A}, \pi, \mathcal{H}, \mathcal{D}_E, J_E, \gamma)$ is a Riemannian spectral triple of signature $(0, -(p+q))$.

The Standard Model

$$A_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad H_f = (H_l \oplus H_q) \oplus (H_{\bar{l}} \oplus H_{\bar{q}})$$

$$H_l = \langle \{\nu_R, e_R, (\nu_L, e_L)\} \rangle$$

$$H_q = \langle \{u_R, d_R, (u_L, d_L)\}_{c=1,2,3} \rangle$$

$$A_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad H_f = (H_l \oplus H_q) \oplus (H_{\bar{l}} \oplus H_{\bar{q}})$$

$$H_l = \langle \{\nu_R, e_R, (\nu_L, e_L)\} \rangle$$

$$H_q = \langle \{u_R, d_R, (u_L, d_L)\}_{c=1,2,3} \rangle$$

$$\pi(\lambda, h, m) = \lambda \oplus \bar{\lambda} \oplus h \text{ on } H_l \text{ and } H_q$$

$$\pi(\lambda, h, m) = \bar{\lambda} \text{ on } H_{\bar{l}} \text{ and } 1_4 \otimes m \text{ on } H_{\bar{q}}$$

$$D_f = \begin{pmatrix} S & T^\dagger \\ T & \bar{S} \end{pmatrix}, \qquad S = \begin{pmatrix} S_l & \\ & S_q \otimes 1_3 \end{pmatrix}$$

$$T\nu_R = Y_R \bar{\nu}_R$$

The Standard Model

$$S_l = \left[\begin{array}{c|c|c|c} & & Y_\nu^\dagger & \\ \hline & & & Y_e^\dagger \\ \hline Y_\nu & & & \\ \hline & Y_e & & \end{array}\right], \quad S_q = \left[\begin{array}{c|c|c|c} & & Y_u^\dagger & \\ \hline & & & Y_d^\dagger \\ \hline Y_u & & & \\ \hline & Y_d & & \end{array}\right]$$

γ_f - chirality grading
J_f - real structure

The Standard Model

$$S_l = \left[\begin{array}{c|c|c|c} & & Y_\nu^\dagger & \\ \hline & & & Y_e^\dagger \\ \hline Y_\nu & & & \\ \hline & Y_e & & \end{array}\right], \quad S_q = \left[\begin{array}{c|c|c|c} & & Y_u^\dagger & \\ \hline & & & Y_d^\dagger \\ \hline Y_u & & & \\ \hline & Y_d & & \end{array}\right]$$

γ_f - chirality grading
J_f - real structure

- The existence of right neutrinos implies nonorientability of the geometry
- It is well known that the above Dirac operator is not unique within the model-building scheme of noncommutative geometry. Even the introduction of more constraints, like the second-order condition or Hodge-duality does not allow to exclude the terms, which would introduce the couplings between lepton and quarks and lead to the leptoquark fields

There exists 0-cycle

$$\beta = \pi(1,1,-1)J_F\pi(1,1,-1)J_F^{-1}$$

that is a $\mathbb{Z}/2\mathbb{Z}$-grading which distinguish between leptons and quarks.

There exists 0-cycle

$$\beta = \pi(1,1,-1)J_F\pi(1,1,-1)J_F^{-1}$$

that is a $\mathbb{Z}/2\mathbb{Z}$-grading which distinguish between leptons and quarks. Moreover, this β makes the geometry to be a finite pseudo-Riemannian spectral triple of signature $(4k, 4k+2 \pmod 8)$ with $k \in \mathbb{N}$.

There exists 0-cycle

$$\beta = \pi(1,1,-1)J_F\pi(1,1,-1)J_F^{-1}$$

that is a $\mathbb{Z}/2\mathbb{Z}$-grading which distinguish between leptons and quarks. Moreover, this β makes the geometry to be a finite pseudo-Riemannian spectral triple of signature $(4k, 4k+2 \pmod 8)$ with $k \in \mathbb{N}$.

$(A_f, H_f, D_f, \gamma_f, J_f, \beta)$ could be seen as a Riemannian restriction of a real even pseudo-Riemannian spectral triple of signature $(0,2)$ (note that this choice is not unique and it is also possible to chose in a consistent way e.g. the signature $(4,6)$).

Take as a Hilbert space $H \cong F \oplus F^*$ with

$$F \ni v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \in M_4(\mathbb{C}).$$

Vectors from H can be represented as $\begin{bmatrix} v \\ w \end{bmatrix}$, with $v, w \in M_4(\mathbb{C})$.

Take as a Hilbert space $H \cong F \oplus F^*$ with

$$F \ni v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \in M_4(\mathbb{C}).$$

Vectors from H can be represented as $\begin{bmatrix} v \\ w \end{bmatrix}$, with $v, w \in M_4(\mathbb{C})$. The real structure is given by

$$J \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w^* \\ v^* \end{bmatrix}.$$

Take as a Hilbert space $H \cong F \oplus F^*$ with

$$F \ni v = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \in M_4(\mathbb{C}).$$

Vectors from H can be represented as $\begin{bmatrix} v \\ w \end{bmatrix}$, with $v, w \in M_4(\mathbb{C})$. The real structure is given by

$$J \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w^* \\ v^* \end{bmatrix}.$$

We can identify $\mathrm{End}_{\mathbb{C}}(H)$ with $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$ and denote by e_{ij} a matrix with the the 1 in position (i, j) and zero everywhere else.

Elements of the algebra $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ are represented by

$$\left[\begin{array}{cc|c} \lambda & & 0 \\ & \bar{\lambda} & \\ \hline 0 & & q \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & m \end{array}\right] \otimes e_{22} \otimes 1,$$

where $\lambda \in \mathbb{C}, q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$.

Elements of the algebra $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ are represented by

$$\left[\begin{array}{cc|c} \lambda & & 0 \\ & \bar{\lambda} & \\ \hline 0 & & q \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & m \end{array}\right] \otimes e_{22} \otimes 1,$$

where $\lambda \in \mathbb{C}, q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$. The grading is of the form

$$\gamma = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1 + 1 \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix}.$$

Elements of the algebra $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ are represented by

$$\left[\begin{array}{cc|c} \lambda & & 0 \\ & \bar{\lambda} & \\ \hline 0 & & q \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & m \end{array}\right] \otimes e_{22} \otimes 1,$$

where $\lambda \in \mathbb{C}, q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$. The grading is of the form

$$\gamma = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes e_{11} \otimes 1 + 1 \otimes e_{22} \otimes \begin{bmatrix} -1_2 & \\ & 1_2 \end{bmatrix}.$$

The Dirac operator is of the form

$$D = D_0 + D_1 + D_R,$$

where $D_1 = JD_0J^{-1}$.

Possible pseudo-Riemannian structures for the Standard Model

We would like to have a spectral triple of KO-dimension 6, with a selfadjoint Dirac operator, but such that commutes with a suitable β that represents the shadow of a pseudo-Riemannian structure.

Possible pseudo-Riemannian structures for the Standard Model

We would like to have a spectral triple of KO-dimension 6, with a selfadjoint Dirac operator, but such that commutes with a suitable β that represents the shadow of a pseudo-Riemannian structure. Let us now take the general form of a Dirac operator that satisfies an order-one condition. We have

$$D_R = e_{11} \otimes (A_{11}e_{21} + A_{11}^* e_{12}) \otimes e_{11}$$

with some complex number A_{11}, and

$$D_0 = \begin{bmatrix} & M \\ M^\dagger & \end{bmatrix} \otimes e_{11} \otimes e_{11} + \begin{bmatrix} & N \\ N^\dagger & \end{bmatrix} \otimes e_{11} \otimes (1 - e_{11}) + \\ + \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \otimes e_{12} \otimes e_{11} + \begin{bmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{bmatrix} \otimes e_{21} \otimes e_{11},$$

where M, N, A, B are 2×2 complex matrices.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.
Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
- $\pi(1, -1, 1)$
- $\pi(-1, 1, 1)$

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.
Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
- $\pi(1, -1, 1)$
- $\pi(-1, 1, 1)$

For the case $\beta = \pi(1, -1, 1) J \pi(1, -1, 1) J^{-1}$ the restrictions for the Dirac operator are $M = N = 0$ and no restriction for A, B.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J\pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.
Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
- $\pi(1, -1, 1)$
- $\pi(-1, 1, 1)$

For the case $\beta = \pi(1, -1, 1)J\pi(1, -1, 1)J^{-1}$ the restrictions for the Dirac operator are $M = N = 0$ and no restriction for A, B. Furthermore, if $\beta = \pi(-1, 1, 1)J\pi(-1, 1, 1)J^{-1}$ then again $M, N, B = 0$ and A has to satisfy $A = A \cdot \mathrm{diag}(1, -1)$.

Possible pseudo-Riemannian structures for the Standard Model

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.
Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
- $\pi(1, -1, 1)$
- $\pi(-1, 1, 1)$

For the case $\beta = \pi(1, -1, 1) J \pi(1, -1, 1) J^{-1}$ the restrictions for the Dirac operator are $M = N = 0$ and no restriction for A, B. Furthermore, if $\beta = \pi(-1, 1, 1) J \pi(-1, 1, 1) J^{-1}$ then again $M, N, B = 0$ and A has to satisfy $A = A \cdot \mathrm{diag}(1, -1)$. It is worth noting that both of these restrictions lead not only to unphysical Dirac operators that do not break the electroweak symmetry but also do not satisfy the Hodge duality.

We look for a β that is a 0-cycle, i.e. a sum of elements of the form

$$\beta = \pi(\lambda_1, q_1, m_1) J \pi(\lambda_2, q_2, m_2) J^{-1},$$

with $\lambda_1, \lambda_2 \in \mathbb{C},\ q_1, q_2 \in \mathbb{H},\ m_1, m_2 \in M_3(\mathbb{C})$.
Up to the trivial rescaling (by -1) we have three possibilities.

- $\pi(1, 1, -1)$
- $\pi(1, -1, 1)$
- $\pi(-1, 1, 1)$

For the case $\beta = \pi(1, -1, 1) J \pi(1, -1, 1) J^{-1}$ the restrictions for the Dirac operator are $M = N = 0$ and no restriction for A, B. Furthermore, if $\beta = \pi(-1, 1, 1) J \pi(-1, 1, 1) J^{-1}$ then again $M, N, B = 0$ and A has to satisfy $A = A \cdot \mathrm{diag}(1, -1)$. It is worth noting that both of these restrictions lead not only to unphysical Dirac operators that do not break the electroweak symmetry but also do not satisfy the Hodge duality.

Finally, with the $\beta = \pi(1, 1, -1) J \pi(1, 1, -1) J^{-1}$ we have no restriction whatsoever for M, N while then $B = 0$ and A needs to satisfy: $A = A \cdot \mathrm{diag}(1, -1)$. That leaves the possibility that A_{11} and A_{21} coefficients are present, providing no significant physical effects, and in particular leading only to terms involving a sterile neutrino.

Summary

- We proposed new definition of the finite pseudo-Riemannian spectral triples
- There is a hope that it can be generalized into infinite case and, as a result, for the full SM spectral triple
- We proposed an alternative explanation of the observed quarks-leptons symmetry which prevents the $SU(3)$-breaking, as a shadow of the pseudo-Riemannian structure
- We proposed that the consistent model-building for the physical interactions and possible extensions of the Standard Model within the noncommutative geometry framework should use possibly the pseudo-Riemannian extension of finite spectral triples. We demonstrated that the pseudo-Riemannian framework allows for more restrictions and, in the discussed case introduces an extra symmetry grading, which we interpreted as the lepton-quark symmetry