Fully solvable mathematical scheme in finding out the right mass and width values of $f_0(500)$ and $\rho^0(770)$ mesons

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September 16, 2018

BALDIN18, JINR Dubna, 21 Sept. 2018

Outline

INTRODUCTION

The lightest hadronic resonance with vacuum quantum numbers of glueballs 0^{++}, $f_0(500)$, is the **most contraversial particle** from the whole spectrum of existing scalar mesons in *C.Patrignani et al. (PDG)* **Chin. Phys. C40 (2016) 100001**
From the first its identification in 1974 many papers with various results have been published up to now and **only recently some clarification of the situation has been achieved** in papers
I.Caprini, G.Colangelo, H.Leutwyler, **Phys. Rev. Lett. 96, 132001 (2006)**
R.Garcia-Martin, R.Kaminski, J.R.Pela'ez, J.Ruiz de Elvira, **Phys. Rev. Lett. 107, 072001 (2011)**

INTRODUCTION

A similar situation is with **another elastic resonance, the $\rho^0(770)$ meson**, the parameters of which in *C.Patrignani et al. (PDG)* **Chin. Phys. C40 (2016) 100001** comes from a description of data on

$$\sigma_{tot}(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2\beta_\pi^3}{3t} \mid F_\pi^{I=1}(t) + Re^{i\phi}\frac{m_\omega^2}{m_\omega^2 - t - im_\omega\Gamma_\omega} \mid^2 \quad (1)$$

at the elastic region - to be considered up to $1GeV^2$, where for the charged pion EM FF $F_\pi^{I=1}(t)$ the **Gounaris-Sakurai (G.-S.)** model

$$F_\pi^{GS}(t) = \frac{m_\rho^2 + m_\rho\Gamma_\rho(\frac{3}{\pi}\frac{m_\pi^2}{q_\rho^2}ln(\frac{m_\rho+2q_\rho}{2m_\pi}) + \frac{m_\rho}{2\pi q_\rho} - \frac{m_\pi^2 m_\rho}{\pi q_\rho^3})}{(m_\rho^2 - t) + \Gamma_\rho(\frac{m_\rho^2}{q_\rho^3})(q^2(h(t) - h(m_\rho^2)) + q_\rho^2 h'(m_\rho^2)(m_\rho^2 - t)) - im_\rho\Gamma_\rho(\frac{q}{q_\rho})^3\frac{m_\rho}{\sqrt{t}}}. \quad (2)$$

has been used.

INTRODUCTION

However, we have clearly demonstrated in the paper
E.Bartos, S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.liptaj,
Phys. Rev. D96, 113004 (2017)
that the Gounaris-Sakurai model is not enough accurate for a correct determination of the $\rho^0(770)$ meson parameters.

Therefore in this presentation we demonstrate the **"fully solvable mathematical scheme"**, by means of which, **together with the most accurate S-wave isoscalar and P-wave isovector $\pi\pi$-scattering phase shifts data**, the **right mass and width values of $f_0(500)$ and $\rho^0(770)$ mesons** are found.

THE PION SCALAR AND CHARGED PION VECTOR ELECTROMAGNETIC FORM FACTORS

The **pion scalar-isoscalar form factor** (FF) $\Gamma_\pi(t)$ is defined by the matrix element of the scalar quark density

$$< \pi^i(p_2) \mid \hat{m}(\bar{u}u + \bar{d}d) \mid \pi^j(p_1) >= \delta^{ij}\Gamma_\pi(t) \quad (3)$$

where $t = (p_2 - p_1)^2$ and $\hat{m} = \frac{1}{2}(m_u + m_d)$.

The **charged pion vector electromagnetic (EM) form factor** (FF) $F^E_\pi(t)$ is defined by the matrix element of the pion EM current J^{EM}_μ as follows

$$< p_2 \mid J^{EM}_\mu(0) \mid p_1 >= eF^E_\pi(t)(p_1 + p_2)_\mu. \quad (4)$$

THE PION SCALAR AND CHARGED PION VECTOR ELECTROMAGNETIC FORM FACTORS

Both EM FFs, $\Gamma_\pi(t)$ and $F^E_\pi(t)$, are **analytic functions in the whole complex t-plane**, besides a cut on the real axis from $4m^2_\pi$ to $+\infty$.

For $t \leq 4m^2_\pi$ they are real, with the asymptotic behaviors

$$\Gamma_\pi(t)_{t\to-\infty} \sim t^{-1} \tag{5}$$

$$F^E_\pi(t)_{t\to-\infty} \sim t^{-1}. \tag{6}$$

THE PION SCALAR AND CHARGED PION VECTOR ELECTROMAGNETIC FORM FACTORS

Discontinuities of FFs under consideration across the cuts are **given by the unitarity conditions**, which in the elastic region $4m_\pi^2 < t < 16m_\pi^2$ lead to the forms

$$Im\Gamma_\pi(t) = A_0^{0*}(t)\Gamma_\pi(t) \quad (7)$$

$$ImF_\pi^E(t) = A_1^{1*}(t)F_\pi^E(t), \quad (8)$$

where $A_0^{0*}(t) = e^{i\delta_0^0(t)} sin\delta_0^0(t)$ is the S-wave isoscalar and $A_1^{1*}(t) = e^{i\delta_1^1(t)} sin\delta_1^1(t)$ is the P-wave isovector $\pi\pi$-scattering amplitude. The $\delta_0^0(t)$ **and** $\delta_1^1(t)$, with threshold behaviors $\delta_0^0(t)_{|q=0} \to a_0^0 q$ $\delta_1^1(t)_{|q=0} \to a_1^1 q^3$, are **S-wave isoscalar and P-wave isovector $\pi\pi$-scattering phase shifts**, respectively.

THE PION SCALAR AND CHARGED PION VECTOR ELECTROMAGNETIC FORM FACTORS

Note: As it is clearly demonstrated by existing experimental $\pi\pi$-scattering data, a **validity of the elastic unitarity conditions** for $\Gamma_\pi(t)$ and $F_\pi^E(t)$ FFs can be **extended up to** $1 GeV^2$.

Just **from the Schwarz reflection principle** it follows that both FFs fulfil the so-called **reality conditions**

$$[\Gamma_\pi(t)]^* = \Gamma_\pi(t^*) \tag{9}$$

$$[F_\pi^E(t)]^* = F_\pi^E(t^*) \tag{10}$$

reflecting the reality of both FFs on the real axis below the lowest threshold $4m_\pi^2$.

THE PION SCALAR AND CHARGED PION VECTOR ELECTROMAGNETIC FORM FACTORS

Since the charge e appears in the definition of $F^E_\pi(t)$ as a prefactor, the charged pion vector EM FF is normalized at $t = 0$ to one

$$F^E_\pi(0) = 1. \tag{11}$$

On the other hand the pion scalar FF $\Gamma_\pi(t)$ is normalized to the **pion sigma term** value

$$\Gamma_\pi(0) = (0.99 \pm 0.02) m^2_\pi \tag{12}$$

to be predicted by the χPT
J.Gasser, Ulf-G.Meissner, **Nucl. Phys, B357, 90 (1991)**.

FULLY SOLVABLE MATHEMATICAL SCHEME

The **analytic properties** of $F_\pi^E(t)$ and $\Gamma_\pi(t)$ in t-plane, together **with asymptotic behaviors** of these FFs, allow **through the Cauchy formula** to derive **dispersion relations with one subtraction** $\Rightarrow$ in combination with the elastic unitarity conditions of FFs under consideration lead to the **phase representations**

$$\Gamma_\pi(0) = P_n(t) exp[\frac{t}{\pi}\int_4^\infty \frac{\delta_0^0(t')}{t'(t'-t)}dt'] \tag{13}$$

$$F_\pi^E(t) = P_n(t) exp[\frac{t}{\pi}\int_4^\infty \frac{\delta_1^1(t')}{t'(t'-t)}dt']. \tag{14}$$

FULLY SOLVABLE MATHEMATICAL SCHEME

Just these phase representations are **starting points** for our **"fully solvable mathematical scheme"**, by means of which one finds out the **right mass and width values of** $f_0(500)$ **(or** σ**-particle) and** $\rho^0(770)$ **meson resonances**.

In this scheme **crucial role play results of the MADRID/CRACOW group**

R.Garcia-Martin, R.Kamin'ski, J.R.Pela'ez, J.Ruiz de Elvira, F.J.Yndurain, **Phys. Rev. D83, 074004 (2011)**

who by means of the GKPY Roy-like equations with an imposed crossing symmetry conditions were able to obtain from existing inaccurate experimental data **the most precise information on behaviors of S- and P-waves** $\pi\pi$**-scattering phase shifts**.

FULLY SOLVABLE MATHEMATICAL SCHEME

On the other side, **not less important are parametrizations of FF phases $\delta_0^0(t)$ and $\delta_1^1(t)$**

$$tg\delta_\Gamma(t) \equiv tg\delta_\pi(t) = \frac{A_1 q + a_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots} \tag{15}$$

which are derived from the analyticity of FFs in q-plane

S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.Liptaj,
Phys. Rev. D94, 054036 (2016)

where **FFs in elastic region have only zeros and poles**, the latter exclusively in the lower half-plane.

FULLY SOLVABLE MATHEMATICAL SCHEME

From the previous parametrization it follows

$$\delta_0^0(t) = arctg \frac{A_1 q + a_3 q^3 + A_5 q^5 + A_7 q^7 + ...}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + ...} \qquad (16)$$

where the **parameter A_1 is identified with the S-wave isoscalar $\pi\pi$-scattering length a_0^0**, and

$$\delta_1^1(t) = arctg \frac{A_3 q^3 + A_5 q^5 + A_7 q^7 + ...}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + ...} \qquad (17)$$

where requiring thresholg behavior of $\delta_1^1(t)_{|q \to 0} = a_1^1 q^3, A_1 \equiv 0$ and the **parameter A_3 is identified with the P-wave isovector $\pi\pi$-scattering length a_1^1.**

FULLY SOLVABLE MATHEMATICAL SCHEME

The **optimal description** of the GKPY data on $\delta_0^0(t)$ with theoretical errors **have been achieved (see Fig.1) by 5 nonzero values of coefficients** in (16)

$$\begin{aligned} A_1 \equiv a_0^0 &= 0.2219 \pm 0.0029 \\ A_2 &= -0.0764 \pm 0.0423 \\ A_3 &= 0.1390 \pm 0.0251 \\ A_4 &= -0.0062 \pm 0.0053 \\ A_5 &= -0.0135 \pm 0.0020 \end{aligned} \tag{18}$$

FULLY SOLVABLE MATHEMATICAL SCHEME

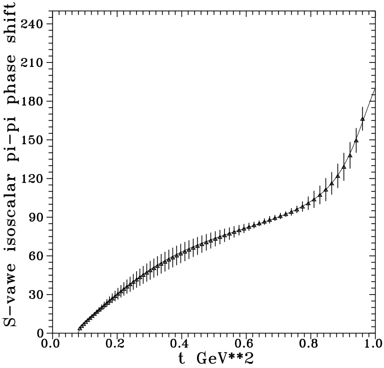

Figure : Optimal description of the GKPY $\delta_0^0(t)$ data

FULLY SOLVABLE MATHEMATICAL SCHEME

Substitution of (16) with coefficients (18) into the phase representation of $\Gamma_\pi(t)$ leads to the integral, which **one can not calculate explicitly**.

Now **Very important step in the "fully solvable mathematical scheme"**!
In the theory of functions of complex variable **there is an equivalent logarithmic form**

$$\delta_0^0(t) = \frac{1}{2i} ln[\frac{(1 + A_2 q^2 + A_4 q^4) + i(A_1 q + A_3 q^3 + A_5 q^5)}{(1 + A_2 q^2 + A_4 q^4) - i(A_1 q + A_3 q^3 + A_5 q^5)}] \quad (19)$$

to "arctg", which leads to a very simple calculation of the corresponding integral by means of the theory of residua.

FULLY SOLVABLE MATHEMATICAL SCHEME

With this aim one finds **roots of the corresponding polynomials in the numerator and denominator of the previous relation**.

$$\begin{aligned} q_1 &= 0.00 - i2.0430 \pm 0.2029 \\ q_2 &= 3.3827 \pm 0.0115 + i0.1744 \pm 0.0340 \\ q_3 &= -3.3827 \pm 0.0115 + i0.1744 \pm 0.0340 \\ q_4 &= 1.4147 \pm 0.0579 + i1.0749 \pm 0.0162 \\ q_5 &= -1.4147 \pm 0.0579 + i1.0749 \pm 0.0162 \end{aligned} \tag{20}$$

$$q_1^* = -q_1; \quad q_2^* = -q_3; \quad q_3^* = -q_2; \quad q_4^* = -q_5; \quad q_5^* = -q_4.$$

FULLY SOLVABLE MATHEMATICAL SCHEME

Then the phase representation of $\Gamma_\pi(t)$ leads to the form

$$\Gamma_\pi(t) = P_n(t)exp[\frac{(q^2+1)}{2\pi i}\int_{-\infty}^{\infty}\frac{q' ln\frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)}dq'] \quad (21)$$

with $q^2 < 0$ i.e. $q = i\sqrt{\frac{4-t}{4}} \equiv ib$.

In order to carry out calculation explicitly, it is **convenient to decompose the integral into two integrals, according to singularities to be placed in the upper and lower half q-plane**.

FULLY SOLVABLE MATHEMATICAL SCHEME

Then the **explicit form of the integral** in the phase representation

$$I = \frac{2\pi i}{(q^2+1)} ln(\frac{(q-q_1^*)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)} \frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)}) \tag{22}$$

is obtained in a straightforward way,

if in the case of the first integral the contour of integration is closed in the lower half plane

and **the second integral is closed in the upper half plane**.

In a such way one avoids rather complicated evaluation of the cut-contributions, which are automatically generated by branch points under logarithms.

FULLY SOLVABLE MATHEMATICAL SCHEME

Then **final form of the pion scalar isoscalar FF** is

$$\Gamma_\pi(t) = P_n(t)\left(\frac{(q-q_1^*)}{(q-q_2^*)(q-q_3^*)(q-q_4^*)(q-q_5^*)}\frac{(i-q_2^*)(i-q_3^*)(i-q_4^*)(i-q_5^*)}{(i-q_1^*)}\right), \quad (23)$$

where $P_n(t)$ is any polynomial normalized at $t = 0$ to one, however, it has not to violate the asymptotic behavior of $\Gamma_\pi(t)$.

FULLY SOLVABLE MATHEMATICAL SCHEME

The pole $q = q_3^*$ on the second Riemann sheet in t-variable corresponds to the $f_0(500)$ meson resonance

S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.Liptaj,
Phys. Rev. D94, 054036 (2016)

with parameters

$$m_{f_0(500)} = (459 \pm 29) MeV \quad (24)$$
$$\Gamma_{f_0(500)} = (517 \pm 77) MeV.$$

The pole $q = q_2^*$ corresponds to $f_0(980)$ meson resonance and its parameters are

$$m_{f_0(980)} = (985 \pm 85) MeV \quad (25)$$
$$\Gamma_{f_0(980)} = (93 \pm 34) MeV.$$

FULLY SOLVABLE MATHEMATICAL SCHEME

The **optimal description** of the GKPY data on $\delta_1^1(t)$ with theoretical errors **have been achieved with** $\chi^2/ndf = 0.0234$ **(see Fig.2) by 4 nonzero values of coefficients in (17)**

$$\begin{aligned} A_2 &= 0.1070 \pm 0.0329 \\ A_3 \equiv a_1^1 &= 0.0321 \pm 0.0008 \\ A_4 &= -0.03825 \pm 0.0030 \\ A_5 &= 0.0003 \pm 0.0002 \end{aligned} \tag{26}$$

FULLY SOLVABLE MATHEMATICAL SCHEME

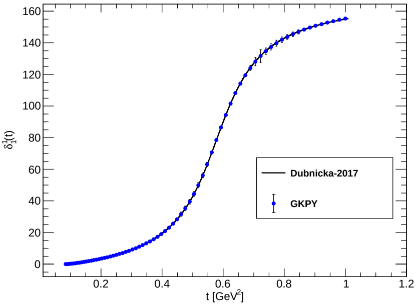

Figure : Optimal description of the GKPY $\delta_1^1(t)$ data

FULLY SOLVABLE MATHEMATICAL SCHEME

The **roots of the corresponding polynomials in the numerator and denominator of the equivalent logarithmic relation**

$$\delta_1^1(t) = \frac{1}{2i} ln[\frac{(1 + A_2q^2 + A_4q^4) + i(A_3q^3 + A_5q^5)}{(1 + A_2q^2 + A_4q^4) - i(A_3q^3 + A_5q^5)}] \quad (27)$$

to

$$\delta_1^1(t) = arctg\frac{A_3q^3 + A_5q^5}{1 + A_2q^2 + A_4q^4} \quad (28)$$

are

$$\begin{aligned} q_1 &= -2.5480 \pm 0.0020 + i0.2752 \pm 0.0016 \\ q_2 &= 2.5480 \pm 0.0020 + i0.2752 \pm 0.0016 \\ q_3 &= 0.0 - i1.8432 \pm 0.0658 \\ q_4 &= 0.0 + i2.146 \pm 0.1054 \\ q_5 &= 0.0 - i139.793 \pm 015254.2 \end{aligned} \quad (29)$$

FULLY SOLVABLE MATHEMATICAL SCHEME

Then the phase representation of $F^E_\pi(t)$ leads to the form

$$F^E_\pi(t) = P_n(t) exp[\frac{(q^2+1)}{2\pi i}\int_{-\infty}^{\infty}\frac{q' ln\frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)}dq'] \quad (30)$$

with $q^2 < 0$ i.e. $q = i\sqrt{\frac{4-t}{4}} \equiv ib$, and **in a completely similar procedure, like in the case of the pion scalar isoscalar FF** $\Gamma_\pi(t)$, one finds the $\rho^0(770)$ meson resonance parameter values *E.Bartos, S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.Liptaj,* **Phys. Rev. D96, 113004 (2017)**
to be

$$\begin{aligned} m_\rho(770) &= (763.56 \pm 0.51) MeV \quad (31) \\ \Gamma_\rho(770) &= (143.09 \pm 0.82) MeV. \end{aligned}$$

CONCLUSIONS

The **parameter values of the scalar** $f_0(500)$ **meson**
S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.Liptaj,
Phys. Rev. D94, 054036 (2016)

$$\begin{aligned} m_{f_0(500)} &= (459 \pm 29) MeV \\ \Gamma_{f_0(500)} &= (517 \pm 77) MeV. \end{aligned} \tag{32}$$

and the **parameter values of the** $\rho^0(770)$ **resonance**,
E.Bartos, S.Dubnicka, A.Z.Dubnickova, R.Kaminski, A.Liptaj,
Phys. Rev. D96, 113004 (2017)

$$\begin{aligned} m_\rho &= (763.56 \pm 0.51) MeV \\ \Gamma_\rho &= (143.09 \pm 0.82) MeV. \end{aligned}$$

CONCLUSIONS

found out by

the **fully solvable mathematical scheme**

making use of the **most accurate information on the S-wave and P-wave** $\pi\pi$ **scattering phase shifts**

given by the **GKPY Roy-like equations with an imposed crossing symmetry conditions**
R.Garcia-Martin, R.Kamin'ski, J.R.Pela'ez, J.Ruiz de Elvira, F.J.Yndurain,
Phys. Rev. D83, 074004 (2011)

can be considered as the **most reliable masses and widths** of particles under consideration.

Thank you for your attention.