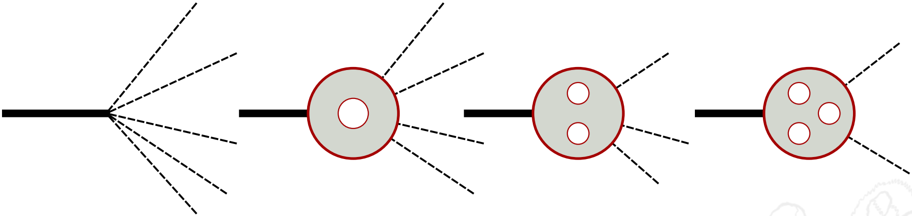
Analytical calculation of phase-space integrals in massless QCD



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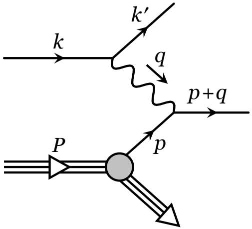
Overview

- **Motivation**
- **Phase-space integrals**
- **Dimensional recurrence relations and $1 \to 5$ PS integrals**
- **Fixing periodic function**
- **Integrals with virtual corrections**

Motivation

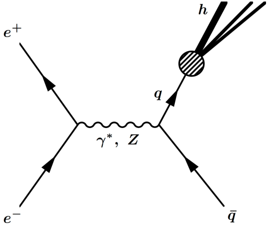
Fully-inclusive and semi-inclusive processes in QCD

- Deep-inelastic scattering



- Sum rules — MINCER
- Fixed moments — MINCER
- Full x or n dependence — $\mathrm{Im}\, T^{\mu\nu}$

- e^+e^- annihilation



- Total crossection — MINCER
- Fixed moments — PS integration/number
- Full x or n dependence — PS int/function

Phase-space integrals

Phase-space integration in a nutshell

Energy positive

$$\int \left(\prod_{i=1}^{n} d^D p_i \delta(p_i^2) \theta(E_i) \right) \delta^{(D)}(q - p_1 - \ldots - p_n) f(p_i \cdot p_j)$$

On-shellness condition

Momentum conservation

Propagators

Important features:

- Very complicted integration domain
- Propagators may contain singularities:

  collinear $(\beta_i \cdot \beta_j) \to 0$, partons emitted at a small angle

  infrared $E_i \to 0$, very low energy massless partons
- The enormous number of momenta components to be integrated directly
- Function $f(\ldots)$ and hence propagators are functions of invariants $s_{ijk\ldots} = (p_i + p_j + p_k + \ldots)^2$ formed by scalar products only

Invariant phase-space integration

Integrals of our interest have form:

$$I_n = \int \left(\prod_i d^D p_i \right) f(p_i \cdot p_j)$$

From momentum integration to eplicit integration over scalar products

$$I_n = \prod_{k=1}^{n-1} \Omega_{D-k} \int \prod_{i<j} ds_{ij} \left(\Delta_n\right)^{\frac{D-n-1}{2}} \Theta\left(\Delta_n\right) \delta\left(1 - s_{12\ldots n}\right) f(s_{ij})$$

We define Gramm determinant for n massless partons momenta:

$$\Delta_n = \frac{(-1)^{n+1}}{2^n} \begin{vmatrix} 0 & s_{12} & \cdots & s_{1n} \\ s_{12} & 0 & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & \cdots & 0 \end{vmatrix}, \quad \Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

Complications due to the Gramm determinant

- Two- and three-particle PS: no constraints from the theta-function $\Theta(\Delta_n)$

$$\Delta_2 = \frac{1}{4}s_{12}^2, \quad \Delta_3 = \frac{1}{4}s_{12}s_{13}s_{23}$$

- Four-particle PS depends on Källén function $\lambda(x,y,z) = (x-y-z)^2 - 4yz$

$$\Delta_4 = -\frac{1}{16}\lambda(s_{12}s_{34}, s_{13}s_{24}, s_{14}s_{23})$$

  PS unit cube paramtrisation exists [Gehrmann-De Rider,Gehrmann,Heinrich'04]

- For five-paricle PS maping on hypercube we need to solve $\Delta_5 = 0$, parametrisation with lots of square roots [Heinrich'06]

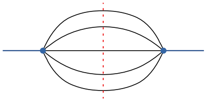
$$\Delta_5 = \frac{1}{16}(s_{14}s_{15}s_{23}s_{25}s_{34} + s_{13}s_{15}s_{24}s_{25}s_{34} - s_{13}s_{14}s_{25}^2s_{34} \\ -s_{15}^2s_{23}s_{24}s_{34} + \ldots - s_{12}s_{13}s_{23}s_{45}^2)$$

Ways to evaluate single-scale integrals

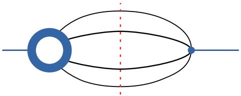
- Using definition of the hypergeometric function, for more complicated integrals using `HyperInt` package
  - ✓ Analytical expression from the begining
  - ✗ Integral is free from singularities, to expand in ε under the integral sign
  - ✗ Only generalized polylogarithms (GPL) and linear reducible denominators
  - ✗ Difficult to manipulate with expressions containing GPL of higher weights
- Mellin-Barnes representation
  - ✓ Can be applied to divergent integrals
  - ✗ Only low dimensioanal integrals can be calculated analytically
- Dimensional Recurrence Relations (DRR)
  - ✗ Difficult to construct homogeneos solution for coupled integrals
  - ✗ Solution is numerical, needs PSLQ and known basis
  - ✓ Precision is very high and many orders of expansion in ε accessible easily

On the way to the final answer

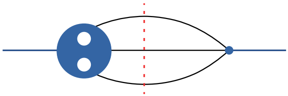
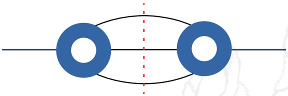
- Five-particle phase-space integrals: **real⊗real**



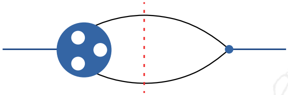
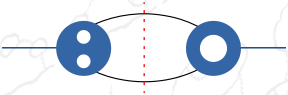
- Four-particle phase-space integrals: **virtual⊗real**



- Three-particle phase-space integrals: **virtual⊗real** and **virtual⊗virtual**

- Two-particle phase-space integrals: **virtual⊗real** and **virtual⊗virtual**

Dimensional recurrence relations and $1 \to 5$ PS integrals

Lowering DRR

Using integral representation trough invariants for arbitrary D we can perform shift $D \to D+2$:

$$I_n^{(D)} = \prod_{k=1}^{n-1} \Omega_{D-k} \int \prod_{i<j} ds_{ij} \left(\Delta_n\right)^{\frac{D-n-1}{2}} \Theta\left(\Delta_n\right) \delta\left(1 - s_{12\ldots n}\right) \left[f(s_{ij})\right]$$

$$I_n^{(D+2)} = \prod_{k=1}^{n-1} \Omega_{D-k} \int \prod_{i<j} ds_{ij} \left(\Delta_n\right)^{\frac{D-n-1}{2}} \Theta\left(\Delta_n\right) \delta\left(1 - s_{12\ldots n}\right) \left[\frac{2\pi}{D} \Delta_n f(s_{ij})\right]$$

Rewriting integrand of $D+2$ dimensional inegral as D-dimensional one with additional factor, we can rewrite $D+2$ dimensional integral as a linear combination of D - dimensional integrals with $f \to f'$:

$$f'(s_{ij}) = \frac{2\pi}{D} \Delta_n f(s_{ij})$$

IBP relations for cut integrals: definition

Reverse unitarity allows us apply to integration of phase-space integrals methods developed for loop integrals [Anastasiou,Melnikov'02]
We define cut propagators

$$\delta(q^2)\theta(q_0) \to \mathcal{C}(q^2) = \frac{1}{2\pi i}\text{Disc}\frac{1}{q^2} = \frac{1}{2\pi i}\left(\frac{1}{q^2+i0} - \frac{1}{q^2-i0}\right)$$

Same differentiation rules as for ordinary propagators:

$$\frac{\partial}{\partial q_\mu}\left[\mathcal{C}(q^2)\right]^a = -2a \cdot q_\mu \left[\mathcal{C}(q^2)\right]^{a+1}$$

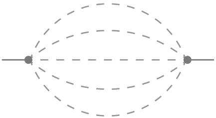
But we nullify integrals with cut propagators in the negative powers

$$\left[\mathcal{C}(q^2)\right]^{-a} = 0, \quad \forall a = 0, 1, 2, \ldots$$

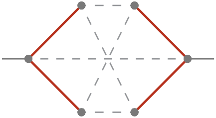
Can relate each PS integral with corresponding loop integral and apply IBP reduction with small modifications

Master integrals basis for PS and loop integrals

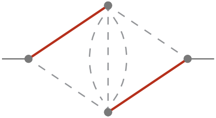
- Four-loop propagator master integrals basis contains 28 elements
  [Baikov,Chetyrkin'10] [Lee,Smirnov,Smirnov'12]
- Not all of them could have five-particle cut, but some could be cut in more than one different way, total number of PS integrals is 31
- From the simplest...



F_1

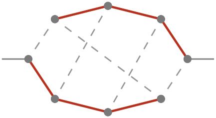


F_2

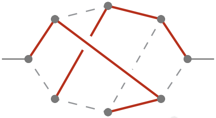


F_3

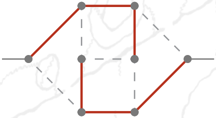
- ...to the most complicated



F_{29}



F_{30}



F_{31}

Constructing DRR system for PS integrals

- Constructed lowering dimensional recurrence relations [Tarasov'96] for all 31 master integrals using package `LiteRed` [Lee'12]
- Integrals with cuts reduced using `FIRE 5` package [Smirnov'14]
- After reduction as in the loop-integral case each sector contains not more than a **single master integral**
- Thus, system have triangular form and drastically simplifies calculations

$$F_i(\nu+1) = c_{ii} F_i(\nu) + \left[\sum_{j=1}^{i-1} c_{ij} F_j(\nu) \right], \quad \nu = \frac{D}{2}$$

- Homogeneous system decouple into the set of single equations

$$\mathcal{H}_i(\nu+1) = c_{ii} \mathcal{H}_i(\nu)$$

- Large number of problems have been solved due to this property using DRR [Tarasov'00] [Lee'09;Lee,Terekhov'10;Lee,Smirnov,Smirnov'10-11]

DRA method: solving DRR system

- General solution of triangular system can be written as

$$F_i(\nu) = \omega_i(\nu)\mathcal{H}_i(\nu) + \mathcal{R}_i(\nu)$$

$\mathcal{H}_i(\nu)$	-	homogeneous solution, from the diagonal matrix element
$\mathcal{R}_i(\nu)$	-	partial solution, depends only on integrals from subsectors
$\omega_i(\nu)$	-	periodic function to be fixed using independent methods

DRA: Dimensional Recurence and Analyticity [Lee'09]

Analize singularities of all the ingredients $\mathcal{H}, \mathcal{R}, \omega, F$ and fix periodic function

- To find solution basic stripe $[\nu, \nu+1)$ should be fixed, proper choice can greatly simplify evaluation
- Position of poles and their multiplicity for function $F_i(\nu)$ on a basic stripe should be known in advance

Constructing main ingridients

- For the case of single integral in sector homogeneous system decouples into first order difference equations

$$\mathcal{H}_i(\nu+1) = c_{ii}(\nu)\mathcal{H}_i(\nu)$$

- For c_{ii} rational function of ν in form:

$$c_{ii}(\nu) = c\frac{(\nu-a_1)(\nu-a_2)\dots(\nu-a_A)}{(\nu-b_1)(\nu-b_2)\dots(\nu-b_B)}$$

- We can write one of the possible solutions explicitly:

$$\mathcal{H}(\nu) = c^\nu \frac{\Gamma(\nu-a_1)\Gamma(\nu-a_2)\dots\Gamma(\nu-a_A)}{\Gamma(\nu-b_1)\Gamma(\nu-b_2)\dots\Gamma(\nu-b_B)}$$

- Partial solution for high precision numerical evaluation can be constructed from the known DRR system and provided set of homogeneous solutions using package `DREAM` [Lee,Mingulov'17]

Last step - to fix periodic function $\omega(\nu)$

Fixing periodic function

From periodic functions to unknown coefficients

- Once we know $\mathcal{H}_i(\nu)$ and $\mathcal{R}(\nu)$ we can analyze their singularities in the fixed stripe, periodic function $\omega(\nu)$ can be thought to be a function of the complex variable $z = e^{2i\pi\nu}$
- If all the functions $\mathcal{H}_i(\nu), \mathcal{R}(\nu), F_i(\nu)$ have only finite number of singular points in the stripe, we can fix $\omega(\nu)$ from finite number of terms of its Laurent series expansion
- Need to know singularities of $F_i(\nu)$ in the stripe, in some cases possible to choose a stripe such $F_i(\nu)$ is holomorphic, e.g.:
  - $\nu \in [-2, 0)$ - fully massive tadpoles, no IR divergencies
  - $\nu \in [6, 8)$ - phase-space integrals, no UV divergencies
- For loop integrals we can use `SDAnalize` from `FIESTA` [Smirnov,Smirnov'11] to find poles of $F_i(\nu)$ and their multiplicity to construct ansatz for $\omega(\nu)$, for poles $z_1, z_2, \ldots$ with multiplicities $a_1, a_2, \ldots$:

$$\omega(\nu) = c_0 + \sum_{k=1}^{a_1} \frac{c_{k,1}}{(e^{2i\pi\nu} - z_1)^k} + \sum_{k=1}^{a_2} \frac{c_{k,2}}{(e^{2i\pi\nu} - z_2)^k} + \ldots$$

Periodical conditions fixing in PS integrals

- Easy to find stripe, where $F_i(\nu)$ holomorphic, hence only single constant need to be fixed
- Furthermore $F_i(\nu)$ holomorphic in the whole infinite plane in positive direction, constant can be fixed from assymptotics at infinity
- Assymptotics at infinity can be obtained using Laplace method for the integral in the form

$$I = \int_{\Omega} dx\, h(x) e^{\lambda \varphi(x)}$$

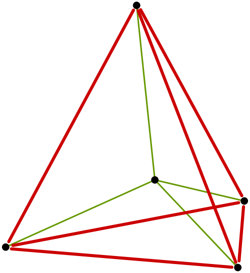
- If $\max \varphi(x) = \varphi(\bar{x})$ and $\bar{x}$ is interior point of Ω, then integral I can be aproximates for $\lambda \to \infty$ by:

$$I = e^{\lambda \varphi(\bar{x})} \left(\frac{2\pi}{\lambda}\right)^{k/2} \frac{h(\bar{x})}{\sqrt{|\det \varphi_{xx}(\bar{x})|}} + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Laplace method for PS integrals

- From the integral over invariants we can obtain asymptotics:

$$F_i(\nu \to \infty) = \left(\prod_{k=1}^{n-1} \Omega_{2\nu-k} \right) \Delta_n(\bar{x})^\nu \left(\frac{\pi}{\nu} \right)^{\frac{n(n-1)-2}{4}} \left(\mathcal{C}_i(\bar{x}) + \mathcal{O}(\nu^{-1}) \right)$$



- Point $\bar{x}$ is a maximum of Δ_n
- n-particle Gram determinant equal to the volume of n-hedron
- In the limit $D \to \infty$ maximal volume corresponds to the reguler n-hedron
- Angles between all pairs of vectors are equal

$$s_{ii} = 0, s_{ij} = \frac{2}{n(n-1)}$$

- All integrals have same assymptotics upto the constant $\mathcal{C}_i$

Asymptotics of the $1 \to 5$ PS integrals

- For the five-particle PS integrals we can find assymptotics of all the homogeneous solutions using function from `DREAM` package
- Asymptotics of the partial solution is equal to asymptotics of integrals from subsectors
- In our case we checked that all $\mathcal{H}_i, i > 1$ are growing exponentially faster then full solution

$$\lim_{\nu \to \infty} \frac{\mathcal{H}_i(\nu)}{F_i(\nu)} = \infty$$

- Only option for periodic function is to be equal zero, we fixed all ingridients and can obtain numerical results with high precision

$$\mathcal{H}_1(\nu) = \frac{\pi^{4\nu}\Gamma(\nu-1)^4}{(2\pi)^4\Gamma(4(\nu-1))\Gamma(5(\nu-1))}, \mathcal{H}_2 = \mathcal{H}_3 = \ldots = \mathcal{H}_{31} = 0;$$

Numerical results and PSLQ reconstruction

- Using DREAM package we obtained numerical values for all 31 integrals with accuracy about 2000 digits
- To reconstruct analytical expression we apply PSLQ algorithm with a basis constructed from multiple zeta values(MZV) up to weight 12
- Sample result for the most complicated integral up to weight 6:

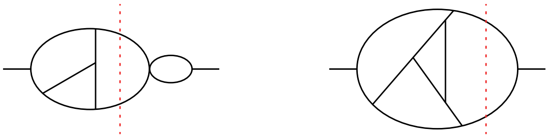
$$
\begin{aligned}
F_{31} = {} & \frac{7}{9\varepsilon^5} - \frac{17}{18\varepsilon^4} + \frac{1}{\varepsilon^3}\left(-\frac{143}{9} - \frac{125}{9}\zeta_2 \right) + \frac{1}{\varepsilon^2}\left(\frac{902}{9} + \frac{133}{6}\zeta_2 - \frac{236}{3}\zeta_3 \right) \\
& + \frac{1}{\varepsilon}\left(-\frac{4190}{9} + \frac{716}{3}\zeta_2 + \frac{1418}{9}\zeta_3 - \frac{265}{6}\zeta_2^2 \right) \\
& + \frac{16892}{9} - \frac{4709}{3}\zeta_2 + \frac{9718}{9}\zeta_3 + \frac{3373}{20}\zeta_2^2 + 1228\zeta_3\zeta_2 - \frac{17612}{9}\zeta_5 \\
& + \varepsilon\left(-\frac{63902}{9} + \frac{22181}{3}\zeta_2 - \frac{68062}{9}\zeta_3 - \frac{377}{5}\zeta_2^2 - \frac{23666}{9}\zeta_3\zeta_2 + \frac{48610}{9}\zeta_5 - \frac{688249}{1890}\zeta_2^3 + \frac{27128}{9}\zeta_3^2 \right)
\end{aligned}
$$

Integrals with virtual corrections

Warm up: two-particle phase-space integrals

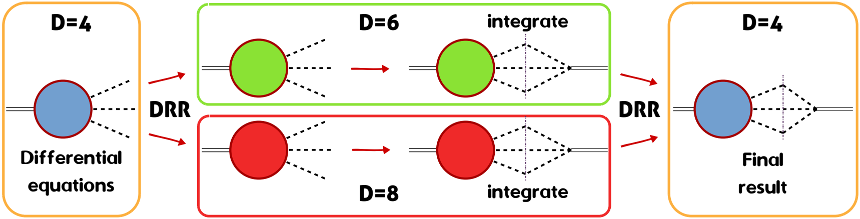
Situation becomes more complicated:

On top of the complicated IR srtucture of phase-space integration, integrals with virtual corrections have UV divergencies from the loop integration



- All **virtual-virtual** integrals are trivial and reducible, due to one-loop part beeing simply one-loop propagator
- To calculate **virtual-real** intergals we integrate three-loop massless form-factor over two-partile PS
  1. We prepare system of DRR for two-particle cut integrals and solve it up to finite number of unknown periodic functions
  2. Using results for three-loop form-factor in the form of the solution of DRR [Lee,Smirnov,Smirnov'10] we integrate it over PS and fix unknown functions

Three-particle phase-space integrals



Calculation flow:

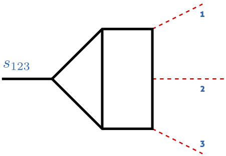
1. Solve **DE** for loop integrals as series in ε near $d = 4 - 2\varepsilon$
2. Using **DRR** transform it to $d = 6 - 2\varepsilon$, where only UV divergencies survive
3. For cross-check transform to $d = 8 - 2\varepsilon$
4. Integrate each term of ε-expansion using `HyperInt` [Panzer'14]
5. With the help of **DRR** for the cut integrals convert them to $d = 4 - 2\varepsilon$

For **virtual**$\otimes$**virtual** contribution virtual parts are known for arbitrary d in terms of hypergeometric functions ${}_2F_1$ and ${}_3F_2$ [Gehrmann,Remiddi'01]

Two-loop boxes with one off-shell leg

Results up to finite-part (weight four)

Expressible through GPL of two variables $y = \frac{s_{13}}{s_{123}}$ and $z = \frac{s_{23}}{s_{123}}$ [Gehrmann,Remiddi'01]



- System of DE reducible to ε-form using `Fuchsia` [Gituliar,Magerya'17]

$$\partial f_i = M_{i,j}(\varepsilon, y, z) f_j \quad \rightarrow \quad \partial g_i = \varepsilon M'_{ij}(y,z) g_j, \quad f_i = T_{ij}(\varepsilon, y, z) g_j$$

- Basis of integrals g_i have uniform transcendental weight, system decouples and can be easily integrated oder by order in ε using properties of GPL

$$g_i\{\varepsilon^n\} = \int M'_{ij} g_j \{\varepsilon^{n-1}\} d\,y + \mathcal{C}_{in}(z)$$

DE for double box: fixing boundary conditions

- **Planar topologies** have only branch points $y = 0$ and $z = 0$, other points $y = 1$, $y = 1 - z$ and $y = -z$ are regular. Regularity reqirement can be used to fix boundary conditions

$$\partial_y f_i = \left(\frac{A_{ij}(y,z)}{1-y} + \frac{B_{ij}(y,z)}{1-y-z} + \frac{C_{ij}(y,z)}{y+z} + R_{ij}(y,z) \right) f_j$$

- Taking limits and nullifying all regular terms we obtain linear systems:

$$0 = (1-y)\partial_y f_i|_{y\to 1} = A_{ij}(y,z) f_j|_{y\to 1}$$

- **Nonplanar topologies** have only branch points $y = 0$,$z = 0$ and $y = 1 - z$, other points $y = 1$ and $y = -z$ are regular, and can be used for initial conditions fixing

Conclusion

1. $1 \to 5$

- Constructed solution of DRR, results are reconstructed using `PSLQ`

2. $1 \to 4$

- Needs further investigation

3. $1 \to 3$

- **virtual-virtual** contribution calculated using DRR for virtual part and direct integration over PS in $d = 6 - 2\varepsilon$
- For **virtual-real** contribution constructed solution for planar two-loop boxes with one off-shell leg to be integrated over PS. Nonplanar topologies to be solved separately.

4. $1 \to 2$

- Constructed solution of DRR, results are reconstructed using `PSLQ`

Thank you for attention!