FEATURES OF RADIATION OF RELATIVISTIC ELECTRONS IN STORAGE RINGS

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The formulation of the problem is due to the fact that the spectral properties of radiation in alternating magnetic fields, as compared with homogeneous fields, differ little, but the angular characteristics depend significantly on betatron oscillations. The dynamics of an electron in the magnetic systems of accelerators is quite complicated, so a number of simplifications have been made here. The asymptotic formulas for the spectral and angular distributions of the radiation intensity with the first quantum correction are oblained by Schwinger's operator method.

For them, as well as for the angular characteristics of synchrotron light, approximate expressions are given, allowing researchers to determine the desired properties of emission at certain points of the orbit. In addition, on a particular example for radition problems, the solution of the Hill equation by the methods of Whittaker and stretched parameters is considered in the linear approximation. As a result, it was revealed that the degree of polarization of the radiation and the angular properties are essentially affected by vertical vibrations of electrons; this coincides with the available experiments. The beam effect is determined by the fact that the averaging over the phases of the electron oscillations is carried out and the mean-square amplitudes are taken into account.

1. Introduction

At present there is a continuing expansion in the construction of synchrotron radiation sources and their utilization in various fields of physics and technology. In view of this, the development of more accurate methods for studying the radiation characteristics of these unique experimental facilities is warranted. The existing theory of synchrotron radiation has heretofore been developed mainly for uniform magnetic fields [1, 2, 3, 4]. In [5, 6], the question of the influence of betatron oscillations on radiation was not specifically considered. As a consequence, the formulas obtained are not generally valid in the lattices of modern accelerators and storage rings. The agreement of conventional theories with experimental data is satisfactory mainly for the parameters of spectral density and total power. In actuality, electrons accelerating in periodic magnetic fields perform transverse oscillations which can noticeably affect the angular and polarization properties of the emitted radiation. In consequence of this, it would appear advisable elaboration to formulate the emission problem from first to last in a non-uniform magnetic field. Such an analysis would, among other things, motivate the development of more compre-

hensive beam diagnostics and enhance the precision of experiments involving polarizable synchrotron radiation.

The effect of betatron oscillations on synchrotron radiation was first discussed in [7] for an axisymmetric magnetic field. Then, we carried out an analysis of the properties of radiation emitted in the FD, FODO, FOFDOD systems, etc. [8, 9, 10]. In the present paper, using the beta function concept, we attempt to extend the theory to more general magnetic structures, including storage rings.

2. Betatron function method

According to present views, the small vertical motions of charged particle in a circular accelerator can be described by the function

$$z = \sqrt{\frac{\beta_z A_z}{\pi}} \cos(\int \frac{ds}{\beta_z} + \delta_0), \tag{1}$$

where A_z is the emittance, β_z is the beta function that depends on the orbital length s, and δ_0 is the initial phase.

Let $\varphi = s/R_0$ be a generalized azimuthal angle, where R_0 is the mean radius (viz., the total orbit length devided by 2π) in magnetic systems with straight sections.

In this case the velocity component is determined from (1) to be

$$v_z = c\sqrt{\frac{A_z}{\pi\beta_z}}\sqrt{1 + (\frac{1}{2}\frac{d\beta_z}{ds})^2} \cos(\int \frac{ds}{\beta_z} + \delta_1 + \delta_0), \tag{2}$$

where $\sin \delta_1 = 1/\sqrt{1 + (d\beta_z/ds)^2/4}$.

In cylindrical co-ordinates, if the particle position is specified by the radius-vector

$$\{(R_0 + \rho)\cos\varphi, \ (R_0 + \rho)\sin\varphi, \ z\},$$

the velocity components are given by

$$v_x = \dot{\rho}\cos\varphi - (R_0 + \rho)\sin\varphi\,\dot{\varphi}, v_y = \dot{\rho}\sin\varphi + (R_0 + \rho)\cos\varphi\,\dot{\varphi}, v_z = \dot{z},$$

where the dot denotes the total derivative with respect to t.

When ρ is added to R_0, the direction of a narrow cone of radiation varies moderately. Thus the radial oscillations influence the angular radiation distributions only slightly and we may assume that the particle moves on an average circular orbit of radius R_0.

To obtain a continuous description of particle motion, the guiding magnetic field will be extended over the entire length of the lattice. In this case the field can be written as

$$H_0(\frac{1}{1 + k} + F(\varphi)), \tag{3}$$

where k is the ratio of lengths of the straight sections to the lengths of the bending magnets, and the periodic function $F(\varphi)$ averages out to zero. Following a Fourier series expansion, the transverse projections of the magnetic field take the form:

$$H_z = H_0(\frac{1}{1 + k} - f(\varphi)\frac{\rho}{R}), \quad H_r = -H_0 f(\varphi)\frac{z}{R}, \tag{4}$$

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where R is the usual magnet bending radius and the step function $f(\varphi)$, introduced in place of the gradient distribution n, has a definite form associated with any given magnetic structure.

In order to get an expression for the angular velocity one can take as a basis the following equation

$$\frac{1}{r}\frac{d}{dt}(mr^2\dot\varphi) = -\frac{e_0}{c}(\dot z H_r - \dot r H_z), \tag{5}$$

where the terms H_r and H_z are generally different for each element of a cell.

After substituting (4) into (5) and integrating we have, in the parabolic approximation

$$r^2\dot\varphi = \omega_0[RR_0 + R\rho + \frac{1}{2(1+k)}\rho^2 + (1+k)\int f(\varphi)(z\dot z - \rho\dot\rho)dt],$$

where $\omega_0 = ce_0 H_0/E$, E is the total electron energy, and $R_0 = (1+k)R$. Integrating here $\dot r$ restored former state r as $R_0 + \rho$. As a result of new value of frequency take into account also straight sections.

The final expression for the angular velocity assumes the form:

$$\dot\varphi = \frac{\omega_0}{1+k}[1 - \frac{\rho}{R_0} + \frac{3}{2}\frac{\rho^2}{R_0^2} + \frac{1}{R^2}\int (z\dot z - \rho\dot\rho)f(\varphi)dt]. \tag{6}$$

We shall now employ the operator method [2, 7] for studying the properties of the emitted synchrotron light. We will suppose that the radiation vector $\vec\kappa = \omega\vec n/c$, where $\vec n = \{0; \sin\theta, \cos\theta\}$ lies in the YZ plane (see also [11]), and θ is the spherical angle.

We define the linear polarization vectors in the wave zone as follows:

$$\vec e_\sigma = \{1, 0, 0\}, \qquad \vec e_\pi = \{0, \cos\theta, -\sin\theta\}.$$

Here, the first vector corresponds to the electric field component lying in the orbital plane and the second is orthogonal to this plane (see Fig.1). In the first quantum approximation the components of the emission intensity can be written as

$$\frac{dW_\sigma}{d^3\kappa} = \frac{ce^2}{(2\pi)^3 R_0}\frac{\nu'}{\nu} \mid \int dt\, v_x \exp(i\frac{\nu'}{\nu}(\omega t - \vec\kappa\vec r)) \mid^2,$$

$$\tag{7}$$

$$\frac{dW_\pi}{d^3\kappa} = \frac{ce^2}{(2\pi)^3 R_0}\frac{\nu'}{\nu} \mid \int dt(v_y \cos\theta - v_z \sin\theta) \exp(i\frac{\nu'}{\nu}(\omega t - \vec\kappa\vec r)) \mid^2,$$

where $\omega = \nu\omega_0/(1+k)$ and $\nu' = \nu(1 + h\,\omega/E)$.

The angle φ can be reckoned from any point of the orbit. Then, the expansion parameters are given by $\varphi \sim m_0 c^2/E, \tau = N\varphi$ (N is the number of magnetic periods in the accelerator or storage ring), $\cos\theta(\theta \sim \pi/2), \rho/R_0$, and z/R_0. Using these, we also take $\sin\varphi \sim \varphi - \varphi^3/6, \sin\theta \sim 1$ in the parameter

$$\Phi = \omega t - \vec\kappa\vec r = \frac{\nu\omega_0}{1+k}[t - \frac{1}{c}(R_0 + \rho)\sin\varphi\sin\theta - \frac{z}{c}\cos\theta].$$
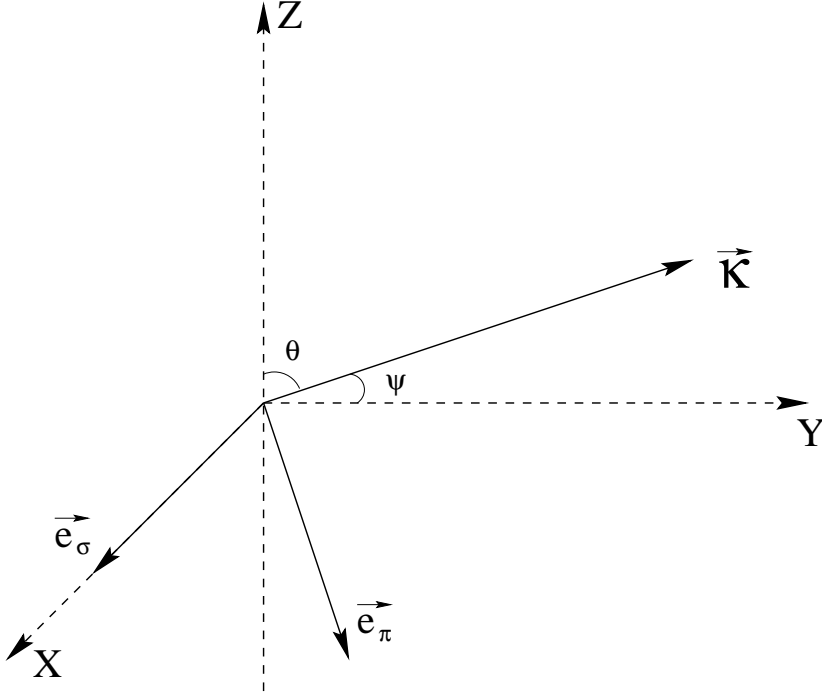
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FIG. 1.   The directions of the
linear polarization components

To determine φ in (6) it is necessary to carry out the integration. In particular, in the zeroth approximation $\varphi = \omega_0 t/(1+k)$.

From the equality

$$v^2 = \dot{\rho}^2 + \dot{z}^2 + R^2\omega_0^2[1 + \frac{\rho^2}{R_0^2} + \frac{2}{R^2}\int(z\dot{z} - \rho\dot{\rho})f(\varphi)dt]$$

we can solve for the quantity $R\omega_0/c$.

The velocity components are

$$\frac{v_x}{c} \approx \frac{\dot{\rho}}{c} - \varphi, \quad \frac{v_y}{c} \approx \beta, \quad v_z = \dot{z}.$$

We now introduce a new variable $u = -v_x/c = \varphi - \dot{\rho}/c$ and carry out expansions of the form:

$$z = z\mid_{\tau=0} + \frac{dz}{d\tau}\mid_{\tau=0} \cdot \tau + ... \approx z_0 + \frac{v_z}{c}R_0\varphi,$$

$$\frac{\omega_0}{1+k}\int dt[\int(z\dot{z} - \rho\dot{\rho})f(\varphi)dt] \approx const + \frac{1}{N}\tau\int dt(z\dot{z} - \rho\dot{\rho})f(\varphi)$$

and so on.

Retaining terms up to the third order, we obtain

$$\Phi = [1 - \beta\sin\theta + \frac{u^2}{6} - \frac{v_z}{c}\cos\theta + \frac{1}{2}(\frac{v_z}{c})^2]u + const.$$

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In the ultrarelativistic case we have $1 - \beta \sin\theta \approx \varepsilon/2$, where $\varepsilon = 1 - \beta^2 \sin^2\theta$.

Integrating Eqs.(7) with respect to the new variable u and averaging over the initial phases we get for the spectral-angular distributions

$$\frac{dW_\sigma(\nu)}{d\Omega} = \frac{ce^2\nu\nu'}{12\pi^4 R_0^2} \int_0^{2\pi} d\delta\, \varepsilon_1^2 \, K_{2/3}^2\left(\frac{\nu'}{3}\varepsilon_1^{3/2}\right),$$

(8)

$$\frac{dW_\pi(\nu)}{d\Omega} = \frac{ce^2\nu\nu'}{12\pi^4 R_0^2} \int_0^{2\pi} d\delta\, \varepsilon_1\, \varepsilon_2\, K_{1/3}^2\left(\frac{\nu'}{3}\varepsilon_1^{3/2}\right),$$

where

$$\varepsilon_1 = 1 - \beta^2 + \varepsilon_2, \qquad \varepsilon_2 = (\cos\theta - \frac{v_z}{c}\mid_{\tau=0})^2,$$

$$\frac{v_z}{c}\mid_{\tau=0} = \alpha\,\cos\delta, \qquad \alpha = \sqrt{\frac{A_z}{\pi}}\left[\frac{1}{\sqrt{\beta_z}}\sqrt{1 + (\frac{1}{2}\frac{d\beta_z}{ds})^2}\right]\mid_{\tau=0}.$$

In (2) in the neighbourhood of the point $\tau = 0$ the constant phase was denoted by δ.

Formulas (8) can mainly be used for storage rings. For their practical application we must bear in mind that the graph of the beta function is usually known for a closed trajectory. Consequently, we can define its value at the point at which emission is recorded, while the derivative can be approximated by the ratio $\triangle\beta_z/\triangle s$.

For bending magnets the graph of the beta function is almost linear; in this case we can use the slope of the linear segment for approximating the derivative.

Introducing the angle ψ reckoned from the orbital plane, we can approximate $\cos\theta$ by ψ.

Let us now discuss the method of calculating the integrals in (8). Initially, we may go over to the Airy function

$$K_{1/3}(\frac{2}{3}x^{3/2}) = \pi\sqrt{\frac{3}{x}}Ai(x), \qquad K_{2/3}(\frac{2}{3}x^{3/2}) = -\frac{\pi\sqrt{3}}{x}Ai'(x)$$

with $x = (\nu/2)^{2/3}\varepsilon$ and employ well-known integral tables [12].

In the case of greatest interest (for small amplitudes of the oscillations or small cross-sections of the bunch) one can use an expansion in terms of the parameter $q^2 = \alpha^2/2\varepsilon$. Then, in the classical approximation we obtain the following expressions:

$$\frac{dW_\sigma(\nu)}{d\Omega} = W\{(Ai')^2 + 2x^2q^2[2xgU + (1+2g)AiAi'] + \frac{1}{2}x^2q^4[12x^2g(1+g)Ai^2 +$$

$$x(3 + 24g + 16x^3g^2)U + 3(1 + 16x^3g + 24x^3g^2)AiAi']\},$$

(9)

$$\frac{dW_\pi(\nu)}{d\Omega} = Wx\{(g+q^2)Ai^2 + 2xgq^2(2xgU + 5AiAi') + 3xq^4AiAi' +$$

$$\frac{1}{2}x^2gq^4[(39 + 16x^3g^2)U + 28xgAi^2 + 8x^2g(14 + 3g)AiAi']\},$$

where

$$W = \frac{2^{1/3}ce^2\nu^{2/3}}{\pi^2 R_0^2}, \qquad g = \psi^2/\varepsilon, \qquad U = xAi^2 + (Ai')^2,$$

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$$\varepsilon = 1 - \beta^2 + \cos^2\theta = \frac{1}{\gamma^2}(1 + \gamma^2\psi^2).$$

We note that the parameter x is the argument both of the function Ai and its derivative.

Assuming $q^2 = 0$ (there are no vertical oscillations) and replacing R_0 by R, these formulas can be transformed into expressions for an uniform magnetic field.

In the neighbourhood of the orbital plane x is the suitable expansion parameter. Here we use the Airy function $V(x) = \sqrt{\pi}Ai(x)$ with the initial Fock conditions: $V(0) = 0.629271, V'(0) = -0.458745$ [13]. This function, along with its derivative, can be expanded in convergent power series. In this case, the braces in (9) must be replaced by, respectively,

$$V'(0)^2\{1 + \frac{1}{3}x^3[2 + 3q^2(2 + 3q^2)(1 + 4g) + 5q^6]\}+$$

$$V(0)V'(0)x^2(1 + 2q^2 + 4q^2g + \frac{3}{2}q^4)$$

and

$$V^2(0)(g + q^2) + V(0)V'(0)x(2q^2 + 3q^4 + 2g + 10gq^2)+$$

$$\frac{1}{2}V'(0)^2x^2[2q^2 + 6q^4 + 5q^6 + g(2 + 20q^2 + 39q^4) + 8q^2g^2].$$

From these formulas we can observe that at $\theta = \pi/2$ the σ-component is less than the same component for a uniform magnetic field, and the magnitude of the π-component is not equal to zero.

Peaks of curves plotted in accordance with formulas (8) will generally be below the graphs that are built to the uniform magnetic field at the same energy E and bending radius R. In the plane of the equilibrium orbit the radiation will not be completely linearly polarized.

3. Angular characteristics of synchrotron light

We will first discuss the spectral properties of the emission. If, in (9), we retain $\cos\theta$ instead of ψ and integrate with respect to the spherical angle θ, we derive spectral formulas almost coinciding with the analogous expressions for a uniform magnetic field. There are corrections to the order of α^2/R_0^2, and similar terms are to the radial vibrations.

Summing expressions (8) over the entire spectrum, we come to the following integrals:

$$\frac{dW_\sigma}{d\Omega} = \frac{7ce^2}{64\pi^2R_0^2}\int_0^{2\pi}d\delta\left(\frac{1}{\varepsilon_1^{5/2}} - \frac{320}{7\sqrt{3}\pi}\frac{1}{\varepsilon_1^4}\frac{\hbar\omega_0}{E}\right), \tag{10}$$

$$\frac{dW_\pi}{d\Omega} = \frac{5ce^2}{64\pi^2R_0^2}\int_0^{2\pi}d\delta\,(\cos\theta - \alpha\cos\delta)^2\left(\frac{1}{\varepsilon_1^{7/2}} - \frac{256}{5\sqrt{3}\pi}\frac{1}{\varepsilon_1^5}\frac{\hbar\omega_0}{E}\right).$$

If we now integrate with respect to θ and δ, we obtain the total intensities:

$$W_\sigma = \frac{7ce^2}{12R_0^2\varepsilon_0^2}\left(1 - \frac{25\sqrt{3}}{7\varepsilon_0^{3/2}}\frac{\hbar\omega_0}{E}\right), \qquad W_\pi = \frac{ce^2}{12R_0^2\varepsilon_0^2}\left(1 - \frac{5\sqrt{3}}{2\varepsilon_0^{3/2}}\frac{\hbar\omega_0}{E}\right).$$

where $\varepsilon_0 = 1 - \beta^2$. Substituting R_0 by R and summing these components we come to the well-known result [2, 3].

Methods of integrating the first terms in (10) with respect to δ were considered in [7]. For the final calculations it is convenient to introduce the additional variables:

$$\varepsilon = \varepsilon_0 + \cos^2\theta, \quad p = \alpha^2/\varepsilon, \quad g = \cos^2\theta/\varepsilon, \quad p_1 = \alpha^2/\varepsilon_0, \quad g_1 = \cos^2\theta/\varepsilon_0,$$

$$f = \varepsilon_0/\varepsilon, \quad \triangle = (1+p)^2 - 4pg, \quad 2r^2 = 1 - (1-p)/\sqrt{\triangle}.$$

First, we need to express $\int_0^{2\pi} d\delta/\sqrt{\varepsilon_1}$ through the complete elliptic integral $\mathbf{K}(r)$, then to differetiate this eguality several times in the partameters; here it is necessary to use the equation

$$x(1-x)\frac{d^2\mathbf{K}}{dx^2} + (1-2x)\frac{d\mathbf{K}}{dx} - \frac{1}{4}\mathbf{K} = 0,$$

taken from [14].

In the ultrarelativistic case the classical part of the angular distributions of the radiation intensity can be written in the form:

$$\frac{dW_\sigma}{d\Omega} = \frac{14W_1}{3\pi\varepsilon^{5/2}\triangle^{5/4}}\left\{(3 + p_1 + 16\frac{pg}{\triangle} - \frac{2p}{\triangle^{1/2}r^2}G_1)\mathbf{K} + \frac{2\triangle^{1/2}}{f}G_1\mathbf{E}\right\},$$

$$\tag{11}$$

$$\frac{dW_\pi}{d\Omega} = \frac{2W_1}{3\pi\varepsilon^{5/2}\triangle^{5/4}}\left\{(G_2 - \frac{p}{\triangle^{1/2}r^2}G_3)\mathbf{K} + \frac{\triangle^{1/2}}{f}G_3\mathbf{E}\right\},$$

where $\mathbf{K}(r)$ and other elliptic integral $\mathbf{E}(r)$ calculated using tables [14], and

$$W_1 = ce^2/32\pi R_0^2, \qquad G_1 = p_1 - g_1 + (2/\triangle)[(p+f)^2 - g^2],$$

$$G_2 = p_1 + \frac{1}{\triangle}[8p(p+f) - 25pg + 15g] + \frac{8pg}{\triangle^2}[9(p-g)^2 - 7f^2 + 2f(p+g)],$$

$$G_3 = 1 + 2(p_1 - g_1) + \frac{1}{\triangle}[4(p^2 - f^2) + 3g(p - 7f) - 7g^2]+$$

$$8\frac{gf}{\triangle^2}[7 - 9p^2 + 2p(g - f)].$$

If the betatron oscillations are small ($p << 1$) we can develop Eqs.(11) in powers of p. In this case we get the more obvious formulas:

$$\frac{dW_\sigma}{d\Omega} = \frac{7W_1}{\varepsilon^{5/2}}(1 - \frac{5}{4}p + \frac{35}{4}pg),$$

$$\frac{dW_\pi}{d\Omega} = \frac{5W_1}{\varepsilon^{5/2}}(g + \frac{1}{2}p - \frac{35}{4}pg + \frac{63}{4}pg^2).$$

Setting $p = 0$ and replacing parameter R_0 with R in W_1 we regain the usual expressions for a uniform magnetic field.

This section of the most well applicable near the critical wavelength of the synchrotron light.

4. Perturbation method

In additional, we would like to attach to the given topic the magnetic systems of cyclic accelerators. As an illustration we shall exemplify a case based on averaging dynamic characteristics of the strong-focusing FODO structure. At this point, we will denote the lengths of bending magnets and straight sections by a and l, respectively.

Let N be the number of periods and $L = 2a + 2l$ be the length of a single cell. The length of closed trajectory is equal to $2\pi R + 2Nl = 2\pi R_0$, where R is the radius of magnetic curvature and R_0 is the so-called mean radius. With this, we again obtain $R_0 = (1 + k)R$, where $k = Nl/\pi R = l/a$.

Since we are interested primarily in small betatron oscillations, the same simplifications concerning the parameters of the main orbital motion can be used again. We suppose that a particle performs a rotation of radius R_0 in an average guiding magnetic field H_0 over one period. Note that in this case the periodic function $F(\varphi)$ entering in (3) is equal to

$$\frac{2}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin 2\nu\tau_1}{\nu} \cos 2\nu(\tau - \tau_1),$$

where $\tau = N\varphi$, $\tau_1 = \pi a/2(a + l)$.

The components of the magnetic field now take the form:

$$H_z = H_0(\frac{a}{a + l} - f(\tau)\frac{\rho}{R}), \qquad H_r = -H_0 f(\tau)\frac{z}{R},$$

where $\rho = r - R_0$,

$$f(\tau) = \frac{4n}{\pi} \sum_{\nu=0}^{\infty} \frac{1}{2\nu + 1} \sin(2\nu + 1)\tau_1 \cos((2\nu + 1)(\tau - \tau_1)).$$

In the linear approximation the betatron oscillations of a charged particle may be described by the following equations:

$$\frac{d^2\rho}{d\tau^2} + \frac{1}{N^2}(1 - (1 + k)^2 f(\tau))\rho = 0, \tag{12}$$

$$\frac{d^2 z}{d\tau^2} + \frac{1}{N^2}(1 + k)^2 f(\tau)z = 0. \tag{13}$$

To complete a description of the full system we must add Eq. (6). According to the Floquet theorem a solution for ρ or z may be expressed in the form $\exp(i\gamma\tau)\,\varphi(\tau)$ provided that $\varphi(\tau + 2\pi) = \varphi(\tau)$.

Replacing ρ and z in (12) and (13), we obtain new differential equations for the periodic functions $\varphi(\tau)$. We choose $1/N$ as the expansion parameter and develop $\varphi(\tau)$ and γ in power series. Equating coefficients of like powers to zero, we come to a chain of differential equations. As part of our procedure we also have to eliminate the secular terms. Finally, we obtain the following asymptotic forms:

$$\rho = A\cos(\frac{\nu_\rho}{N}\tau + \chi_0)(1 - S_1) - A\sin(\frac{\nu_\rho}{N}\tau + \chi_0)S_2, \tag{14}$$

$$z = B\cos(\frac{\nu_z}{N}\tau + \delta_0)(1 + S_1) + B\nu_z\sin(\frac{\nu_z}{N}\tau + \delta_0)S_2, \tag{15}$$

where

$$S_1 = G\sum_{\nu=0}^{\infty} q_\nu \cos(2\nu + 1)(\tau - \tau_1), \qquad g_\nu = \frac{\sin(2\nu + 1)\tau_1}{(2\nu + 1)^3},$$

$$S_2 = \frac{2G}{N} \sum_{\nu=o}^{\infty} \frac{g_\nu}{2\nu+1} \sin(2\nu+1)(\tau - \tau_1), \qquad \nu_z = \frac{\pi n}{2\sqrt{3}N} \sqrt{1 + 4k + 3k^2},$$

$$G = 4n(1+k)^2/\pi N^2, \qquad \nu_\rho = \sqrt{1 + \nu_z^2}.$$

Here, A and B are the amplitudes of the dominant sinusoidal motions, and χ_0 and δ_0 are the initial phases. Moreover, the frequency ν_z is the same as shown in [15] for this structure.

Eq. (15) can now be written in the form:

$$z = B\sqrt{1 + 2S_1} \cos(\frac{\nu_z}{N}\tau + \delta_1 + \delta_0) \tag{16}$$

with $\sin \delta_1 \approx \nu_z S_2$.

Comparing Eq.(1) with Eq.(16), we can see that $\beta_z A_z/\pi = B^2(1 + 2S_1)$. After averaging, we have $\bar{\beta}_z A_z/\pi = B^2$. Bearing in mind that $\bar{\beta}_z = R_0/\nu_z$, we obtain the approximation $B \approx \sqrt{A_z R_0/\pi\nu_z}$.

Thus, we have $\beta_z = R_0(1 + 2S_1)/\nu_z$ and

$$\frac{d\beta_z}{ds} = -\frac{8n(1+k)^2}{\pi N \nu_z} \sum_{\nu=0}^{\infty} g_\nu (2\nu+1) \sin(2\nu+1)(\tau - \tau_1).$$

Using Eq.(2), the parameter α entering into (8) can, finally, be determined by the substitution of values

$$\alpha = \sqrt{\frac{A_z \nu_z}{\pi R_0}} \cdot \sqrt{1 + \frac{\pi^2 n^2 (1+k)^2}{4N^2 \nu_z^2} - \frac{\pi^2 n k}{2N^2}}.$$

This makes it possible to investigate the angular properties of the emitted radiation.

V. Comparison with experiments

An example of spectral-angular distributions of radiation in the FODO lattice is shown in Fig.2. Here the angle $\theta = 90°$ corresponds to the orbital plane. As required we can easily recount degrees in radians on the horizontal axis. The indicated plots demonstrate the effect of betatron oscillations on the angular radiation properties.

Special experiments were carried out on this issui for a long time on the electron synchrotrons with different energies [3, 16]. In general, we can note a good agreement with the given theory of these works. All authors emphasize the important role of the axial electron oscillations in the formation of the spectral-angular properties.

Basic formulas (8) determine the properties of the radiation for modern accelerators. Here the parameters $\cos\theta$ or ψ, which is often used in literature, add in ε_2 a scalar term, formed by variable vector of particle velocity. Thereby the radiative scattering increases.

That can give these expressions at large amplitudes of oscillations? At higher oscillation amplitudes the π-component exhibits a peak instead of the minimum at $\theta = \pi/2$. Furthermore, it is expected that the σ-component will attain a small local minimum in the orbital plane for the extremal amplitudes and, in addition, the π-component will generate symmetric local hollows.

Thus, based on our theoretical study we propose to search these effects in experiments on existing machines with a higher degree of accuracy than previously. We are talking only

about the storage rings. Moreover, these tests were at fixed wavelengths and, apparently, experiments should be carried out also for the total angular distributions.

Because the radiation, that we consider, is incoherent at wavelengths which cause the greatest practical interest our results can be applied to the electron beams. Specifically, the cross-section of a typical bunch from which the radiation is emitted constitutes an ensemble of charged particles with various amplitudes. Recognizing this, we can re-interpret the amplitude parameter B in our formulas as a mean square value using following arguments. We take into account the longitudinal beam distribution, specify the injector scatter of electrons and carry out averaging over initial phases.
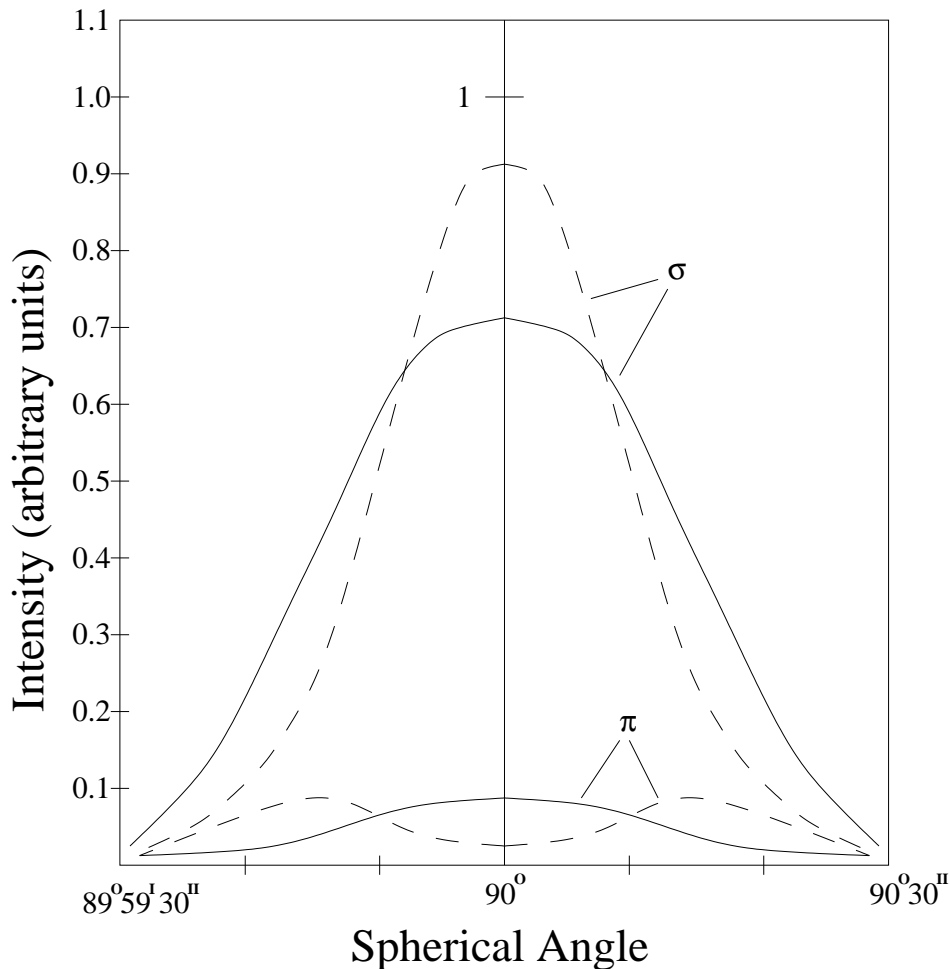
6. Acknowledgements

FIG. 2.   Spectral-angular disrtibutions of components of linear polarization. We have taken the energy $E = 5\ GeV$, the amplitudes $B = 0.2\ mm$ (broken curves) and $B = 0.4\ mm$ (continious curves), $R = 30\ m$, $k = 0.6$, $N = 24$, $n = 70$, and the radiation wavelength $\lambda = 1\mathring{A}$. On the vertical axis the unit denotes a maximum of the σ-component for the uniform magnetic field.

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