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"RELATIVISTIC NUCLEAR PHYSICS & QUANTUM CHROMODYNAMICS"

Fractality in Hadron Interactions:
A Conservation Law and Quantization of Fractal Dimensions

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Motivation & Goals

- Systematic analysis of inclusive spectra in pp, pA and AA collisions
  to search for general features of
  - hadron structure
  - constituent interactions
  - fragmentation and other processes

  over a wide scale range.
- Development of a unified approach to description of particle production
  using principles of self-similarity, fractality, and locality
  of hadron interactions at constituent level.
- Search for signatures of a phase transition in nuclear matter
  exploiting scaling properties in suitable representation of data.
- Search for new principles, symmetries and conservation laws
  which govern physics at small distances.

Principles & Symmetries

- Universal principles are usually reflected as regularities in measured observables. They can be expressed as scalings in different representations of data.
- z-scaling of differential cross sections of inclusive particles as a tool to study principles and symmetries that influence production processes at constituent level.
- z-reprezentation of transverse momentum spectra based on principles of
  - *Self-similatity*
  - *Fractality*
  - *Locality.*

There exists a symmetry inherent to them:
Symmetry with respect to structural degrees of freedom
- structural relativity

M.V. Tokarev and I. Z.: in Investigation of Properties of Nuclear Matter at High Temperatures and Baryon Densities
Dubna, Russia, 2007, edited by Sissakian, A.N. - Soifer, V.A,
ISBN 5-9530-0166-5, p. 99-136.

Fundamental principles and symmetries

"Fundamental symmetry principles dictate basic laws of physics, control structure of matter, and define the fundamental forces in Nature."

L.M. Lederman

Self-similarity - property of physical phenomena and the principle to construct theories.

Fractality - concept widely used in physics. …..non-integer dimensions,
fractal objects (some fractals posses property of self-similarity)

Multifractality - characterized by many non-integer dimensions

Universal principles: - reflected as regularities in measured observables.
- expressed as scalings in different representations of data.

"***Scaling***" and "***Universality***" were developed to understand critical phenomena.
Systems near phase transitions or a critical point (CP) exhibiting self-similar properties are invariant under transformation of scale. The scaling is usually described by a *power law*.

Critical exponents in the power laws are defined *by symmetry of interaction* and ***dimension of space*** only.

The notions of scaling and universality have also been applied for particle production far from a phase transition or a CP. The system should reveal *discontinuity in some characteristics* describing its behavior nearby the phase boundary or CP.

H.Stanley, G.Barenblatt,…

Self-similarity principle

- Dropping of certain quantities out of description of physical picture of a self-similar system.
- Construction of self-similarity parameters as simple combinations of suitable physical quantities.

Examples of self-similarity parameters:

Reynolds number in hydrodynamics:

$$R=U\rho/\mu$$

U-velocity of the fluid

ρ-density of the fluid

μ-viscosity of the fluid

Point explosion:

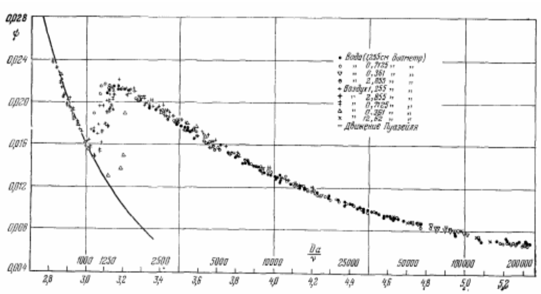
$$\Pi=r(Et^2/\rho)^{-1/5}$$

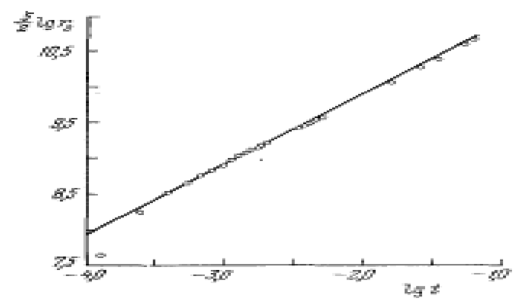
r-radius of the front wave

E-energy of the explosion

t-elapsed time

ρ-density of the environment





Self-similarity in Inclusive Reactions

Differential cross section $Ed^3\sigma/dp^3$ for production of an inclusive particle with mass m depends on:

1. reaction characteristics (A_1, A_2, P_1, P_2)
2. particle characteristics (m, p, θ)
3. structural and dynamical characteristics ($\delta, \varepsilon, ..dN/d\eta..$) of the reaction $M_1 + M_2 \rightarrow m + X$

The assumption of self-similarity of hadron interactions transforms to requirement of simultaneous description of inclusive spectra by a scaling function $\psi(z)$. Due to the property of self-similarity, it should be achieved by grouping suitable characteristics of the inclusive process into a relevant self-similarity parameter z .

We search for a solution $\psi(z) \sim Ed^3\sigma/dp^3$ reflecting *self-similarity*, *locality*, and *fractality* of hadron interactions which depends *in a universal way* on an adequate, physically meaningful, but still simple self-similarity variable z:

$$\psi(z) = \frac{1}{N\sigma_{in}} \frac{d\sigma}{dz}$$

Self-similarity types

G.I. Barenblatt (1978)

I. type:

Self-similar solutions $F_\sigma(\alpha,\beta,\gamma,..)$ expressed via scaling function $\Phi(\Pi_1,\Pi_2, ...)$ depending on self-similarity parameters $\Pi_1(\alpha,\beta,\gamma,..)$, $\Pi_2(\alpha,\beta,\gamma,..)$ ....
(F_σ, α, β, γ – dimensional quantities; Φ, Π_1, Π_2 – dimensionless functions)

V.S. Stavinsky (1972): cumulative particle production
$F_\sigma(\alpha, \beta, \gamma) = Ed^3\sigma/dp^3$; $\alpha, \beta, \gamma = p, \theta, \sqrt{s}$
$\Phi(\Pi_i) = \exp(\Pi_i/c)$; $\Pi_i = 1 - x_i$; x_1, x_2 - cumulative numbers
$\Phi(\Pi_0) = \exp(-\Pi_0/c)$; $\Pi_0 = \sqrt{(x_1P_1 + x_2P_2)^2}/m_N$
…but universality is broken by *power asymptotic* at high p_T !!!

II. type (intermediate asymptotics):

If $\Phi(\Pi_1,\Pi_2, ..)$ does not converge but has *power asymptotic* for extreme $\Pi_1,\Pi_2,..$, then self-similar solutions F_σ can be expressed via $\psi(z,...)$, $z = \Pi_0/\Pi_i^{\Delta_i}$

A.M. Baldin (1998):
Hypothesis of self-similarity in Relativistic nuclear physics:
... search for $\Phi(\Pi_1,\Pi_2, ...)$ or eventually for $\psi(z,…)$.
…parameters Δ_i have to be found from experiment.

$z = ?$

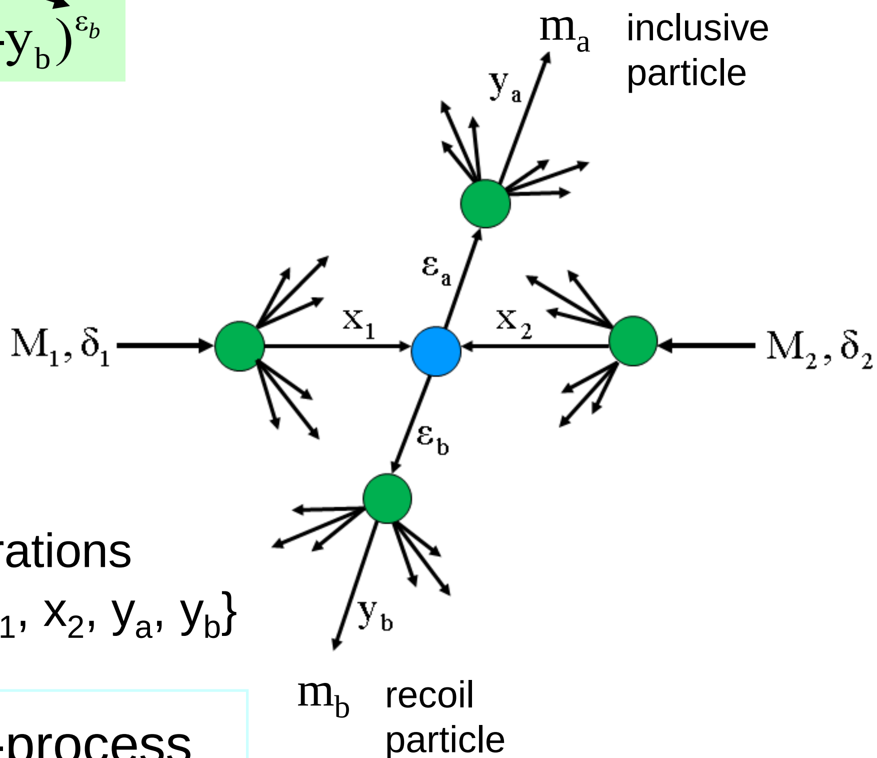
$\psi(z) = ?$

(Functional) self-similarity of II. type & variable z

$$z \cong \frac{s_{\perp}^{1/2}}{\Omega} \leftarrow \Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

$\Pi_0 \approx \sqrt{(x_1P_1+x_2P_2)^2} - \Sigma m_i$ Δ_i

Momentum fractions $\{x_1, x_2, y_a, y_b\}$ define constituent sub-process



z - self-similarity parameter of II. type
- expressed via momentum fractions x_i, y_i
- fractal measure

Parameters = fractal dimensions:
δ_1, δ_2 - structure of M_1, M_2
ε_a, ε_b - fragmentation processes

Ω ~ relative number of all constituent configurations containing the sub-process defined by $\{x_1, x_2, y_a, y_b\}$

Ω^{-1} ~ resolution at which constituent sub-process can be singled out of the inclusive reaction.

fractal property of z: $z(\Omega) \to \infty$ if $\Omega^{-1} \to \infty \ (x, y \to 1)$

Fractality in Hadron Interactions

Final-state phase space in the high-energy and high-p_T limit is a fractal…
B. Andersson, P. Dahlquist, G. Gustafsson

QCD branching structure of parton cascades shows self-similar nature and leads to QCD anomalous dimension of the phase space….
M.I. Dremin, B.B. Levtchenko

…to see fractality of phase-space experimentally, important role of energy and entropy distributions was emphasized…. J.D. Bjorken

Evidences of fractality in high energy physics:

- intermittency of spectra of secondary particles
  A. Bialas, R. Peshanski, M.I. Dremin, W. Kittel…..
- fractality of the emitting source O.V. Utyuzh, G. Wilk, Z. Wlodarczyk
- fractal structure of thermodynamic functions A. Deppman,…
- fractal structure of proton T. Lastovicka
- and others…

Fractality of Hadron Constituents

Collisions of hadrons and nuclei at high energies are assumed as collisions of hadron constituents - objects with inexhaustible (parton) structure at small scales.

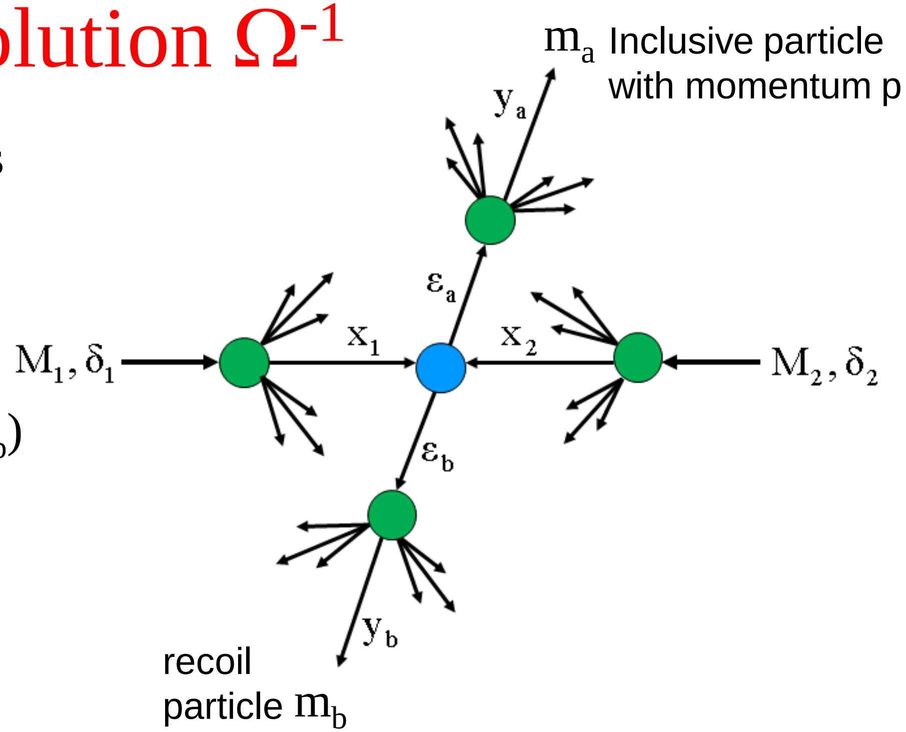
We consider hadrons and nuclei as extended objects which have fractal properties with respect to increasing resolution concerning the parton content involved.

(Objects consisting of "subtle nets" of quarks, anti-quarks and gluons which emit other (anti)quarks and gluons at small scales and those in turn generate particles of the same sort at even smaller scales etc.).

Assumption:

Hadron constituent sub-structure does not exhaust with increasing resolution.

Variable z & minimal resolution Ω^{-1}



Gross features of the single particle distributions

$$M_1+M_2 \Rightarrow m + X$$

in terms of the constituent sub-process

(Locality of hadron interactions)

$$(x_1M_1) + (x_2M_2) \Rightarrow (m_a/y_a) + (x_1M_1+x_2M_2+m_b/y_b)$$

(*) $$(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Ω - relative No. of constituent configurations which contain a sub-process defined by x_1, x_2, y_a, y_b

Fractality: $\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$ - fractal dimensions

$$z = z_0 \Omega_{max}^{-1}$$

z - fractal measure $\quad x_1, x_2, y_a, y_b \to 1 \quad \Omega_{max}^{-1} \to \infty \quad z \to \infty$

Ω_{max}^{-1} - min. resolution w.r.t. all sub-processes satisfying (*)

$$z = \frac{s_\perp^{1/2}}{W_{max}}$$

$$W_{max} = (dN_{ch}/d\eta|_0)^c \cdot \Omega_{max}$$

$$W(x_1, x_2, y_a, y_b) = (dN_{ch}/d\eta|_0)^c \cdot \Omega(x_1, x_2, y_a, y_b)$$

$s_\perp^{1/2}$ - transverse kinetic energy consumed on production of m_a & m_b

W - relative No. of all configurations which can lead to production of m_a & m_b

Variable $z = \frac{s_\perp^{1/2}}{W_{max}}$ & Energy $s_\perp^{1/2}$

$$W_{max} = (dN_{ch}/d\eta|_0)^c \cdot \Omega_{max}$$

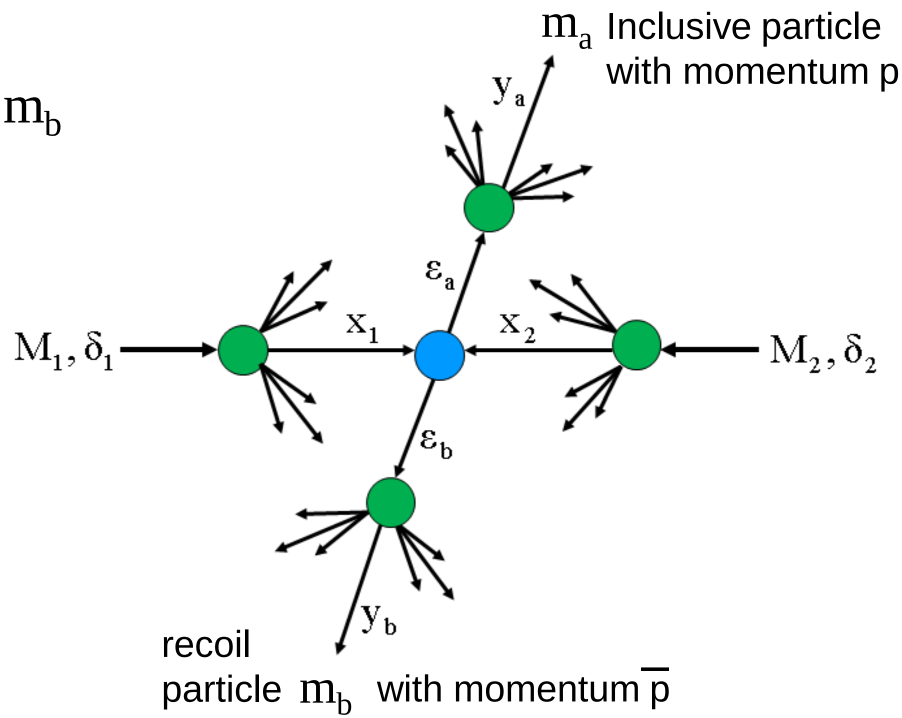
$s_\perp^{1/2}$ - transverse kinetic energy consumed on production of m_a & m_b

$$s_\perp^{1/2} = T_a + T_b$$

$$T_a = y_a\left(\sqrt{s_\lambda} - M_1\lambda_1 - M_2\lambda_2\right) - m_a$$

$$T_b = y_b\left(\sqrt{s_\chi} - M_1\chi_1 - M_2\chi_2\right) - m_b$$

$$s_\lambda = (\lambda_1 P_1 + \lambda_2 P_2)^2 \quad s_\chi = (\chi_1 P_1 + \chi_2 P_2)^2$$



$$T_a \cong \sqrt{p_T^2 + m_a^2} - m_a \qquad T_b \cong \sqrt{\bar{p}_T^2 + m_b^2} - m_b \qquad p_T/y_a = \bar{p}_T/y_b$$

Constituent sub-process:

$$(\lambda_1 + \chi_1) + (\lambda_2 + \chi_2) \to (\lambda_1 + \lambda_2) + (\chi_1 + \chi_2) \qquad x_i = \lambda_i + \chi_i$$

Scale transformation of z & $\psi(z)$

Scaling variable:

$$z = \frac{s_\perp^{1/2}}{W_{max}}$$

Scaling function:

$$\psi(z) = \frac{1}{N\sigma_{in}}\frac{d\sigma}{dz}$$

$$z' = z/W_0 \quad \psi'(z') = W_0\psi(z)$$

W_0 - absolute number of constituent configurations
(drops out of the z-scaling).

$W_0=W_0(F)$ - depends on type (F) of the inclusive particle (*m*).

Scaling functions for different hadrons collapse
to a single curve using the transformation

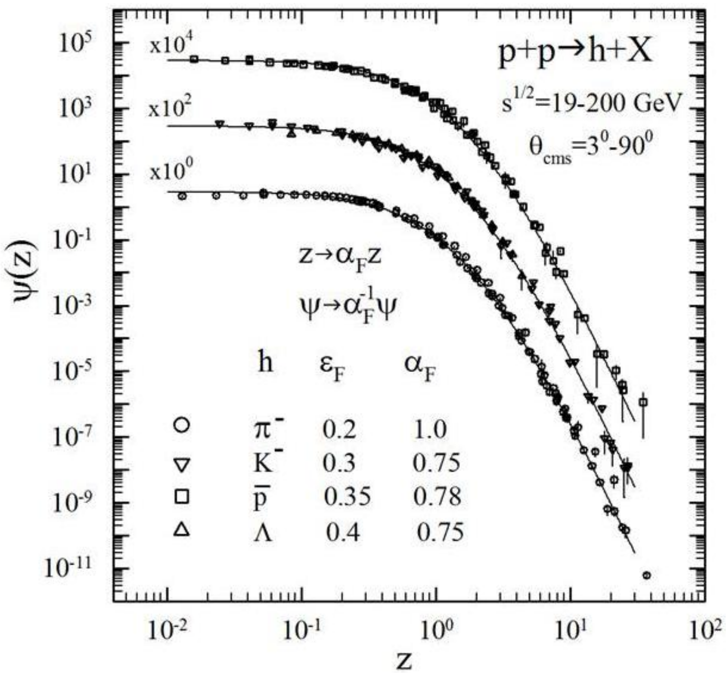
$$z \rightarrow \alpha_F z \quad \psi \rightarrow \alpha_F^{-1}\psi$$

$\alpha_F = W_0(F)/W_0(\pi)$ for the corresponding particle type (F)

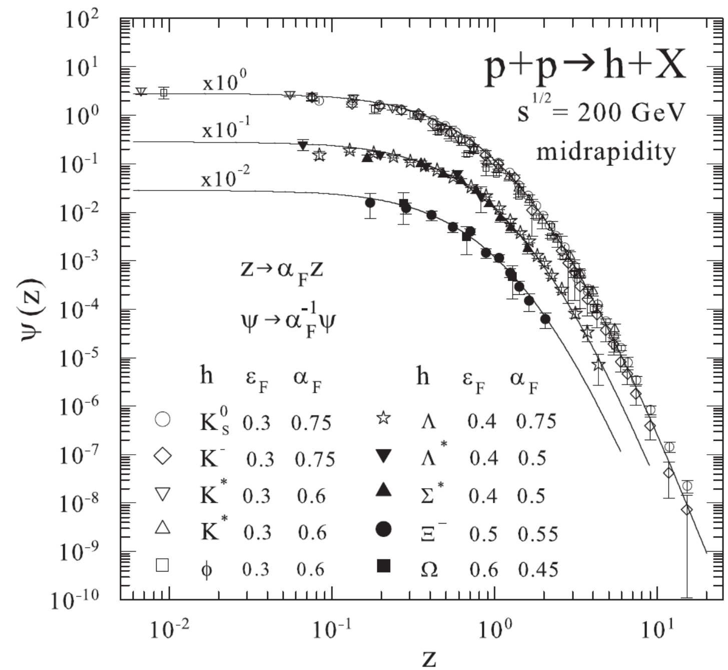
The transformation preserves the normalization $\int_0^\infty \psi(z)dz = 1$

z-scaling in pp Collisions at RHIC

π^-, K^- $\bar{p}$, Λ



K_S^0, K^-, K^*, ϕ, Λ, Ξ, Ω, Σ^*, Λ^*



- Energy & angular independence
- Flavor independence
- Saturation for $z < 0.1$
- Power law $\Psi(z) \sim z^{-\beta}$ for high $z > 4$

FNAL ISR:
PRD 19 (1979) 764
NPB 100 (1975) 237
NPB 106 (1976) 1
PLB 64 (1976) 111
NPB 116 (1976) 77
NPB 56 (1973) 333
PRD 40 (1989) 2777

STAR:
PRL 97 (2006) 132301
PLB 612 (2005) 181
PRC 71 (2005) 064902
PRC 75 (2007) 064901
PRL 108 (2012) 072302
PLB 616 (2005) 8
PLB 637 (2006) 161

PHENIX:
PRD 83 (2011) 052004
PRC 90 (2014) 054905

Fractal self-similarity of hadron interactions

Numerous analyses of inclusive reactions show that
production cross sections of inclusive particles can be described
by a universal scaling form using data z-presentation

z-scaling reflects principles of locality, self-similarity, and fractality

Locality: collisions of hadrons and nuclei
are considered as local interactions of their constituents

Self-similarity: interactions of the constituents are mutually similar.

Fractality: self-similarity of the interactions over a wide scale range.
(relative number of configurations depends on fractal dimensions)

Assumption of fractal self-similarity of hadron interactions includes a new symmetry
motivated by basic property of QCD diagrams:
(q, $\bar{q}$, g) can emit other (q, $\bar{q}$, g) at small scales and those can generate particles
of the same sort at even smaller scale etc….

⇨ There should exist conservation of a scale dependent quantity
characterizing hadron interactions at a constituent level

Variable z & Entropy S

$$z = \frac{s_{\perp}^{1/2}}{W_{max}}$$

$$W_{max} = (dN_{ch}/d\eta|_0)^c \cdot \Omega_{max}$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Scale transform:

$z' = z/W_0$

$S_{max} = \ln W_{max} + \ln W_0$

Entropy: Thermodynamical: Statistical: $S = \ln W + \ln W_0$

$$S = c_V \ln T \quad + \quad R \ln V \quad + \quad const.$$

$$S = c \cdot \ln(dN/d\eta|_0) + \ln[(1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}] + \ln W_0$$

- $dN/d\eta|_0$ characterizes "temperature" of the colliding system.
- local equilibrium ⇨ $dN/d\eta|_0 \sim T^3$ (for high T and small μ)
- c - "specific heat" of the produced medium.
- $\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$ – fractal dimensions in space of momentum fractions $\{x_1,x_2,y_a,y_b\}$
- Entropy S increases with $dN/d\eta|_0$ and decreases with increasing resolution Ω^{-1}

Max. entropy $S(x_1,x_2,y_a,y_b)$ ⇔ Max. number of configurations $W(x_1,x_2,y_a,y_b)$

under condition: $(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2$ ⇨ Ω_{max} ⇨ z

Maximum entropy principle with a kinematic constraint

Entropy S_Ω:

$$S_\Omega(x_1, x_2, y_a, y_b) = \ln\Omega(x_1, x_2, y_a, y_b) + \ln\Omega_0$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

Kinematic constraint:

$$(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2$$

$$x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0 = 0$$

$$\lambda_i = \frac{\kappa_i}{y_a} + \frac{\nu_i}{y_b} \qquad \kappa_i = \frac{(P_j p)}{(P_1P_2) - M_1M_2}$$

$$\lambda_0 = \frac{\nu_b}{y_b^2} - \frac{\nu_a}{y_a^2} \qquad \nu_i = \frac{M_j m_b}{(P_1P_2) - M_1M_2}$$

$$i, j = 1, 2 \qquad \nu_{a,b} = \frac{0.5 m_{a,b}^2}{(P_1P_2) - M_1M_2}$$

Maximization of the functional

$$\Phi(x_1,x_2,y_a,y_b) = \Omega(x_1,x_2,y_a,y_b) + \beta(x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0)$$

with a Lagrange multiplicator β.

Invariant combinations of momentum fractions

Conditions for maximal Entropy S_Ω:

$$\frac{\partial\Phi}{\partial x_1} = -\frac{\delta_1\Omega}{1-x_1} + \beta(x_2-\lambda_2) = 0 \qquad \frac{\partial\Phi}{\partial x_2} = -\frac{\delta_2\Omega}{1-x_2} + \beta(x_1-\lambda_1) = 0$$

$$\frac{\partial\Phi}{\partial y_a} = -\frac{\varepsilon_a\Omega}{1-y_a} + \beta\left(\kappa_2\frac{x_1}{y_a^2} + \kappa_1\frac{x_2}{y_a^2} - \nu_a\frac{2}{y_a^3}\right) = 0 \qquad \frac{\partial\Phi}{\partial y_b} = -\frac{\varepsilon_b\Omega}{1-y_b} + \beta\left(\nu_2\frac{x_1}{y_b^2} + \nu_1\frac{x_2}{y_b^2} + \nu_b\frac{2}{y_b^3}\right) = 0$$

$$\Phi(x_1,x_2,y_a,y_b) = \Omega(x_1,x_2,y_a,y_b) + \beta(x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0)$$

Kinematic constraint: $x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0 = 0$

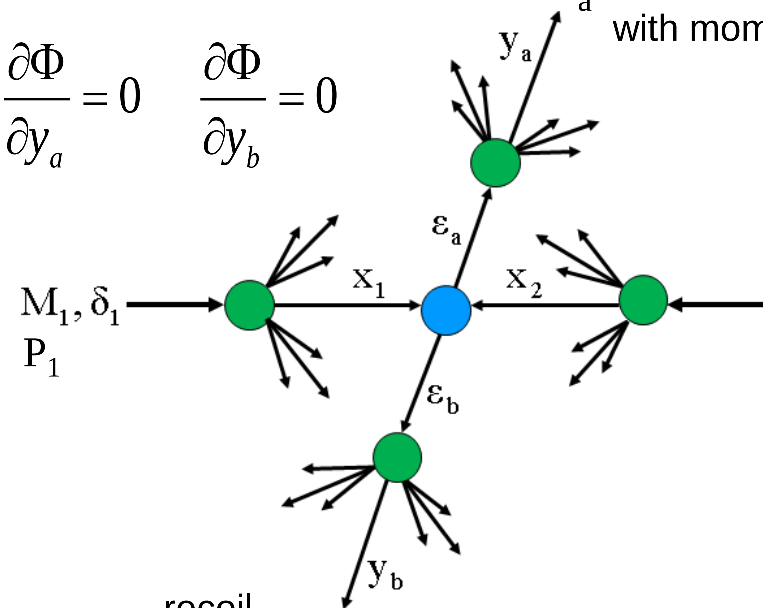
$$\frac{\delta_1 x_1}{1-x_1} + \frac{\delta_2 x_2}{1-x_2} = \frac{\beta}{\Omega}\left[(x_2-\lambda_2)x_1\right] + \frac{\beta}{\Omega}\left[(x_1-\lambda_1)x_2\right] = \frac{\beta}{\Omega}\left[x_1x_2+\lambda_0\right]$$

$$\frac{\varepsilon_a y_a}{1-y_a} + \frac{\varepsilon_b y_b}{1-y_b} = \frac{\beta}{\Omega}\left[\lambda_2x_1+\lambda_1x_2+2\lambda_0\right] = \frac{\beta}{\Omega}\left[x_1x_2+\lambda_0\right]$$

$$\Rightarrow \quad \frac{\delta_1 x_1}{1-x_1} + \frac{\delta_2 x_2}{1-x_2} = \frac{\varepsilon_a y_a}{1-y_a} + \frac{\varepsilon_b y_b}{1-y_b}$$

A conservation law from maximum entropy

Max. Entropy ⇒ $\frac{\partial \Phi}{\partial x_1} = 0 \quad \frac{\partial \Phi}{\partial x_2} = 0 \quad \frac{\partial \Phi}{\partial y_a} = 0 \quad \frac{\partial \Phi}{\partial y_b} = 0$



Solution (numerical only):

$$x_1 = x_1\left(P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b\right)$$
$$x_2 = x_2\left(P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b\right)$$
$$y_a = y_a\left(P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b\right)$$
$$y_b = y_b\left(P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b\right)$$

Conservation law:

$$\frac{\delta_1 x_1}{1-x_1} + \frac{\delta_2 x_2}{1-x_2} = \frac{\varepsilon_a y_a}{1-y_a} + \frac{\varepsilon_b y_b}{1-y_b}$$

for arbitrary $P_1, P_2, p, \delta_1, \delta_2, \varepsilon_a, \varepsilon_b$!!!

Conserved quantity:

$$C(\zeta) = Dg(\zeta) \qquad g(\zeta) = \frac{\zeta}{1-\zeta}$$

The conserved quantity $C(\zeta)$

$$C(\zeta) = D\frac{\zeta}{1-\zeta}$$

$D(=\delta_1, \delta_2, \varepsilon_a, \varepsilon_b)$ – fractal dimension

$\zeta(=x_1, x_2, y_a, y_b)$ – momentum fraction

$C(\zeta)$ characterizes:

- property of a fractal-like object (or fractal-like process) with fractal dimension D to form a "structural aggregate" with certain degree of local compactness which carries the momentum fraction ζ.
- ability of the fractal systems to create (structural) constituents
- cumulative property of internal structure of the colliding hadrons/nuclei
- aggregation property of fragmentation processes

$C(\zeta)$ is proportional to fractal dimension D of the respective fractal system.
The larger momentum fraction ζ carries a structural constituent (or an aggregated part) of the fractal-like system, the larger value of $C(\zeta)$ it has.

$C(\zeta)$ – "cumulativity" ("fractal cumulativity")
of a fractal-like structure with fractal dimension D
carried by its constituent with the momentum fraction ζ

Composition rule for cumulativity $C(\zeta)$

$$C(\zeta'') = C(\zeta) + C(\zeta') + D^{-1}C(\zeta)C(\zeta')$$

$$C(\zeta) = Dg(\zeta)$$

$$g(\zeta) = \frac{\zeta}{1-\zeta}$$

$$g'' = g + g' + gg'$$

$$(1-\zeta'') = (1-\zeta)(1-\zeta')$$

Composition rule for $C(\zeta)$ leads to q-exponential type of the distributions of the Tsallis-Pareto form with non-extensivity parameter q-1~1/D

- Property typical for fractals with a fractal dimension D
- Associative property
- Different ζ .... different levels of resolution

Cumulativity C(ζ) & Energy E(β)

$$C(\zeta) = Dg(\zeta) \qquad g(\zeta) = \frac{\zeta}{1-\zeta}$$

ζ – momentum fraction
D – fractal dimension

$$E(\beta) = M\gamma(\beta) \qquad \gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}$$

β – velocity fraction
M – mass

Conservation law:

$$\frac{\delta_1 x_1}{1-x_1} + \frac{\delta_2 x_2}{1-x_2} = \frac{\varepsilon_a y_a}{1-y_a} + \frac{\varepsilon_b y_b}{1-y_b}$$

$$\frac{M_1}{\sqrt{1-\beta_1^2}} + \frac{M_2}{\sqrt{1-\beta_2^2}} = \frac{M_a}{\sqrt{1-\beta_a^2}} + \frac{M_b}{\sqrt{1-\beta_b^2}}$$

Composition rule:

$$g'' = g + g' + gg'$$

$$(1-\zeta'') = (1-\zeta)(1-\zeta')$$

semigroup: $0 < \zeta \le 1$

$$\gamma'' = \sqrt{\gamma^2 - 1}\sqrt{\gamma'^2 - 1} + \gamma\gamma'$$

$$\frac{(1-\beta'')}{(1+\beta'')} = \frac{(1-\beta)}{(1+\beta)}\frac{(1-\beta')}{(1+\beta')}$$

Lorentz group: $-1 \le \beta \le 1$

Cumulativity C(ζ) & Energy E(β)

$$C(\zeta)=Dg(\zeta) \qquad g(\zeta)=\frac{\zeta}{1-\zeta}$$

Analogies:

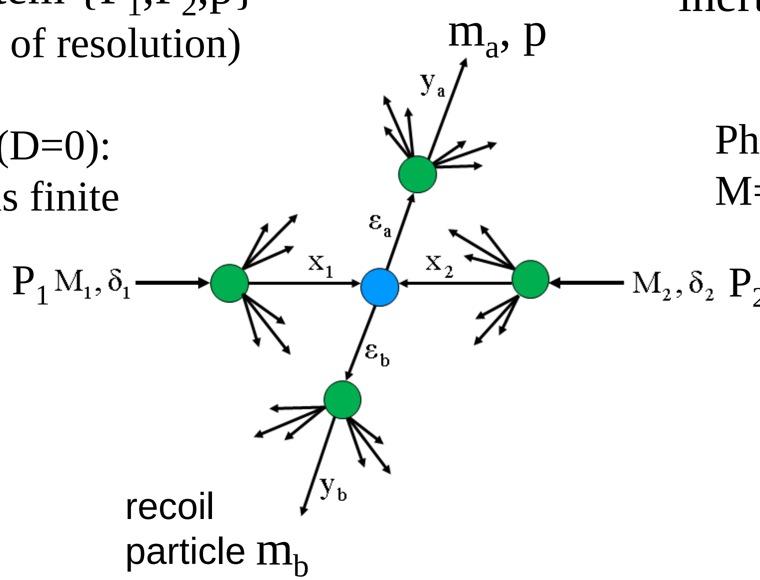
$$E(\beta)=M\gamma(\beta) \qquad \gamma(\beta)=\frac{1}{\sqrt{1-\beta^2}}$$

$C(\zeta)$ – depends on resolution dependent reference system $\{P_1,P_2,p\}$ (State of resolution)

$E(\beta)$ – depends on motion dependent inertial system $\{V_1,V_2,V_3\}$ (State of motion)

Non-structural objects (D=0):
$D=0 \rightarrow \zeta=1$, but $C(1)$ is finite

Photon (M=0):
$M=0 \rightarrow \beta=1$, but $E(1)$ is finite



Differences:

$C(\zeta)$ – **relativistic invariant w.r.t. motion**
scale of ζ – is not absolute
D – can depend on other characteristics...

Conservation law

$$C_1(x_1)+C_2(x_2)=C_a(y_a)+C_b(y_b)$$

does not depend on motion !!!
(It depends only on resolution.)

Quantization of fractal dimensions

Quantization of fractal dimensions $\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$

$$S_\Omega = \ln\Omega + \ln\Omega_0 \qquad \Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

$$\underbrace{(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2}_{x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0 = 0}$$

$$\lambda_i = \frac{\kappa_i}{y_a} + \frac{\nu_i}{y_b} \qquad \lambda_0 = \frac{\nu_b}{y_b^2} - \frac{\nu_a}{y_a^2} \qquad i, j = 1,2$$

$$\kappa_i = \frac{(P_j p)}{(P_1P_2) - M_1M_2} \qquad \nu_i = \frac{M_j m_b}{(P_1P_2) - M_1M_2} \qquad \nu_{a,b} = \frac{0.5 m_{a,b}^2}{(P_1P_2) - M_1M_2}$$

Maximum of the functional

$$\Phi(x_1,x_2,y_a,y_b) = \Omega(x_1,x_2,y_a,y_b) + \beta(x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0)$$

$$\frac{\partial\Phi}{\partial x_1} = -\frac{\delta_1\Omega}{1-x_1} + \beta(x_2-\lambda_2) = 0 \qquad \frac{\partial\Phi}{\partial x_2} = -\frac{\delta_2\Omega}{1-x_2} + \beta(x_1-\lambda_1) = 0$$

$$\frac{\partial\Phi}{\partial y_a} = -\frac{\varepsilon_a\Omega}{1-y_a} + \beta\left(\kappa_2\frac{x_1}{y_a^2} + \kappa_1\frac{x_2}{y_a^2} - \nu_a\frac{2}{y_a^3}\right) = 0 \qquad \frac{\partial\Phi}{\partial y_b} = -\frac{\varepsilon_b\Omega}{1-y_b} + \beta\left(\nu_2\frac{x_1}{y_b^2} + \nu_1\frac{x_2}{y_b^2} + \nu_b\frac{2}{y_b^3}\right) = 0$$

Quantization of $\delta_1, \delta_2, \varepsilon_a, \varepsilon_b$ manifests itself most prominently near fractal limit

⇨ find the solution in the region $x_1, x_2, y_a, y_b \to 1$ and write down explicit expression for entropy S_Ω in the fractal limit

Conditions for momentum fractions from max. entropy

Constrained maximum for entropy:

$$\frac{\partial\Phi}{\partial x_1}=0 \quad \frac{\partial\Phi}{\partial x_2}=0 \quad \frac{\partial\Phi}{\partial y_a}=0 \quad \frac{\partial\Phi}{\partial y_b}=0 \qquad x_1x_2 - x_1\lambda_2 - x_2\lambda_1 - \lambda_0 = 0$$

$$F_1\left(x_1,x_2,y_a,y_b,\kappa_1,\kappa_2\right)\equiv x_1x_2-\kappa_2\frac{x_1}{y_a}-\kappa_1\frac{x_2}{y_a}-\nu_2\frac{x_1}{y_b}-\nu_1\frac{x_2}{y_b}-\nu_b\frac{1}{y_b^2}+\nu_a\frac{1}{y_a^2}=0$$

$$F_2\left(x_1,x_2,y_a,y_b,\kappa_1,\kappa_2\right)\equiv\left[x_1x_2-\kappa_2\frac{x_1}{y_a}-\nu_2\frac{x_1}{y_b}\right]\frac{\left(1-x_1\right)}{\delta_1x_1}-\left[x_1x_2-\kappa_1\frac{x_2}{y_a}-\nu_1\frac{x_2}{y_b}\right]\frac{\left(1-x_2\right)}{\delta_2x_2}=0$$

$$G_1\left(x_1,x_2,y_a,y_b\right)\equiv\left[\frac{\varepsilon_ay_a}{1-y_a}+\frac{\varepsilon_by_b}{1-y_b}\right]^{-1}-\left[\frac{\delta_1x_1}{1-x_1}+\frac{\delta_2x_2}{1-x_2}\right]^{-1}=0$$

$$G_2\left(x_1,x_2,y_a,y_b\right)\equiv\left[\nu_2\frac{x_1}{y_b}+\nu_1\frac{x_2}{y_b}+\frac{2\nu_b}{y_b^2}\right]\frac{\left(1-y_b\right)}{\varepsilon_by_b}-\left[x_1x_2-\nu_2\frac{x_1}{y_b}-\nu_1\frac{x_2}{y_b}-\frac{\nu_a}{y_a^2}-\frac{\nu_b}{y_b^2}\right]\frac{\left(1-y_a\right)}{\varepsilon_ay_a}=0$$

Solution: $x_1=x_1\left(\kappa_1,\kappa_2\right) \quad x_2=x_2\left(\kappa_1,\kappa_2\right) \quad y_a=y_a\left(\kappa_1,\kappa_2\right) \quad y_b=y_b\left(\kappa_1,\kappa_2\right)$

Parameterized via: $\kappa_i=\dfrac{\left(P_jp\right)}{\left(P_1P_2\right)-M_1M_2}$

Conditions for momentum fractions near fractal limit

Fractal limit (L): $x_1=x_2=y_a=y_b=1$

Conditions for momentum fractions in the region $x_1, x_2, y_a, y_b \to 1$:

$$\left.\frac{\partial F_i}{\partial x_1}\right|_L (1-x_1)+\left.\frac{\partial F_i}{\partial x_2}\right|_L (1-x_2)+\left.\frac{\partial F_i}{\partial y_a}\right|_L (1-y_a)+\left.\frac{\partial F_i}{\partial y_b}\right|_L (1-y_b)+\left.\frac{\partial F_i}{\partial \kappa_1}\right|_L (\overline{\kappa_1}-\kappa_1)+\left.\frac{\partial F_i}{\partial \kappa_2}\right|_L (\overline{\kappa_2}-\kappa_2)=0$$

$$\left.\frac{\partial G_i}{\partial x_1}\right|_L (1-x_1)+\left.\frac{\partial G_i}{\partial x_2}\right|_L (1-x_2)+\left.\frac{\partial G_i}{\partial y_a}\right|_L (1-y_a)+\left.\frac{\partial G_i}{\partial y_b}\right|_L (1-y_b)=0$$

$$1-e_1-e_2=(\overline{\lambda_1}+\overline{\lambda_0})(1-x_1)+(\overline{\lambda_2}+\overline{\lambda_0})(1-x_2)+(1-\nu)(1-y_a)+(\nu+\overline{\lambda_0})(1-y_b)$$

$$0=\delta_1^{-1}(\overline{\lambda_1}+\overline{\lambda_0})^2(1-x_1)+\delta_2^{-1}(\overline{\lambda_2}+\overline{\lambda_0})^2(1-x_2)-\varepsilon_a^{-1}(1-\nu)^2(1-y_a)-\varepsilon_b^{-1}(\nu+\overline{\lambda_0})^2(1-y_b)$$

$$0=-\delta_1^{-1}(\overline{\lambda_1}+\overline{\lambda_0})(1-x_1)+\delta_2^{-1}(\overline{\lambda_2}+\overline{\lambda_0})(1-x_2)$$

$$0=\varepsilon_a^{-1}(1-\nu)(1-y_a)-\varepsilon_b^{-1}(\nu+\overline{\lambda_0})(1-y_b)$$

where: $e_1+e_2=\kappa_1+\kappa_2+\nu_1+\nu_2+\overline{\lambda_0}$ Over-lined symbols calculated at fractal limit (L)

Momentum fractions near fractal limit $z(\Omega) \to \infty \quad \Omega \to 0 \ (x, y \to 1)$

Maximum entropy principle ⇨ solution in the region $x_1, x_2, y_a, y_b \to 1$:

$$1 - x_1 = \frac{(1 - e_1 - e_2)}{(\bar{\lambda}_1 + \bar{\lambda}_0)} \frac{\delta_1}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}, \qquad 1 - y_a = \frac{(1 - e_1 - e_2)}{(1 - \nu)} \frac{\varepsilon_a}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}$$

$$1 - x_2 = \frac{(1 - e_1 - e_2)}{(\bar{\lambda}_2 + \bar{\lambda}_0)} \frac{\delta_2}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}, \qquad 1 - y_b = \frac{(1 - e_1 - e_2)}{(\nu + \bar{\lambda}_0)} \frac{\varepsilon_b}{(\delta_1 + \delta_2 + \varepsilon_a + \varepsilon_b)}$$

Over-lined symbols calculated at kinematic limit

$$\bar{\lambda}_1 = \bar{\kappa}_1 + \nu_1 = \frac{(P_2 \bar{p}) + M_2 m_b}{(P_1 P_2) - M_1 M_2} \qquad \nu = \nu_1 + \nu_2 + \nu_a + \nu_b = \frac{(M_1 + M_2) m_b + 0.5(m_a^2 + m_b^2)}{(P_1 P_2) - M_1 M_2}$$

$$\bar{\lambda}_2 = \bar{\kappa}_2 + \nu_2 = \frac{(P_1 \bar{p}) + M_1 m_b}{(P_1 P_2) - M_1 M_2} \qquad \bar{\lambda}_0 = \nu_b - \nu_a = \frac{0.5(m_b^2 - m_a^2)}{(P_1 P_2) - M_1 M_2}$$

$$e_1 + e_2 = \kappa_1 + \kappa_2 + \nu_1 + \nu_2 + \bar{\lambda}_0 = \frac{(P_1 p) + (P_2 p) + (M_1 + M_2) m_b}{(P_1 P_2) - M_1 M_2} + \bar{\lambda}_0 \to 1$$

Substitute the solution into the expression: $\Omega = (1 - x_1)^{\delta_1} (1 - x_2)^{\delta_2} (1 - y_a)^{\varepsilon_a} (1 - y_b)^{\varepsilon_b}$
to obtain value of maximal entropy $S_\Omega = \ln \Omega + \ln \Omega_0$

Entropy S_Ω near fractal limit $z(\Omega) \to \infty \quad (x, y \to 1)$

Maximum entropy principle:

note minus sign

$$S_\Omega = (\delta_1+\delta_2+\varepsilon_a+\varepsilon_b)\ln(1-e_1-e_2) + \ln\Omega_0 - S_\Gamma$$
$$-\delta_1\ln\left(\overline{\lambda_1}+\overline{\lambda_0}\right) - \delta_2\ln\left(\overline{\lambda_2}+\overline{\lambda_0}\right) - \varepsilon_a\ln(1+\nu) - \varepsilon_b\ln\left(\nu+\overline{\lambda_0}\right)$$

$$S_\Gamma = (\delta_1+\delta_2+\varepsilon_a+\varepsilon_b)\ln(\delta_1+\delta_2+\varepsilon_a+\varepsilon_b) - \delta_1\ln\delta_1 - \delta_2\ln\delta_2 - \varepsilon_a\ln\varepsilon_a - \varepsilon_b\ln\varepsilon_b$$

Entropy S_Γ depends *solely* on fractal dimensions

$$S_\Gamma = \delta\left[\left(1+\frac{\varepsilon}{\delta}\right)\ln\left(1+\frac{\varepsilon}{\delta}\right) - \frac{\varepsilon}{\delta}\ln\frac{\varepsilon}{\delta}\right]$$
$$+\delta_1\left[\left(1+\frac{\delta_2}{\delta_1}\right)\ln\left(1+\frac{\delta_2}{\delta_1}\right) - \frac{\delta_2}{\delta_1}\ln\frac{\delta_2}{\delta_1}\right]$$
$$+\varepsilon_a\left[\left(1+\frac{\varepsilon_b}{\varepsilon_a}\right)\ln\left(1+\frac{\varepsilon_b}{\varepsilon_a}\right) - \frac{\varepsilon_b}{\varepsilon_a}\ln\frac{\varepsilon_b}{\varepsilon_a}\right]$$

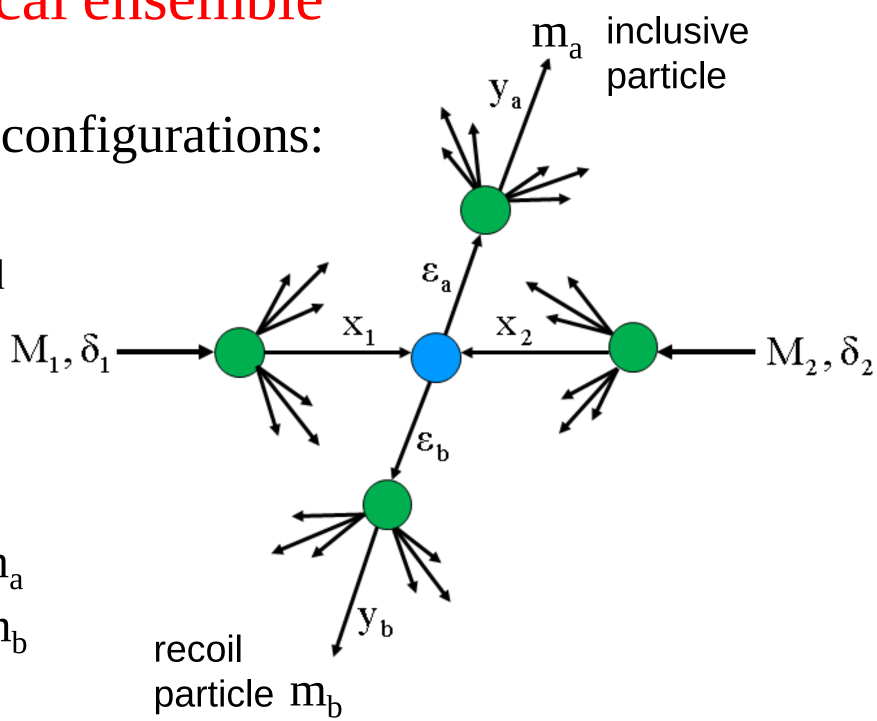
$$\delta \equiv \delta_1 + \delta_2 \qquad \varepsilon \equiv \varepsilon_a + \varepsilon_b$$

S_Γ - entropy of a statistical ensemble

Statistical ensemble of the interacting fractal configurations:

Large collection of the interacting fractals

- with random configurations $\{x_1, x_2, y_a, y_b, ...\}$ and
- with the same fractal dimensions $\{\delta_1, \delta_2, \varepsilon_a, \varepsilon_b\}$



$n_{\delta 1}$ - configurations of internal structure of M_1

$n_{\delta 2}$ - configurations of internal structure of M_2

$n_{\varepsilon a}$ - configurations of fragmentation process to m_a

$n_{\varepsilon b}$ - configurations of fragmentation process to m_b

Entropy S_Γ of a single "average" fractal configuration of the system:

$$S_I\left(\frac{\varepsilon}{\delta}\right)+S_I\left(\frac{\delta_2}{\delta_1}\right)+S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$S_I = \mathrm{d}\left[(1+r)\ln(1+r)-r\ln r\right]$$

$$\delta \equiv \delta_1+\delta_2 \qquad \varepsilon \equiv \varepsilon_a+\varepsilon_b$$

Entropy of the whole statistical ensemble:

$$S_\Gamma = n_\delta S_I\left(\frac{\varepsilon}{\delta}\right)+n_{\delta 1} S_I\left(\frac{\delta_2}{\delta_1}\right)+n_{\varepsilon a} S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$n_\delta \equiv n_{\delta 1}+n_{\delta 2}$$

Quantization of fractal dimensions δ_1, δ_2, ε_a, ε_b

The expression for the entropy S_Γ

$$S_\Gamma = n_\delta S_I\left(\frac{\varepsilon}{\delta}\right) + n_{\delta 1} S_I\left(\frac{\delta_2}{\delta_1}\right) + n_{\varepsilon a} S_I\left(\frac{\varepsilon_b}{\varepsilon_a}\right)$$

$$\delta \equiv \delta_1 + \delta_2$$

$$\varepsilon \equiv \varepsilon_a + \varepsilon_b$$

$$S_I = \mathrm{d}\left[(1+r)\ln(1+r) - r\ln r\right]$$

allows to draw physical consequences provided the fractal dimensions have quantum nature:

$$n_\delta \equiv n_{\delta 1} + n_{\delta 2}$$

$$n_\varepsilon \equiv n_{\varepsilon a} + n_{\varepsilon b}$$

$$\delta_1 = \mathrm{d}\cdot n_{\delta 1} \quad \delta_2 = \mathrm{d}\cdot n_{\delta 2} \quad \varepsilon_1 = \mathrm{d}\cdot n_{\varepsilon a} \quad \varepsilon_b = \mathrm{d}\cdot n_{\varepsilon b}$$

$$\Rightarrow \quad S_\Gamma = n_\delta S_I\left(\frac{n_\varepsilon}{n_\delta}\right) + n_{\delta 1} S_I\left(\frac{n_{\delta 2}}{n_{\delta 1}}\right) + n_{\varepsilon a} S_I\left(\frac{n_{\varepsilon b}}{n_{\varepsilon a}}\right)$$

S_Γ can be interpreted as the logarithm of number of ways in which fractal dimensions of the interacting fractal structures can be composed from the identical dimensional quanta, each of the size d.

Statistical interpretation of the entropy S_Γ

The entropy
$$S_\Gamma = n_\delta S_I\left(\frac{n_\varepsilon}{n_\delta}\right) + n_{\delta 1} S_I\left(\frac{n_{\delta 2}}{n_{\delta 1}}\right) + n_{\varepsilon a} S_I\left(\frac{n_{\varepsilon b}}{n_{\varepsilon a}}\right) \quad (*)$$

$$n_\delta \equiv n_{\delta 1} + n_{\delta 2}$$
$$n_\varepsilon \equiv n_{\varepsilon a} + n_{\varepsilon b}$$

$$S_I = \mathrm{d}\left[(1+r)\ln(1+r) - r\ln r\right]$$

as logarithm of the number of different ways
how identical dimensional quanta can be shared
among fractal dimensions of the interacting fractal structures.

$n \equiv n_\delta + n_\varepsilon$ - overall number of dimensional quanta each of the size d distributed
between $n_\delta \equiv n_{\delta 1} + n_{\delta 2}$ quanta of fractal dimensions in the initial state
and $n_\varepsilon \equiv n_{\varepsilon a} + n_{\varepsilon b}$ quanta of fractal dimensions in the final state

Different arrangements of such distributions:

$$\Gamma_{\delta,\varepsilon} = \frac{(n_\delta + n_\varepsilon)!}{n_\delta!\ n_\varepsilon!} \qquad \Gamma_{\delta_1,\delta_2} = \frac{(n_{\delta 1} + n_{\delta 2})!}{n_{\delta 1}!\ n_{\delta 2}!} \qquad \Gamma_{\varepsilon_a,\varepsilon_b} = \frac{(n_{\varepsilon a} + n_{\varepsilon b})!}{n_{\varepsilon a}!\ n_{\varepsilon b}!}$$

$$S_\Gamma = \mathrm{d}\ln\left(\Gamma_{\delta_1,\delta_2,\varepsilon_a,\varepsilon_b}\right)$$

$$\Gamma_{\delta_1,\delta_2,\varepsilon_a,\varepsilon_b} \equiv \Gamma_{\delta,\varepsilon}\Gamma_{\delta_1,\delta_2}\Gamma_{\varepsilon_a,\varepsilon_b} = \frac{(n_{\delta 1} + n_{\delta 1} + n_{\varepsilon a} + n_{\varepsilon b})!}{n_{\delta 1}!\,n_{\delta 2}!\,n_{\varepsilon a}!\,n_{\varepsilon b}!}$$

for large $n_{\delta 1},\ n_{\delta 2},\ n_{\varepsilon a},\ n_{\varepsilon b}$ and $\ln n! \simeq n\ln n - n$ this gives (*)

Conservation of the number of quanta of fractal cumulativity C(ζ)

Quantization of fractal dimensions $D = d \cdot n$ results in quantum character of fractal cumulativity $C(\zeta)$:

$$C(\zeta) = D \frac{\zeta}{1-\zeta}$$

Conservation law for the fractal cumulativity in units of dimensional quantum d :

$$\frac{n_{\delta 1} x_1}{1 - x_1} + \frac{n_{\delta 2} x_2}{1 - x_2} = \frac{n_{\varepsilon a} y_a}{1 - y_a} + \frac{n_{\varepsilon b} y_b}{1 - y_b}$$

The number of quanta of fractal cumulativity is conserved at any resolution given by arbitrary momenta P_1, P_2, and p of the colliding and inclusive particles.

The quantization is based on assumption of

- fractal self-similarity of internal hadron structure,
- fractal nature of fragmentation processes, and
- locality of hadron interactions at a constituent level

up to the kinematic limit.

Symmetric reactions:

For a symmetric inclusive reaction consider $M_1 = M_2 \equiv M \quad m_a = m_b \equiv m \quad \vartheta_{cms} = 90^0$

$$\delta_1 = \delta_2 \equiv \frac{\delta}{2} \qquad \varepsilon_1 = \varepsilon_2 \equiv \frac{\varepsilon}{2}$$

Entropy S_Ω near fractal limit $z(\Omega) \to \infty,\ \Omega^{-1} \to \infty\ (x, y \to 1)$:

$$S_\Omega = (\delta + \varepsilon)\ln(1 - e_1 - e_2) + \ln\Omega_0 - \delta\left[\left(1 + \frac{\varepsilon}{\delta}\right)\ln\left(1 + \frac{\varepsilon}{\delta}\right) - \frac{\varepsilon}{\delta}\ln\frac{\varepsilon}{\delta}\right] - \frac{\varepsilon}{2}\ln\left(1 + \frac{p_f^2}{p_i^2}\right) - \frac{\varepsilon}{2}\ln\left(1 - \frac{p_f^2}{p_i^2}\right)$$

$$e_1 + e_2 = \frac{2E\sqrt{s} + 4mM}{s - 4M^2}$$

E – energy of inclusive particle

$$p_f^2 \equiv s - 4(M + m)^2$$

$$p_i^2 \equiv s - 4M^2$$

$$\delta = \mathrm{d} \cdot n_\delta \qquad \varepsilon = \mathrm{d} \cdot n_\varepsilon$$

Conservation of cumulativity:

$$\frac{2\delta x}{1 - x} = \frac{\varepsilon y_a}{1 - y_a} + \frac{\varepsilon y_b}{1 - y_b}$$

$$x_1 = x_2 \equiv x$$

Epilogue

- z-Scaling is a specific feature of high-p_T particle production established in p-(anti)p collisions at the U70, ISR, SppS, Tevatron and RHIC. It reflects self-similarity, locality, and fractality of hadron interactions at a constituent level.
- The scaling behavior was confirmed also for inclusive production of direct photons, jets, heavy quarkonia and top quark.
- Hypothesis of self-similarity and fractality was tested in AA collisions using z-presentation of spectra of charged hadrons and pions.
- Analysis of numerous experimental data indicates universality as well as energy and multiplicity independence of the scaling function $\psi(z)$.
- The variable z depends on multiplicity density, “heat capacity”, and entropy of constituent configurations of the interacting system.
- We present new insight into some aspects of the theory of z-scaling and show what kind of physics can stand behind it and what type of physical problems could be addressed by this approach.

Summary

- Based on principles of self-similarity, locality, and fractality of hadron interactions at constituent level, we demonstrated that z-scaling construction reflects conservation of new quantity, named here “cumulativity” (or fractal cumulativity)
- The conservation law follows from general ideas. It holds at any level of resolution given by arbitrary momenta and masses of the colliding objects and arbitrary momenta and types of the inclusive particles.
- According to the Noether’s theorem, there must be a continuous symmetry, a scale dependent translation symmetry, which guaranties the conservation law for the fractal cumulativity.
- The cumulativity $C(\zeta)$ is subject to a composition rule connecting $C(\zeta)$ at different scales. It leads to distributions of the Tsallis-Pareto type with non-extensivity parameters depending on fractal dimensions.
- It was demonstrated that fractal dimensions can be interpreted as quantities which have quantum nature.
- It was shown that the quantization of fractal dimensions results in preservation of the number of quanta of fractal cumulativity.

Further motivation:

Structural relativity: symmetry at small scales

Momentum fractions x_1, x_2, y_a, y_b

$$S = c \cdot \ln(dN/d\eta|_0) + \ln\Omega + \ln W_0$$

Maximal entropy S = maximum of

$$(x_1P_1+x_2P_2-p/y_a)^2 = (x_1M_1+x_2M_2+m_b/y_b)^2$$

$$\Omega = (1-x_1)^{\delta_1}(1-x_2)^{\delta_2}(1-y_a)^{\varepsilon_a}(1-y_b)^{\varepsilon_b}$$

$$\Rightarrow \quad x_i = \lambda_i + \chi_i \quad \Rightarrow \quad \Omega = \Omega_{max}$$

$$\lambda_i \cong \frac{(P_j q)}{(P_1 P_2)} \simeq \frac{q_0 \pm q_z}{\sqrt{s}/2}$$

$$p = y_a q$$

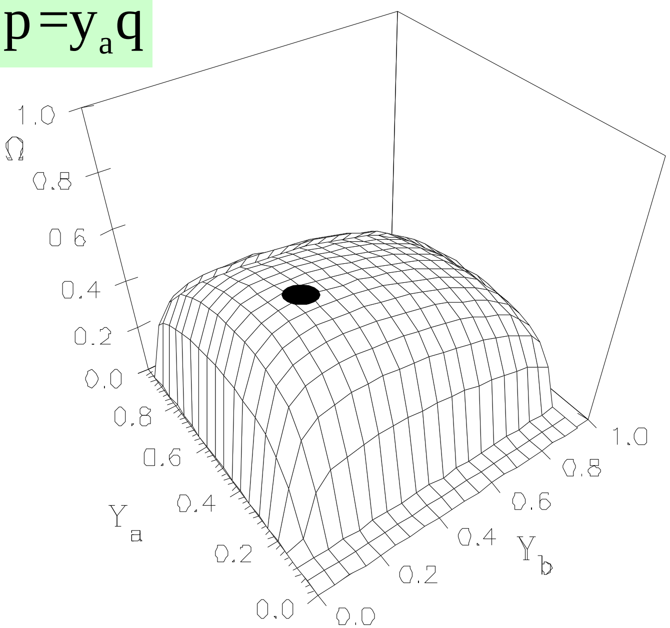
$$\chi_i = \sqrt{\mu_i^2 + \omega_i^2} \mp \omega_i$$

$$\omega_i = \mu_i U$$

$$U(\xi) = \frac{\alpha - 1}{2\sqrt{\alpha}}\xi$$

$$\alpha = \delta_2/\delta_1, \quad 0 \le \xi \le 1$$

μ_i, ξ are simple functions of λ_1 and λ_2



Ω_{max} is calculated numerically
for every momentum p of inclusive particle.

Fractal self-similarity and Structural relativity

Maximal entropy: ⇨ $x_i = \lambda_i + \chi_i$

$$\lambda_i \cong \frac{(P_j q)}{(P_1 P_2)}$$

$$\chi_i = \sqrt{\mu_i^2 + \omega_i^2} \mp \omega_i \qquad \omega_i = \mu_i U$$

$$U(\xi) = \frac{\alpha - 1}{2\sqrt{\alpha}} \xi$$

$$\alpha = \delta_2/\delta_1, \quad 0 \le \xi \le 1$$

δ_1, δ_2 – fractal dimensions

ξ – characterizes resolution

$$\xi \cong \sqrt{\frac{\lambda_1 \lambda_2}{(1-\lambda_1)(1-\lambda_1)}}$$

u - structural “velocity”

$$\frac{u}{\sqrt{1-u^2}} \equiv U(\xi) = \frac{\alpha - 1}{2\sqrt{\alpha}} \xi$$

Lorentz transform:

$$\chi_1 - \chi_2 = \frac{1}{\sqrt{1-u^2}} \left[(\mu_1 - \mu_2) - u(\mu_1 + \mu_2) \right]$$

$$\chi_1 + \chi_2 = \frac{1}{\sqrt{1-u^2}} \left[(\mu_1 + \mu_2) - u(\mu_1 - \mu_2) \right]$$

$$\chi_1 \chi_2 = \mu_1 \mu_2 \ne \text{function}(u)$$

Momentum fractions
w.r.t. fractal structures of A, B:

$$\chi_i \equiv \frac{(P_j \bar{q}^A)}{(P_1 P_2)} \qquad \mu_i \equiv \frac{(P_j \bar{q}^B)}{(P_1 P_2)}$$

"Structural velocity" **u** at small scales

Momentum fractions: $x_1 = \lambda_1 + \chi_1 \quad x_2 = \lambda_2 + \chi_2$

$$\chi_i = \sqrt{\mu_i^2 + \omega_i^2} \mp \omega_i$$

$$\omega_i = \mu_i U$$

$$\frac{u}{\sqrt{1-u^2}} \equiv U(\xi) = \frac{\alpha - 1}{2\sqrt{\alpha}}\xi$$

$\xi \to 1$ at fractal limit

$$\frac{u}{\sqrt{1-u^2}} = \frac{\alpha - 1}{2\sqrt{\alpha}}$$

$$u = \frac{\alpha - 1}{\alpha + 1}$$

$$\alpha = \delta_2/\delta_1, \quad 0 \le \xi \le 1$$

$$\xi \cong \sqrt{\frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_1)}}$$

$$\lambda_i \cong \frac{(P_j q)}{(P_1 P_2)}$$

$$u = \frac{u' + u''}{1 + u'u''} \qquad \alpha = \alpha'\alpha''$$

at fractal (kinematic) limit

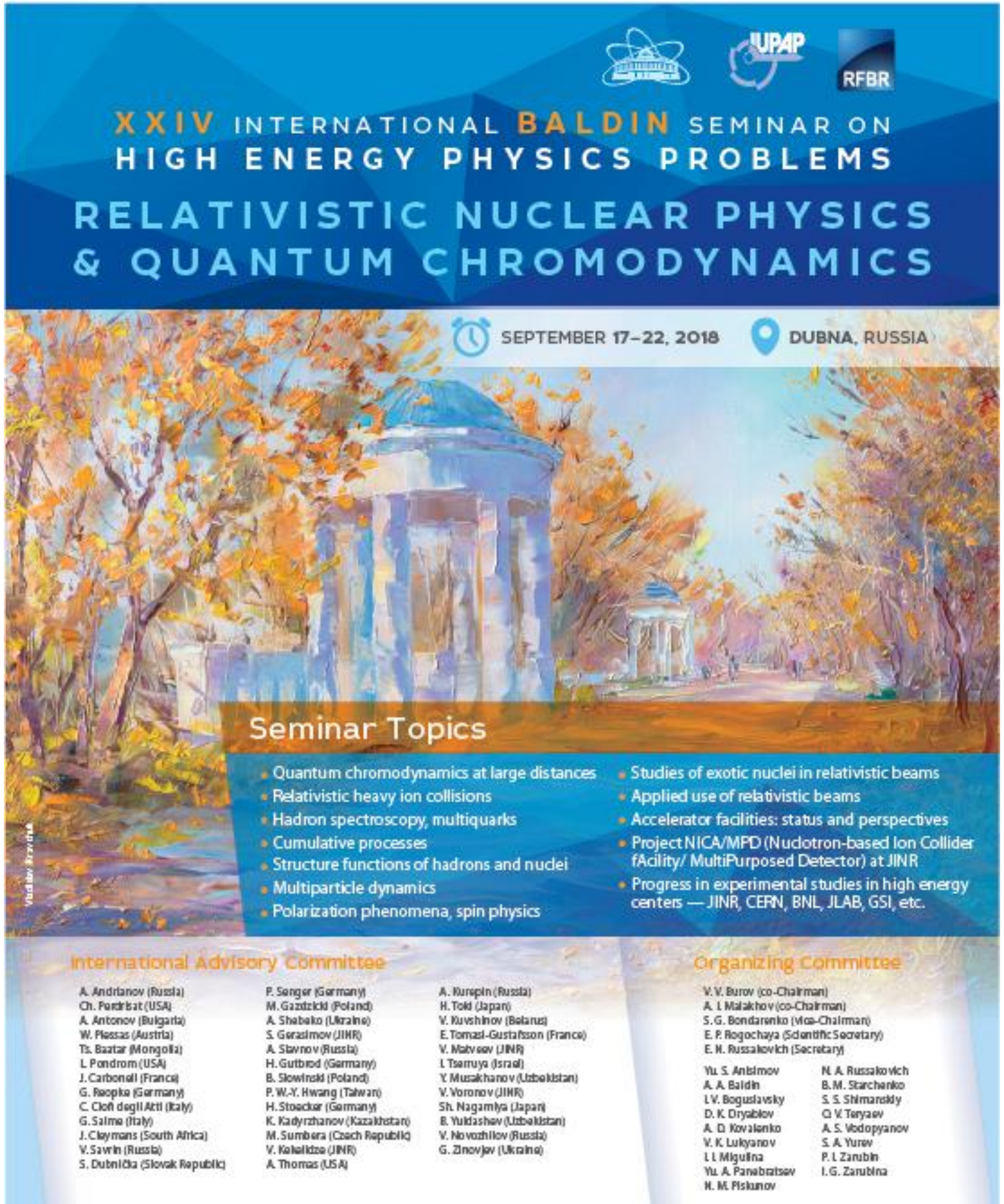
Relativistic composition of "structural velocities" **u** is given by multiplicative composition of ratios of fractal dimensions δ_i.

Outlook and further motivation

- Ratio of fractal dimensions $\alpha=\delta_2/\delta_1$ determines magnitude of the quantity $U(\xi)$ which is "4-velocity parameter" in elementary equations of structural relativity.
- The quantization of δ_1 and δ_2 is associated with quantization of the "structural velocity" $U(\xi)$ and results in quantum character of the structural relativity.
- The asymptotic values of $U(\zeta\to 1)$ (region near fractal limit) can be connected with induced anisotropy of 4-momentum space at small distances which, due to the quantum character of $\delta_1 \neq \delta_2$, should be quantized as well.
- The quantization concerns metric changes connected with "structural velocity" $U(\xi)$.
- We consider that quantum nature of fractal dimensions has connection to quantization of metric structures at small distances and motivates us to further study in this direction ….

The z-scaling approach can be an effective tool to search for and study of new symmetries, conservation laws and quantum properties of hadron structure and fragmentation processes especially at small distances.

The measurements of particle spectra with high p_T at the energies of the future accelerators FAIR (GSI) and NICA (JINR) will be extremely suitable for studying the regime of large fractal cumulativities and can contribute to verification of quantum nature of fractality in the interactions of hadrons and nuclei.



Thank You For Attention !

Back-up slides

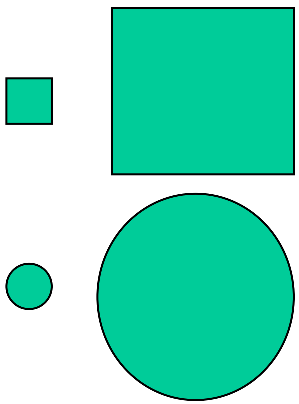
Similarity, self-similarity, fractality

1. Two geometrical objects are called similar if one is the result of a uniform scaling (enlarging or shrinking) of the other.

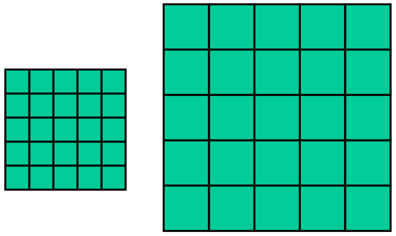
2. Object is called self-similar if it is composed of parts similar to it as a whole.

3. Object is called (self) similar fractal, if it consists of parts like him as a whole on any scale.

Similar objects

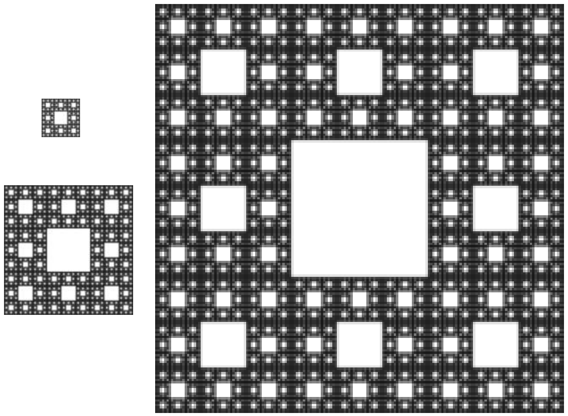


Self-similar object



$D = 2, D_T = 2$

Fractal

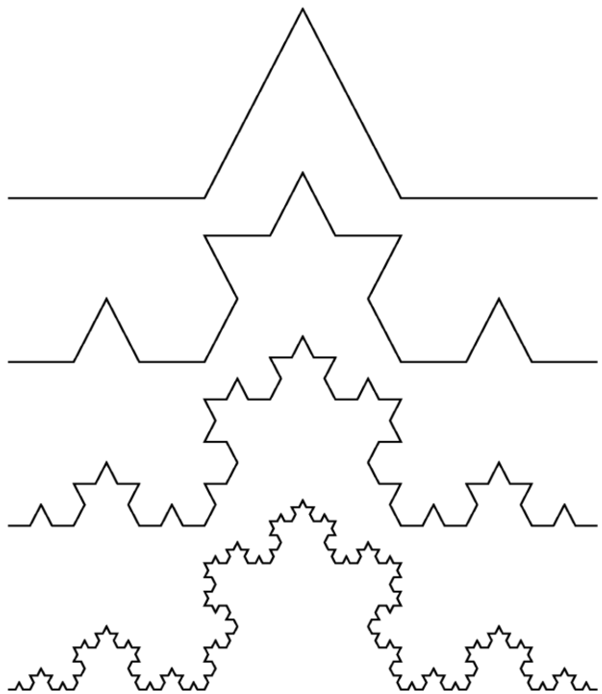


$D = \ln 8/\ln 3 \approx 1.89, \quad D_T = 1$

Example of a Fractal Curve

Swedish mathematician
Nils Fabian Helge von Koch

$$p = 4, \quad q = 3$$



$$z = z_0 \cdot \varepsilon^{-\delta}$$

Curve length is a measure attributed to the fractal curve

$$z(\varepsilon)\big|_{\varepsilon^{-1} \to \infty} \to \infty$$

- $\varepsilon^{-1} = q^n$ resolution
- $D = \mathrm{Ln}(p)/\mathrm{Ln}(q)$ fractal dimension
- $D_T = 1$ topological dimension
- $\delta = D - D_T$ anomalous fractal dimension