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Bound states of purely relativistic nature

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• Schrödinger equation with Coulomb potential

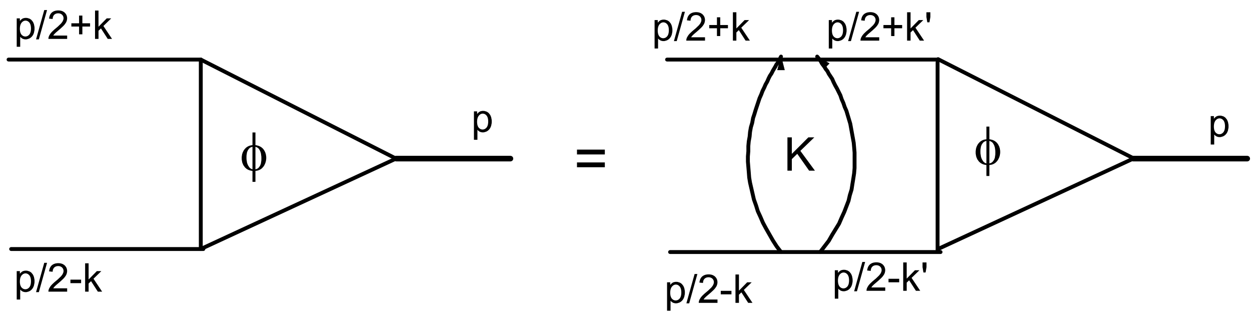
$$\Delta\psi + \frac{2m}{\hbar^2}[E - V(r)] = 0,$$
$$V(r) = -\frac{\alpha}{r}$$

System of two particles, each has mass m
Binding energies (well-known solution):

$$E_n = -\frac{\alpha^2 m}{4n^2}$$ – Balmer series.

• Bethe-Salpeter bound state equation

E.E. Salpeter, H. Bethe, 1951



$$\Phi(k,p) = \frac{(-g^2)i^2}{\left((\frac{p}{2}+k)^2-m^2+i\epsilon\right)\left((\frac{p}{2}-k)^2-m^2+i\epsilon\right)}$$

$$\times \int \frac{d^4k'}{(2\pi)^4}\frac{i\Phi(k',p)}{[(k-k')^2-\mu^2+i\epsilon]}, \quad \mu=0$$

• Abnormal solutions

In 1954, G.C. Wick and R.E. Cutkosky,
still for massless exchange $\mu = 0$,
solved BS equation and reproduced Balmer series.
In addition, they found another series,
which is absent in the Schrödinger equation.

These new solutions, which disappear in the
non-relativistic limit, were called
"abnormal" solutions.

• Solving BS equation

Nakanishi, 1963

Integral representation for the BS amplitude:

$$\Phi(k,p)=\int_{-1}^{1}dz'\int_0^{\infty}d\gamma'\frac{g(\gamma',z')}{\left[\gamma'+m^2-\frac{1}{4}M^2-k^2-p\cdot k\;z'-i\epsilon\right]^3}.$$

For massless exchange:

$$g(\gamma,z)\to g(z)\delta(\gamma)\to g(z)$$

– does not depend on γ.

• Equation for $g(z)$

$\mu = 0 \to$ G.C. Wick, R.E. Cutkosky, 1954

$$g_n''(z) + \frac{2(n-1)z}{(1-z^2)} g_n'(z) - \frac{n(n-1)}{(1-z^2)} g_n(z)$$
$$+\frac{\alpha}{\pi} \frac{1}{(1-z^2)(1-\eta^2+\eta^2 z^2)} g_n(z) = 0$$
$$\eta = \frac{M}{2m} = 1 - \frac{B}{2m}, \quad g_n(\pm 1) = 0$$

For each n – there is infinite set of solutions which can be enumerated by another quantum number: $k = 0, 1, 2, 3, \ldots$.

• Spectrum

In general: $E = E_{nk}, \quad n = 1, 2, 3 \ldots, \quad k = 0, 1, 2, 3, \ldots$

If $k = 0$, the normal Balmer series is reproduced (with a relativistic correction):

$$E_n = -\frac{\alpha^2 m}{4n^2}\left(1 + \frac{4}{\pi}\alpha \log \alpha\right)$$

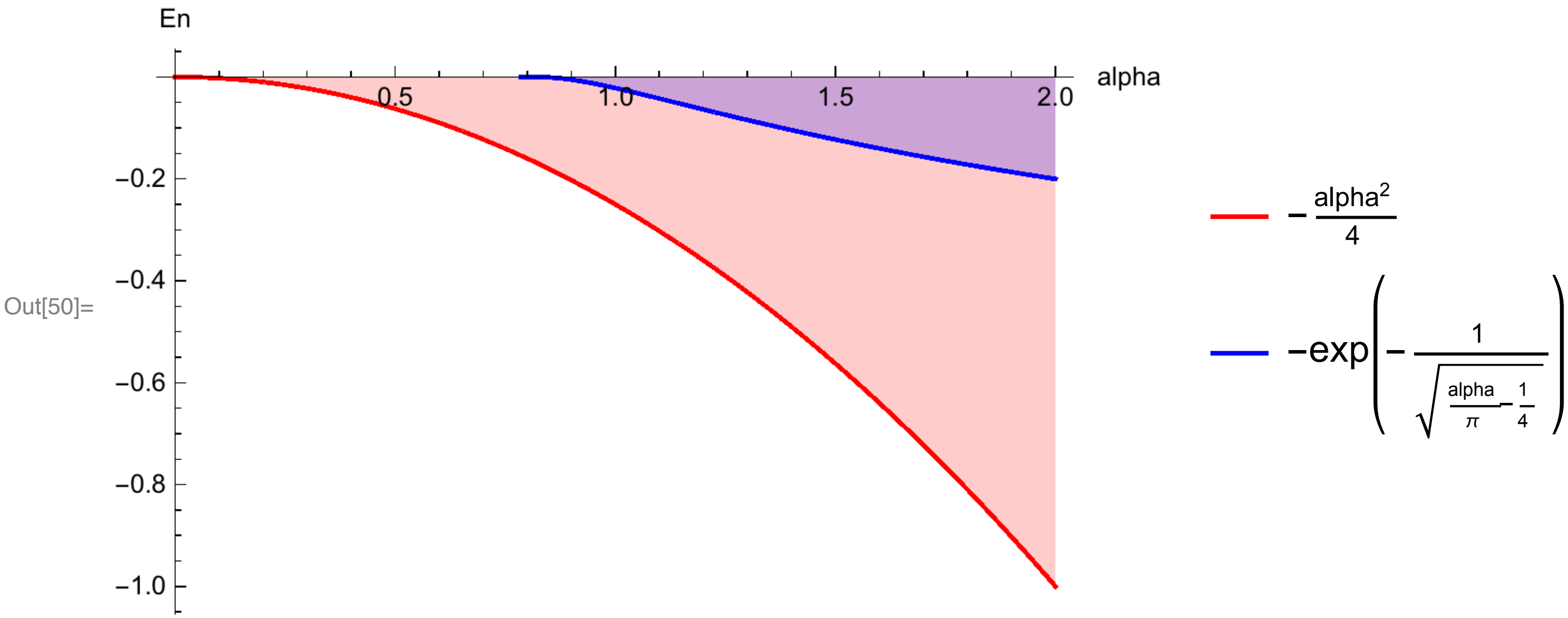
However, for each given n - another (abnormal) series with $k = 1, 2, 3, \ldots$:

$$E_k = -m \exp\left(-\frac{2\pi k}{\sqrt{\frac{\alpha}{\pi} - \frac{1}{4}}}\right), \quad k = 1, 2, 3 \ldots, \quad \alpha > \frac{\pi}{4}.$$

– this formula is valid when $\alpha \to \frac{\pi}{4}, \quad E_k \to 0$.

k is number of nodes in the (abnormal) solution $g_k(z)$.

• Energy spectrum (still for $\mu = 0$)

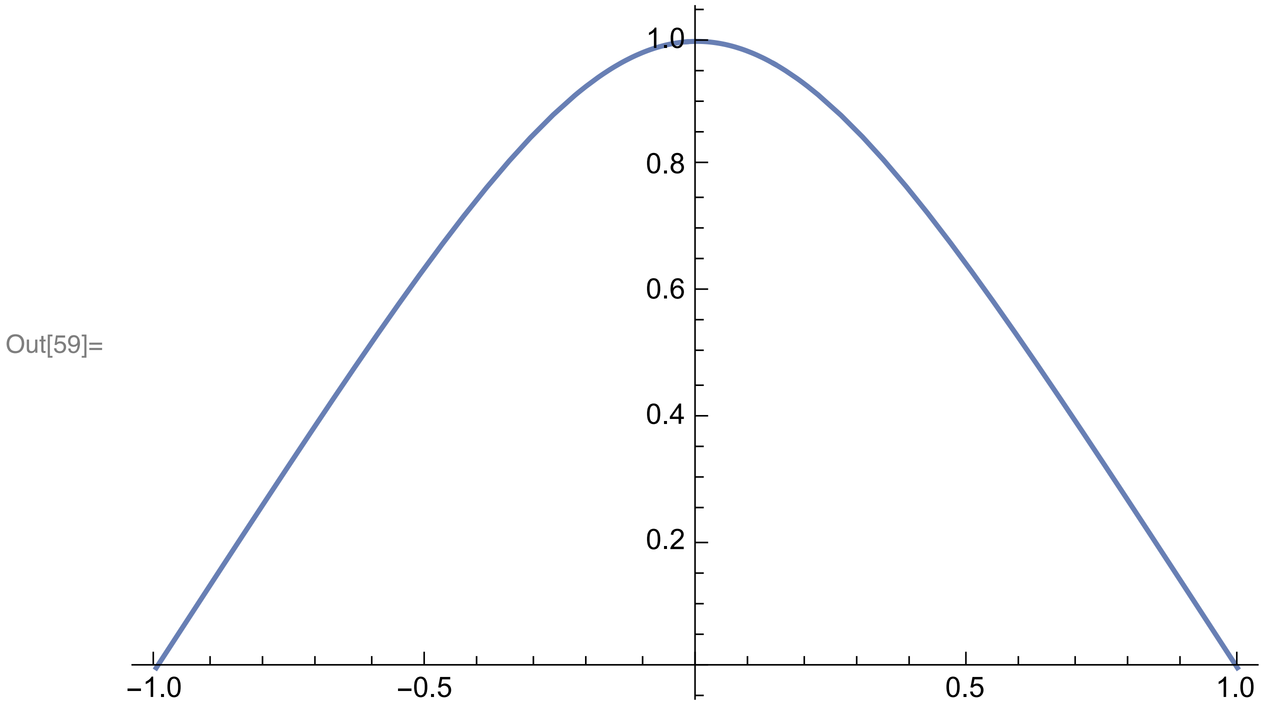


The binding energies for normal and abnormal states.

Abnormal states are not predicted by the Schrödinger equation, but they are predicted by the BS one! Therefore they have purely relativistic origin.
Normal solutions $g(z)$ vs. z have no nodes, abnormal solutions have k nodes.
For even k they are symmetric, for odd k they are antisymmetric.

• Spectral function $g(z)$

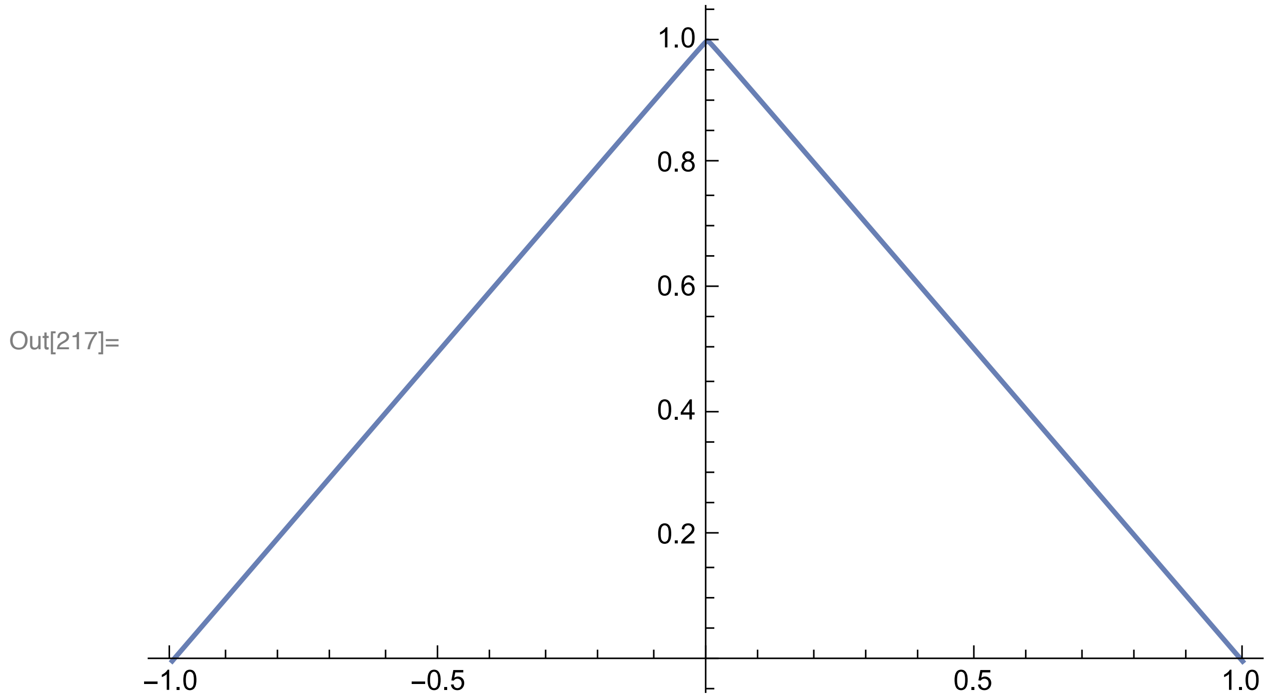
Normal (ground) state, $n = 1$, $k = 0$



$B = 0.2$, $\alpha = 1.786$

• Spectral function $g(z)$

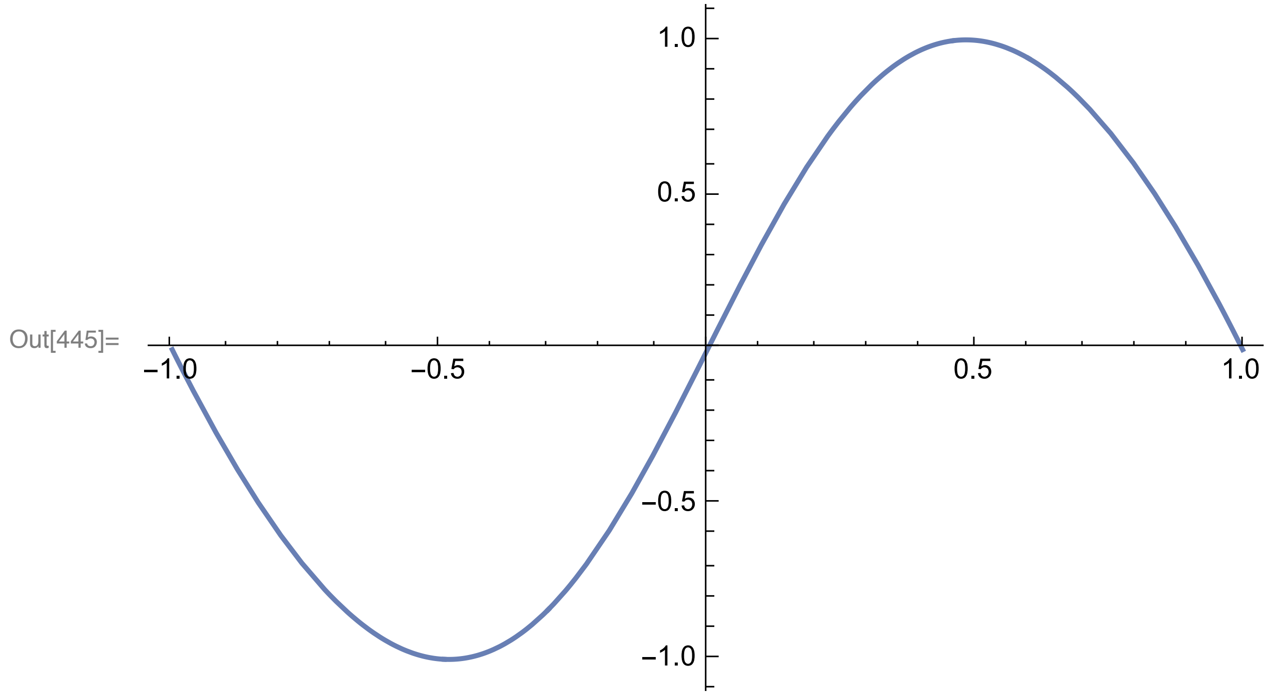
Normal (ground) state, small binding energy, $n = 1$, $k = 0$



$B = 10^{-6}$, $\alpha = 0.0020169$, $g(z) = 1 - |z|$

• Spectral function $g(z)$

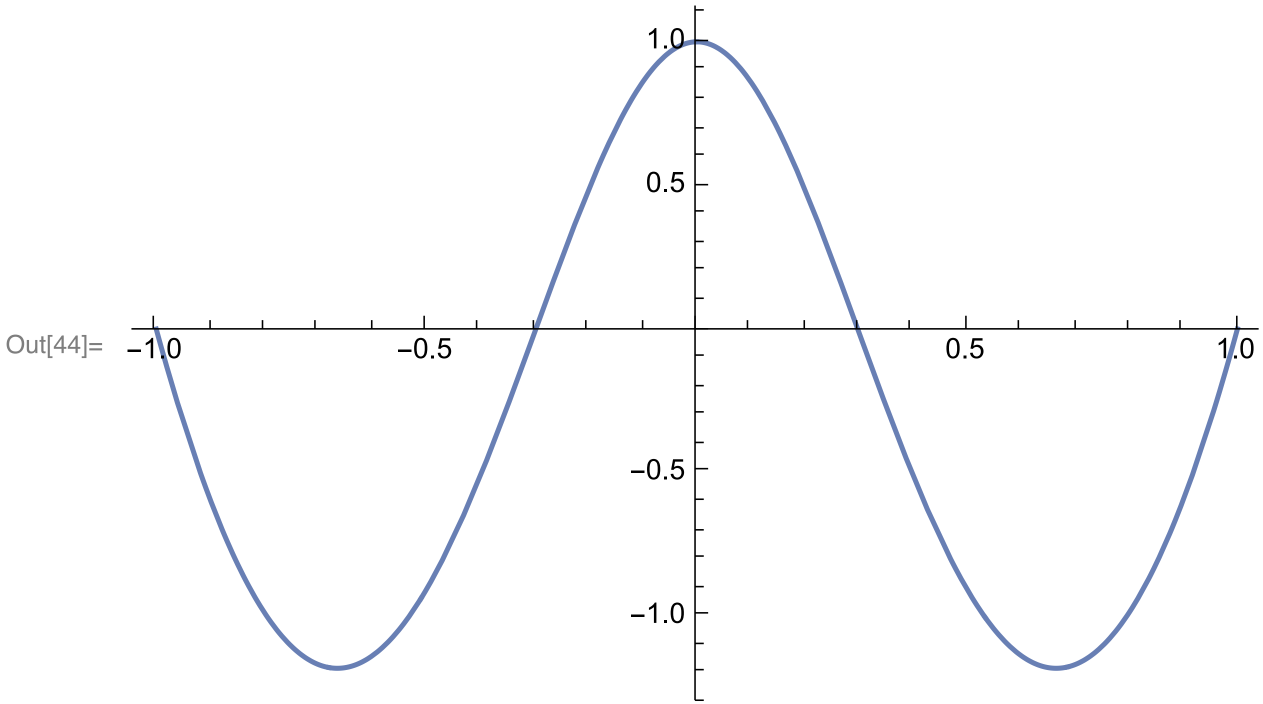
One node, abnormal state, antisymmetric, $n = 1$, $k = 1$



$B = 0.2$, $\alpha = 8.25$

• Spectral function $g(z)$

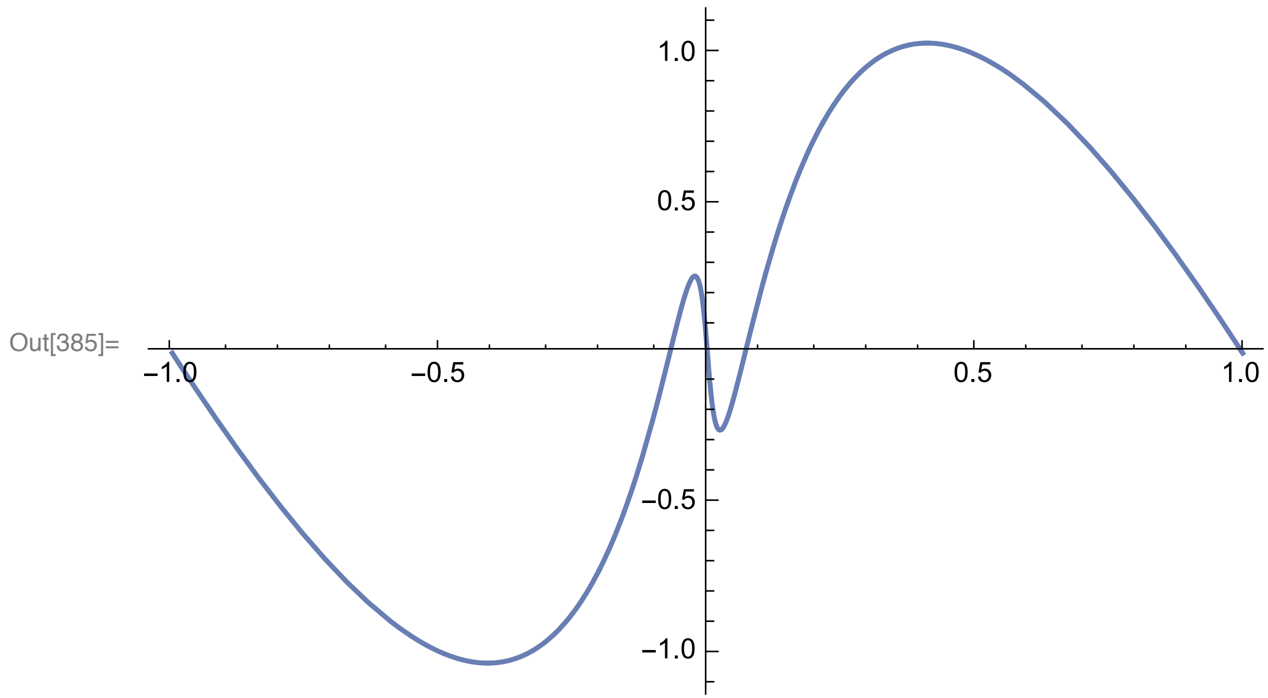
Two nodes, abnormal state, symmetric, $n = 1$, $k = 2$



$B = 0.2$, $\alpha = 17.19$

• Spectral function $g(z)$

Three nodes, abnormal state, antisymmetric, $n = 1$, $k = 3$



$B = 10^{-4}$, $\alpha = 2.63$

• Conclusion in the case $\mu = 0$

Abnormal states for massless exchange,
as solutions of the BS equation, certainly exist.

However, minimal coupling constant
(for point-like particle!)
seems too large: $\alpha_{min} = \frac{\pi}{4} \Rightarrow Z = 107.$
What about abnormal states for finite range
(strong) interaction?

To explore them is our aim.

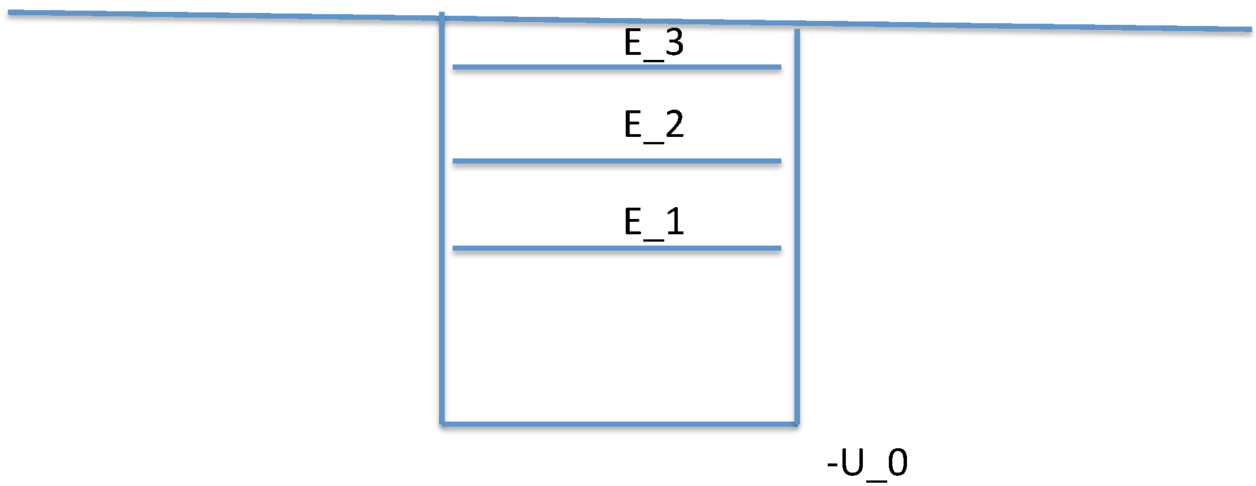
• Finite-range interaction, $\mu \neq 0$

- The number of states is normally finite.
- When the interaction decreases, the bound states disappear one after other:
  - first, the highest excited state disappears;
  - the next (lower) excited state disappears;
  - so on...
  - down to the ground state.

Any state disappears when finite-range interaction weakens.

There is its own critical coupling constant for each state.

• Example: potential wall



$$U_{crit} = \frac{\pi^2}{8mr_0^2}$$

Reminder: Nakanishi representation

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\!\cdot\! k\; z' - i\epsilon\right]^3}.$$

Massive case $\mu \neq 0$, $g(\gamma,z)$ depends on two variables γ, z.

Substitute $\Phi(k,p)$ in the BS equation and find equation for $g(\gamma,z)$

• Deriving equation for $g(\gamma, z)$, $\mu \neq 0$

- For a ladder kernel, in rather complicated form (K. Kusaka, A.G. Williams, 1995)
- For arbitrary BS kernel, but in the form with two-integrals, V.A. Karmanov and J. Carbonell, 2006:

$$\int_0^\infty \frac{g(\gamma', z) d\gamma'}{\left[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2\right]^2} = \int_0^\infty d\gamma' \int_{-1}^{1} dz'\, V(\gamma, z; \gamma', z') g(\gamma', z')$$

- For the ladder kernel, in the canonical form (one integral in r.h..s.), T. Frederico, G. Salmè and M. Viviani, 2014
- For arbitrary BS kernel, in the canonical form, J. Carbonell, T. Frederico, V.A. Karmanov, 2017

Canonical form: $g(\gamma, z) = \int_0^\infty d\gamma' \int_{-1}^{1} dz' N(\gamma, z; \gamma', z') g(\gamma', z')$

Equation for $g(\gamma, z)$ was analyzed analytically and solved numerically. More solutions (symmetric and antisymmetric) were found than in the corresponding Schrödinger equation. Some of them survive in the non-relativistic limit (and turn into the solutions of the Schrödinger equation), some of them (the abnormal ones) disappear in the non-relativistic limit.

The abnormal states for massive exchange $\mu \neq 0$ exist!

• Question

If the "abnormal" states exist for a finite-range interaction, how to distinguish them from the "normal" ones?

In other words: solving BS equation, we find an energy level. What is its nature: normal or abnormal?

• Non-relativistic limit

Relativity exists since the speed of light c is finite

and the same in any frame.

Non-relativistic limit is $c \to \infty$.

We should restore c in the BS equation and take the limit $c \to \infty$ (analytically and/or numerically).

Restoring c:

$$m \to mc^2, \; M \to Mc^2, \; \alpha = \frac{e^2}{\hbar c} \to \frac{\alpha}{c},$$

G. Wanders, *Limite non-relativiste d'une équation de Bethe-Salpeter*,

Helvetica Physica Acta, 1957 (in French).

• Derivation of Schrödinger equation from the BS one (in the limit $c \to \infty$)

(Equal time) wave function:

$$\psi(\vec{p}) = \int_{-\infty}^{\infty} \Phi(\vec{p}, p_0) dp_0 = \int_0^{\infty} \frac{g(\gamma, 0) d\gamma}{(\gamma + \vec{p}^{\,2} + \kappa^2)}$$

Since equation for g is known, one can derive equation for $\psi(\vec{p})$ (normal state):

$$\psi(\vec{p}) = \frac{4m\pi\alpha}{(\vec{p}^{\,2} + \kappa^2)} \int \frac{d^3q}{(2\pi)^3} \frac{1}{((\vec{p} - \vec{q})^2 + \mu^2)} \psi(\vec{q})$$

$$\Rightarrow \quad \left(\frac{\hat{\vec{p}}^{\,2}}{m} + V(r) \right) \psi(\vec{r}) = E\psi(\vec{r}), \quad V(r) = -\frac{\alpha}{r} e^{-\mu r}$$

• Dependence $\alpha(c)$ for a normal solution

We repeat the calculations for a set of values of speed of light $1 \leq c \leq 10$ and find the dependence $\alpha(c)$.

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(*   m=1, mu=0.15, B=0.1  *)
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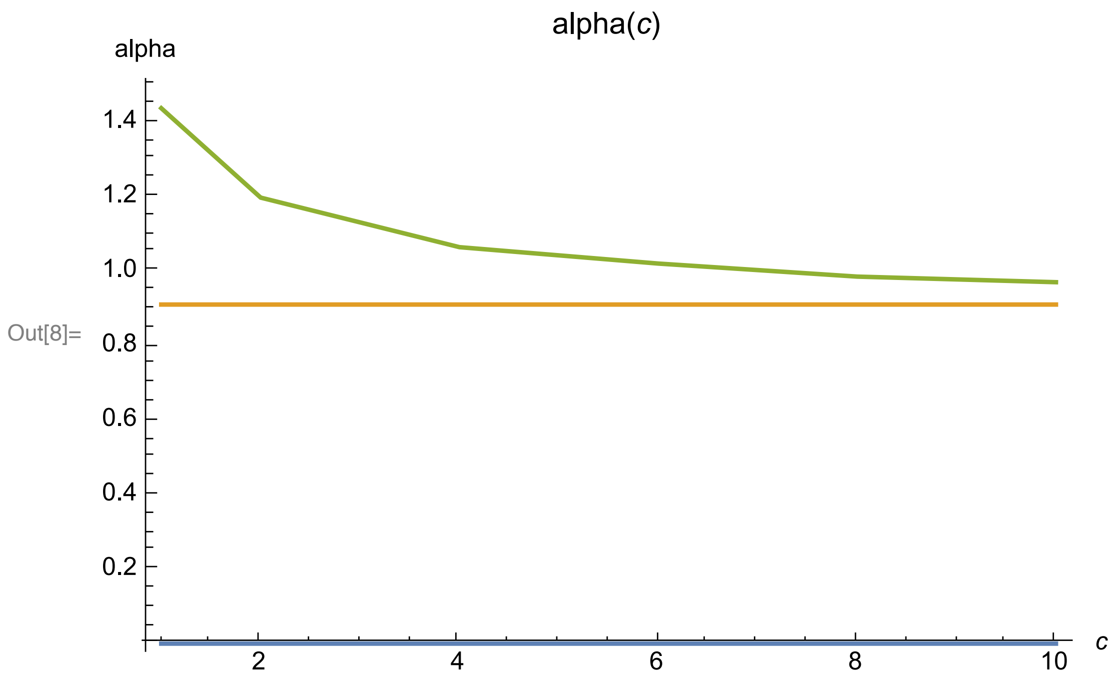
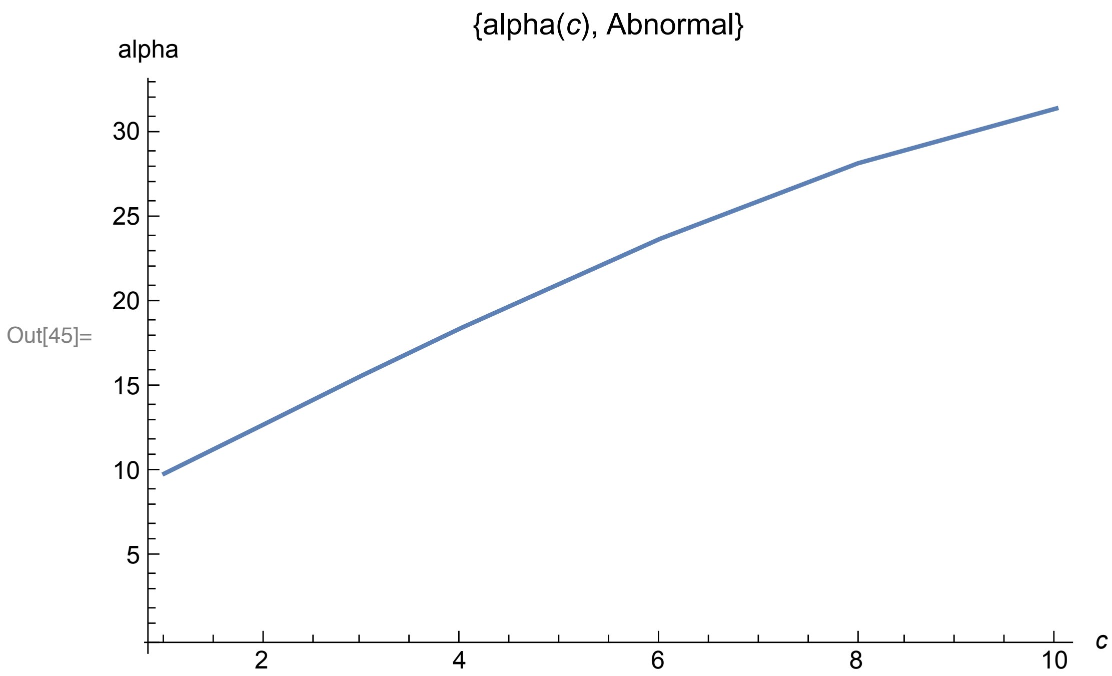


Dependence of the coupling constant α (for a normal state, $\mu = 0.15,\ B = 0.1$) on speed of light c

• Dependence $\alpha(c)$ for an abnormal state



Dependence of the coupling constant α
(for an abnormal state, $\mu = 0.15$, $B = 0.1$) on speed of light c

For normal state: $\alpha(c \to \infty) \to$ finite (nonrelativistic) limit.
For abnormal state: $\alpha(c \to \infty)$ increases without any limit.

• Some properties of $g(\gamma, x)$

Introduce new variable x:

$$x = \frac{1}{2}(1+z), \quad -1 \leq z \leq 1 \Rightarrow 0 \leq x \leq 1$$

$$\psi(\gamma, z) \ \Rightarrow \psi(\gamma, x)$$

• Domain of $g(\gamma, x)$

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(* Domain: 0<gamma<infinity, 0<x<1   *)
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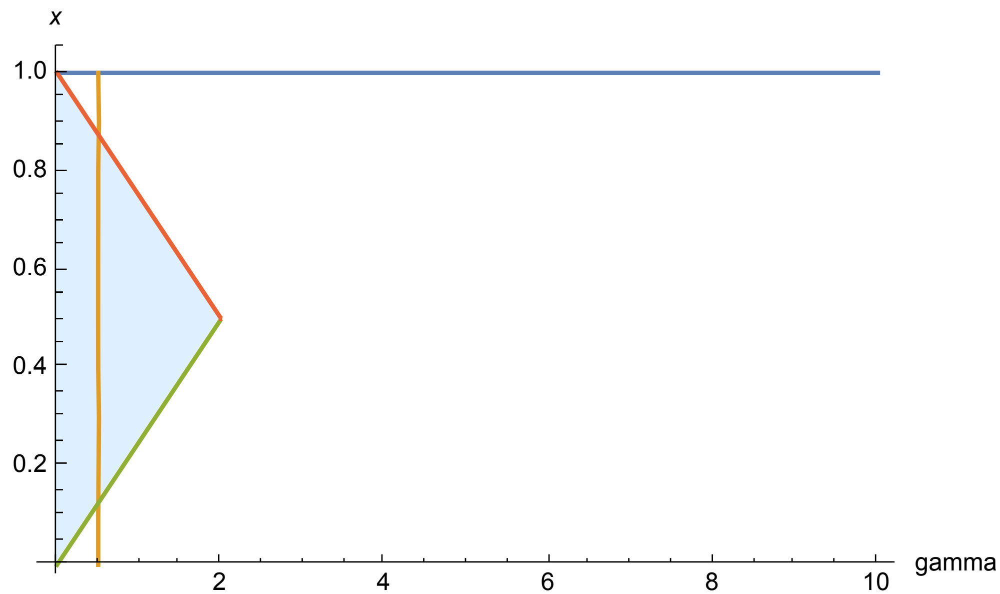
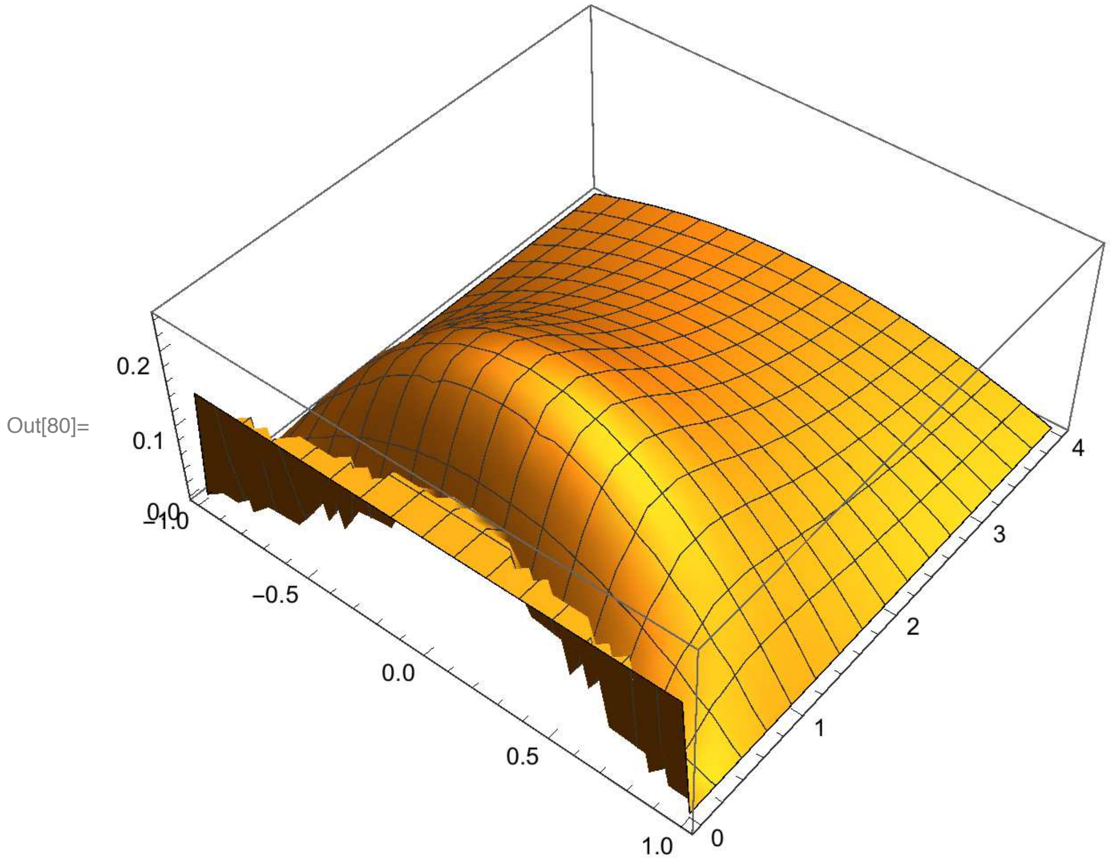


Domain $0 \leq x \leq 1, \quad 0 \leq \gamma, < \infty$ where $g(\gamma, x)$ is defined.

Light blue is domain where $g(\gamma, x) = const$:

$$\gamma \leq \gamma_0(x) \sim \begin{cases} \mu x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \mu(1-x), & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

• 3D plot of $g(\gamma, z)$



Weight function $g(\gamma, z)$

If $\gamma < \gamma_0 \approx (1 - |z|)2\mu\sqrt{mB}$, then $g(\gamma, z) = const$

(vs. γ and vs. z).

If $g(\gamma, z)$ is antisymmetric, then $const = 0$.

• Numerical solution $g(\gamma, x)$ $(c = 1)$

$$x = \frac{1}{2}(1 + z), \quad -1 \leq z \leq 1 \Rightarrow 0 \leq x \leq 1$$

```
(* alpha=1.4375 Normalized g(gamma=0.17,x=0.5)=1 *)
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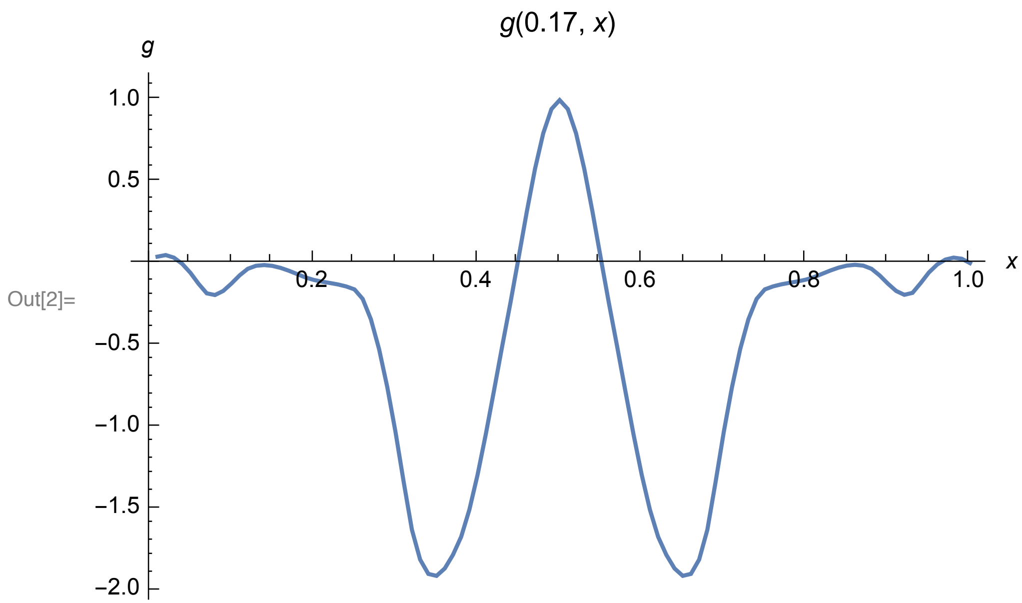
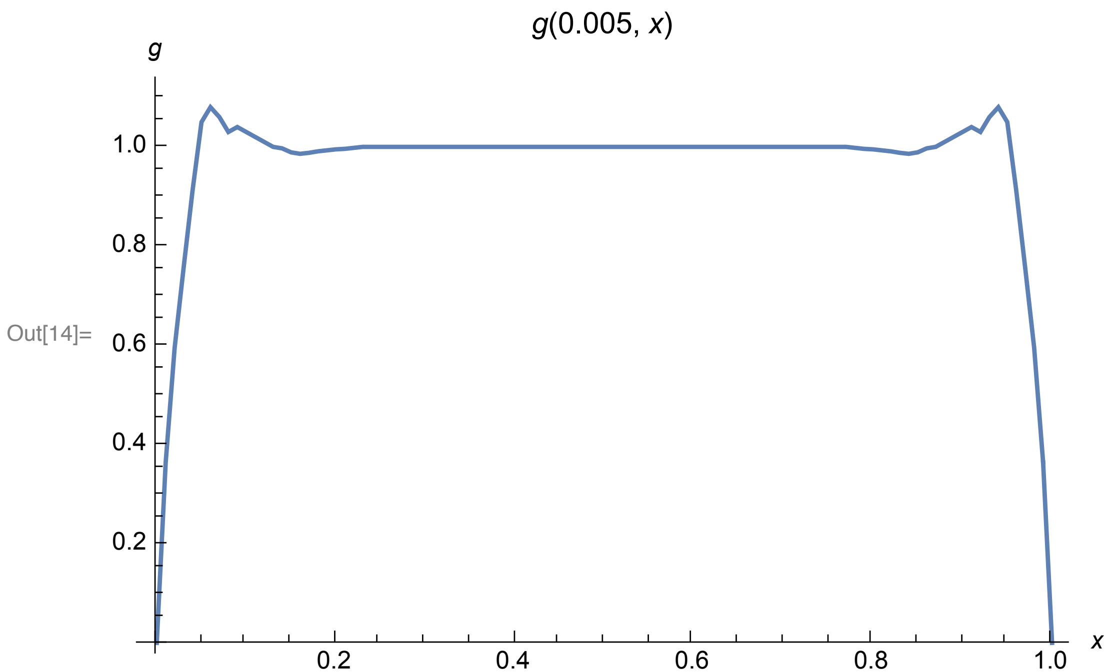


Weight function $g(\gamma = 0.17, x)$

- $g(\gamma = 0.005, x)$

Normal state



Weight function $g(\gamma = 0.005, x)$

• $g(\gamma = 0.005, x)$

Antisymmetric abnormal state

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(* alpha=9.8882 Normalized g(gamma=0.01,x=0.025)=1 *)
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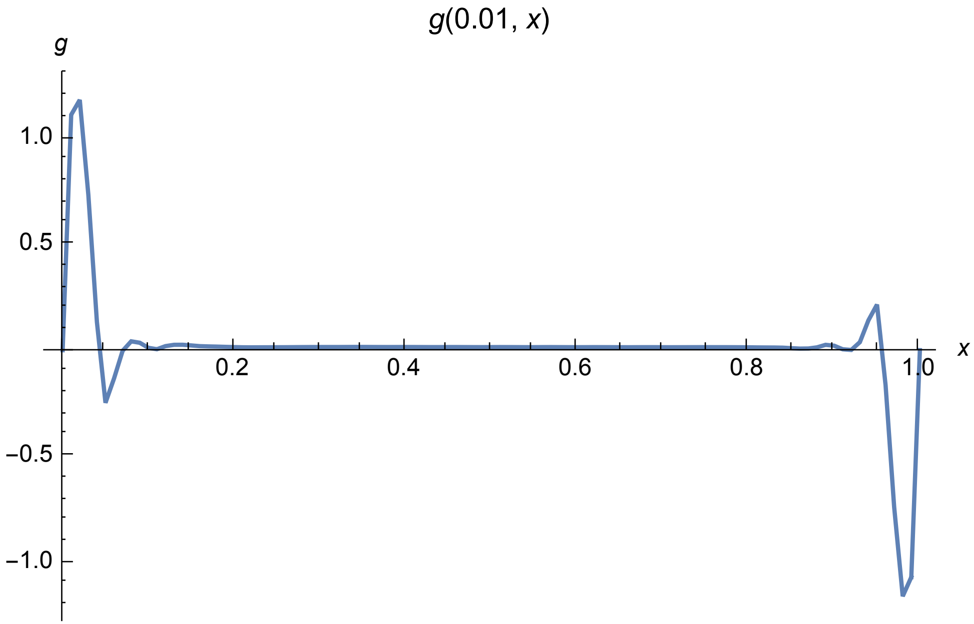

Weight function $g(\gamma = 0.005, x)$

- $g(\gamma = 0.005, x)$

Symmetric abnormal state

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(* alpha=11.2859 Normalized g(gamma=0.005,x=0.4)=1 *)
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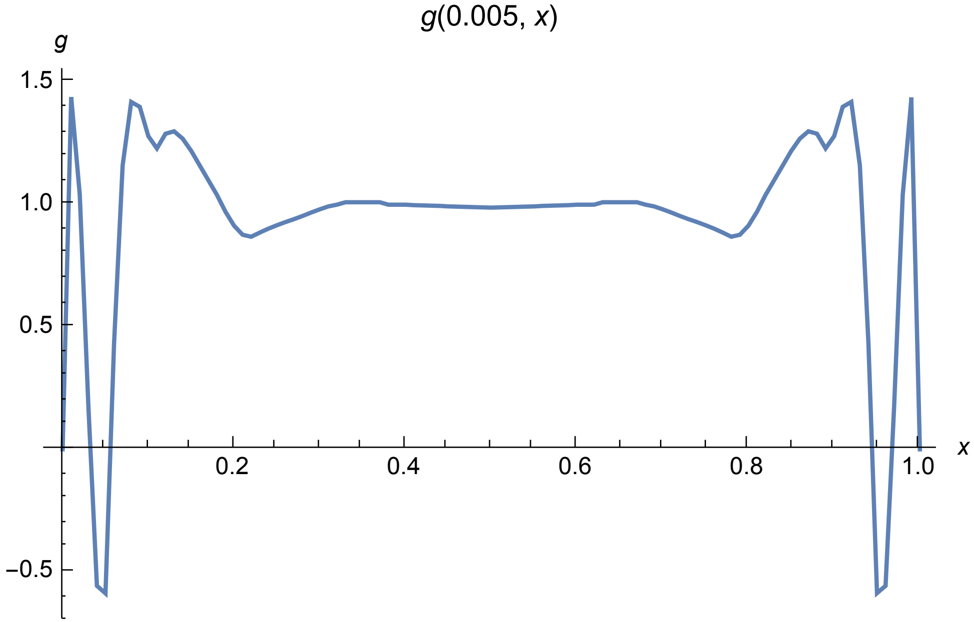

Weight function $g(\gamma = 0.005, x)$

• Criterion

$g(\gamma < \gamma_0(x), x)$ is always constant
(both for normal and abnormal states).
Outside, $g(\gamma, x) \to 0$ when $x \to 0,\ 1$.

For normal states:
in vicinity of $x = 0, 1$, it has no nodes.
For abnormal states:
in vicinity of $x = 0, 1$, it has nodes.

In this way, one can distinguish abnormal states
from the normal ones,
like in the massless case, but in the domain $g(\gamma < \gamma_0(x), x \approx 0, 1)$.

This is a phenomenological observation,
based on numerical calculations
(never any exclusions were found!),
but not yet a theorem.

• Conclusions

- BS equation predicts the states having
  pure relativistic origin
  (not given by the Schrödinger equation).
  Analogy: Dirac equation predicts antiparticles.

- We know how to distinguish them from the normal ones
  (by behavior of solutions $g(\gamma, x)$ vs. x at small γ).

- This is important for understanding the
  relativistic few-body systems

- It is interesting to analyse, from this point of view,
  the spectra of particles.

Thank you!