$K(E_{10})$ as an R symmetry

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Based on:

A. Kleinschmidt, HN: JHEP1308(2013)041; 1504.01586;1602.04116

A. Kleinschmidt, HN and N.K.Chidambaram: PRD91(2015)8,085039

and earlier work with T. Damour, M. Henneaux and A. Kleinschmidt

Main Points

- **Duality symmetries more important than space-time symmetries (general covariance, supersymmetry,...)**
- E_{10}**: a *symmetry based* proposal for (de-)emergence of space (and time) near cosmological singularity.**
- **Fermions transform under 'R symmetry'** $K(E_{10})$**.**
- **Distinction between space-time bosons and fermions meaningless in 'pre-geometric' regime ?**
- **Understanding** $K(E_{10})$**: perhaps the key challenge?**
- **Exploiting the identity** $\mathbf{3 \times 16 = 56 - 8}$**, or:**
  Is there a role to play for $K(E_{10})$ **in 'real' physics?**

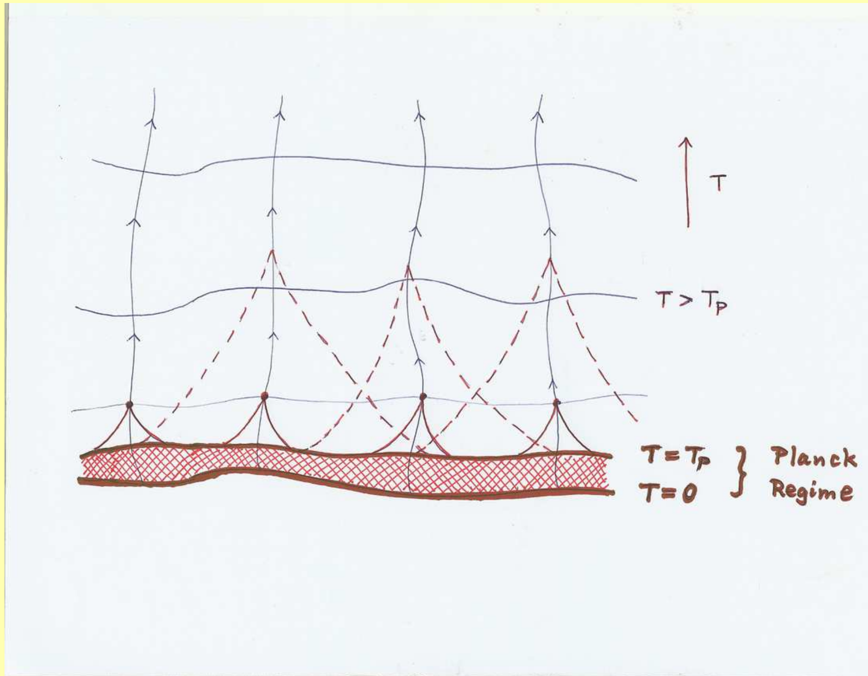
Exceptionality and Maximal Supergravity

- **Maximal theories:** $E_{n(n)}$ **for** $D = 11 - n$ `[Cremmer,Julia(1979)]`
- $E_n(\mathbb{Z})$ **conjectured to be a symmetry of non-perturbative string theory** $\equiv$ **M theory.** `[Hull,Townsend; Green et al.]`

Below $D = 3$ **symmetries become** ***infinite-dimensional*:**

- $E_{9(9)} \equiv E_8^{(1)}$**: a solution generating symmetry acting on** $\mathcal{M} = E_{9(9)}/K(E_9)$ **= moduli space of colliding plane wave solutions of maximal** $D = 2$ **supergravity.**
- **... suggests** $E_{10(10)}$ **for** $D = 1$**: no space, only time?**
- **Expect coset structure** $E_{n(n)}/K(E_n)$ **to persist also for infinite-dimensional case** ($n \geq 9$)**.**

BKL and Spacelike Singularities



For $T \to 0$ spatial points decouple and the system is effectively described by a continuous *superposition of one-dimensional systems* $\to$ effective dimensional reduction to $D = 1$! [Belinski,Khalatnikov,Lifshitz (1972)]

Habitat of Quantum Gravity?

- **Cosmological evolution as one-dimensional motion in the moduli space of 3-geometries** [Wheeler,DeWitt,...]

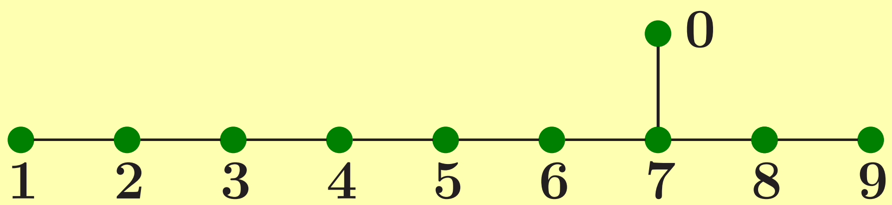
$$\mathcal{M} \equiv \mathcal{G}^{(3)} = \frac{\{\text{spatial metrics } g_{ij}(\mathbf{x})\}}{\{\text{spatial diffeomorphisms}\}}$$

- **Formal canonical quantization $\rightarrow$ WDW equation.**
- **Unification of space-time, matter and gravitation: $\mathcal{M}$ should incorporate matter degrees of freedom in a natural manner (not simply $\mathcal{M} = \mathcal{G}^{(3)} \times \mathcal{M}_{matter}$).**
- **Can we understand and 'simplify' $\mathcal{M}$ by means of embedding into a group theoretical coset $G/K(G)$?**
- **Main conjecture: $G = E_{10}$ and $K(G) = K(E_{10})$**

What is E_{10}?

The nice thing about it is that no one knows .... [Murat Günaydin, unpublished]

E_{10} is the 'group' associated with the Kac-Moody Lie algebra $\mathfrak{g} \equiv \mathfrak{e}_{10}$ defined via the Dynkin diagram [e.g. Kac]



Defined by generators $\{e_i, f_i, h_i\}$ and relations via Cartan matrix A_{ij} ('Chevalley-Serre presentation')

$$
\begin{aligned}
[h_i, h_j] &= 0, & [e_i, f_j] &= \delta_{ij} h_i, \\
[h_i, e_j] &= A_{ij} e_j, & [h_i, f_j] &= -A_{ij} f_j, \\
(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j &= 0 & (\mathrm{ad}\, f_i)^{1-A_{ij}} f_j &= 0.
\end{aligned}
$$

$\mathfrak{e}_{10}$ is the free Lie algebra generated by $\{e_i, f_i, h_i\}$ modulo these relations $\rightarrow$ infinite dimensional as A_{ij} is *indefinite* $\rightarrow$ Lie algebra of *exponential growth* !

$SL(10)$ level decomposition of E_{10}

- **Decomposition w.r.t. $SL(10)$ subgroup in terms of $SL(10)$ tensors $\rightarrow$ *level expansion***

$$\alpha = \ell\alpha_0 + \sum_{j=1}^{9} m^j \alpha_j \quad \Rightarrow \quad E_{10} = \bigoplus_{\ell\in\mathbb{Z}} E_{10}^{(\ell)}$$

- **Up to $\ell \leq 3$ basic fields of $D = 11$ SUGRA together with their magnetic duals (spatial components)**

$\ell = 0$	G_{mn}	**Graviton**
$\ell = 1$	A_{mnp}	**3-form**
$\ell = 2$	$A_{m_1\ldots m_6}$	**dual 6-form**
$\ell = 3$	$h_{m_1\ldots m_8\|n}$	**dual graviton**

- **Analysis up to level $\ell \leq 28$ yields 4 400 752 653 representations (Young tableaux) of $SL(10)$** [Fischbacher,HN:0301017]

- **Lie algebra structure (structure constants, etc.) understood only up to $\ell \leq 4$. Also: no matter where you stop it will get even more complicated beyond!**

The $E_{10}/K(E_{10})$ σ-model

Basic Idea: map evolution according to $D = 11$ SUGRA equations of motion onto null geodesic motion of a point particle on $E_{10}/K(E_{10})$ coset manifold [DHN:0207267]

$$\mathcal{V}(t) = \exp\left(h_{ab}(t) S^{ab} + \frac{1}{3!} A_{abc}(t) E^{abc} + \frac{1}{6!} A_{abcdef}(t) E^{abcdef} + \cdots \right)$$

and then work out Cartan form $\partial_t \mathcal{V}\mathcal{V}^{-1} = Q + P$ with associated σ-model $\rightarrow$ $E_{10}/K(E_{10})$ σ-model dynamics up to $\ell \leq 3$ matches with supergravity equations of motion when truncated to first order spatial gradients.

Conjecture: information about spatial dependence gets 'spread' all over E_{10} Lie algebra. More specifically:

Infinite tower of σ-model fields $\leftrightarrow$ SUGRA fields and their non-local descendants (duals) *at fixed spatial point?*

Hint: level expansion contains complete set of *gradient representations* for all $D = 11$ fields and their duals.

Some practical concerns

[Cf.: Kleinschmidt,HN,Chidambaram:1411.5893]

Use Cartan-Weyl basis, with $[H_\mathtt{a}, H_\mathtt{b}] = 0$ (CSA)

$$[H_\mathtt{a}, E^r_\alpha] = \alpha_\mathtt{a} E^r_\alpha \quad , \quad [E^r_\alpha, E^s_\beta] = \begin{cases} \sum_t c^{rst}_{\alpha,\beta} E^t_{\alpha+\beta} & \textbf{if } \alpha + \beta \in \Delta, \\ \delta^{rs} \alpha^\mathtt{a} H_\mathtt{a} & \textbf{if } \alpha = -\beta, \\ 0 & \textbf{otherwise} \end{cases}$$

(Triangular) parametrization of Kac-Moody group via

$$\mathcal{V}(q^\mathtt{a}(t), A^r_\alpha(t)) \;\; \text{“}=\text{”} \;\; \exp(q^\mathtt{a}(t) H_\mathtt{a}) \exp\left(\sum_{\alpha>0} \sum_{r=1}^{\mathrm{mult}(\alpha)} A^r_\alpha(t) E^r_\alpha \right)$$

does *not* work for imaginary roots α, because E^r_α are not locally nilpotent $\Rightarrow$ exponentiate only real roots ?

$\rightarrow$ blurs association of physical degrees of freedom with Lie algebra elements associated to imaginary roots !

Nevertheless, we can write (in triangular gauge)

$$\partial \mathcal{V}\mathcal{V}^{-1}(t) = \pi^{\mathsf{a}}(t) H_{\mathsf{a}} + \sum_{\alpha>0} \sum_{r=1}^{\mathrm{mult}(\alpha)} P_\alpha^r(t) E_\alpha^r$$

with nice canonical brackets

$$\{\pi^{\mathsf{a}}\,,\,\pi^{\mathsf{b}}\} = 0\,, \quad \{\pi^{\mathsf{a}}\,,\,P_\alpha^r\} = \alpha_{\mathsf{a}} P_\alpha^r\,, \quad \{P_\alpha^r\,,\,P_\beta^s\} = \sum_t c_{\alpha,\beta}^{rst} P_{\alpha+\beta}^t\ (??)$$

$\rightarrow$ **still to be checked (modified?) for imaginary roots.**

$\Rightarrow$ **'good' canonical variables to couple to fermions !**

Suspicion: consistent incorporation of fermions is one crucial missing piece of the puzzle ...

... and possibly requires novel kind of bosonization.

[cf. Witten (1984), Goddard,Nahm,Olive (1985)]

Fermions and $K(E_{10})$

... probably a key issue for further progress...

Important point: maximal supersymmetric theories *not* based on (hypothetical) superextensions of E_n:

- **There is no proper superextension of E_n for any n.**
- **For $D \geq 3$ supergravity fermions transform in *maximal compact subgroup* $K(E_n) \subset E_{n(n)}$, e.g.**

  $K(E_7) \equiv SU(8)$ **fermions $\in$ $\mathbf{8}$ and $\mathbf{56}$**

  $K(E_8) \equiv Spin(16)/Z_2$ **fermions $\in$ $\mathbf{16}_v$ and $\mathbf{128}_c$**

- **The associated (double-valued) fermion representations are not 'liftable' to E_n representations**
- **Expect all of this to remain true for $E_9\,,\,E_{10}\,,\ldots$**

What is $K(E_{10})$?

The nice thing about it is that no one knows .... [HN, unpublished]

For E_{10}, the 'maximal compact' subalgebra is defined as the fixed point algebra of the Chevalley involution

$$\omega(e_j) = -f_j\ , \quad \omega(f_j) = -e_j\ , \quad \omega(h_j) = -h_j$$

together with invariance property $[\omega(x), \omega(y)] = \omega([x, y])$

$$\Rightarrow \quad E_{10} = K(E_{10}) \oplus K(E_{10})^{\perp}\ , \quad x = \omega(x) \textbf{ for } x \in K(E_{10})$$

This definition is analogous to the corresponding one for the finite-dimensional case, e.g. $x = \omega(x) \in \mathfrak{so}(n) \subset \mathfrak{sl}(n)$ for $\omega(x) = -x^T$, with corresponding decomposition $\mathfrak{sl}(n) = \mathfrak{so}(n) \oplus \mathfrak{so}(n)^{\perp}$

Consequently, $K(E_{10})$ is generated by

$$x_i := e_i - f_i = \omega(x_i) \qquad i, j, ... = 1, ..., 10$$

with Berman-Serre relations

$$\begin{aligned} \big[x_i\,,\,x_j\big] &= 0 && \textit{if } i \textit{ and } j \textit{ are non-adjacent} \\ \big[x_i,[x_i,x_j]\big] + x_j &= 0 && \textit{if } i \textit{ and } j \textit{ are adjacent} \end{aligned}$$

Theorem: ***each set of $\{x_i\}$ satisfying the above relations provides a realization of $K(E_{10})$.*** [S.Berman(1989)]

Involutory subalgebra $K(E_{10}) \subset E_{10}$ is spanned by $\{J_\alpha^r\}$

$$J_\alpha^r \equiv E_\alpha^r - E_{-\alpha}^r \,, \quad \alpha \in \Delta_+(E_{10})\,, \quad r = 1, ..., \mathrm{mult}(\alpha)$$

But: $K(E_{10})$ is ∞-dimensional and a very strange beast!

- $K(E_{10})$ has finite-dimensional (unfaithful) representations
- $\Rightarrow K(E_{10})$ is ***not*** simple ($\equiv$ has non-trivial ideals)
- No faithful (infinite-dimensional) representations are known !

[**Idem for $K(E_9)$!** [Julia,HN(1996); Samtleben,HN(2004)]]

Unfaithful representations

$\Longleftrightarrow$ **existence of non-trivial ideals $\mathfrak{i}_V$ in $K(E_{10})$!**

More precisely: for unfaithful representation V the associated ideal is

$$\mathfrak{i}_V := \Big\{ x \in K(E_{10}) \,\Big|\, x \cdot v = 0 \;\forall v \in V \Big\} \subset K(E_{10})$$

For known examples, $\mathfrak{i}_V$ has *finite* co-dimension in $K(E_{10})$

$\Rightarrow$ $\mathfrak{i}_V^\perp \equiv K(E_{10}) \ominus \mathfrak{i}_V$ **is *not* a subalgebra of $K(E_{10})$!**

... but rather a *distribution space* [Kleinschmidt,Palmkvist,HN:JHEP(2007)051]

Analysis of fermionic sector of $D=11$ SUGRA $\Rightarrow$

Spin-$\frac{1}{2}$ ('Dirac representation' V_D): [deBuyl,Henneaux,Paulot(2005)]

$$J^{(0)}_{ab}\chi = \frac{1}{2}\Gamma_{ab}\chi, \qquad J^{(1)}_{abc}\chi = \frac{1}{2}\Gamma_{abc}\chi$$

Spin-$\frac{3}{2}$ ('Rarita-Schwinger representation' V_{RS}) [DKN,dBHP(2006)]

$$J^{(0)}_{ab}\psi_c = \frac{1}{2}\Gamma_{ab}\psi_c + 2\delta_c^{[a}\psi^{b]}\;, \qquad J^{(1)}_{abc}\psi_d = \frac{1}{2}\Gamma_{abc}\psi_d + 4\delta_d^{[a}\Gamma^b\psi^{c]} - \Gamma_d{}^{[ab}\psi^{c]}.$$

In both examples multiple commutators generate full $K(E_{10})$ algebra:

$$\left[J^{(1)}_{abc}\,,\,J^{(1)}_{def}\right] = J^{(2)}_{abcdef} + \delta^{[de}_{[ab}J^{(0)\,f]}_{c]} \qquad etc.$$

Quotient algebras:

$$K(E_{10})/\mathfrak{i}_{V_D} = \mathfrak{so}(32) \not\subset K(E_{10})$$
$$K(E_{10})/\mathfrak{i}_{V_{RS}} = \mathfrak{so}(288,32) \not\subset K(E_{10})$$

Rarita-Schwinger equation can be reformulated as a (kind of) '$K(E_{10})$ covariant Dirac equation'. [DKN: 0606105]

Subalgebras of $K(E_{10})$ [cf. Kleinschmidt,HN:1602.04116]

(a)	$\mathfrak{so}(10)$	**SUGRA in** $D=11$
(b)	$\mathfrak{so}(2)\oplus\mathfrak{so}(16)$	**SUGRA in** $D=3$
(c)	$\mathfrak{so}(9)\oplus\mathfrak{so}(2)$	**IIB SUGRA in** $D=10$
(d)	$\mathfrak{so}(9)\oplus\mathfrak{so}(9)$	**mIIA SUGRA in** $D=10$

Decomposing the spin-$\frac{3}{2}$ representation

$$\begin{aligned}
\mathbf{320} \quad &\xrightarrow{a} \quad \mathbf{288} \oplus \mathbf{32} \\
&\xrightarrow{b} \quad \left(\frac{1}{2}, \mathbf{128}_c\right) \oplus \left(\frac{1}{2}, \mathbf{16}_v\right) \oplus \left(\frac{3}{2}, \mathbf{16}_v\right) \\
&\xrightarrow{c} \quad \left(\mathbf{16}, \frac{3}{2}\right) \oplus \left(\mathbf{128}, \frac{1}{2}\right) \oplus \left(\mathbf{16}, \frac{1}{2}\right) \\
&\xrightarrow{d} \quad (\mathbf{9}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{9}) \oplus (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{1})
\end{aligned}$$

In particular: decompositions of $K(E_{10})$ w.r.t. $\mathfrak{so}(10)$, $\mathfrak{so}(9) \oplus \mathfrak{so}(2)$ and $\mathfrak{so}(9) \oplus \mathfrak{so}(9)$ yield correct fermion assignments for $D = 11$, mIIA and IIB supergravity.

$\Rightarrow$ **$K(E_{10})$ unifies known R symmetries.** `[KN: hep-th/0603205]`

Γ-matrices for $K(E_{10})$

Wall basis for roots $\alpha = \sum p_{\mathsf{a}} e^{\mathsf{a}}$, $\beta = \sum q_{\mathsf{a}} e^{\mathsf{a}}$ **with simple roots**

$$\alpha_1 = (1\,-\!100000000)\,, \cdots\,,\ \alpha_9 = (000000001\,-\!1)\,,\ \alpha_0 = (0000000111)$$

and $\alpha \cdot \beta = G^{\mathsf{ab}} p_{\mathsf{a}} q_{\mathsf{b}} \ \Rightarrow\ \alpha_i \cdot \alpha_j = A_{ij}$ **(**$\equiv$ **Cartan matrix of** E_{10}**).**

For any E_{10} **root** α **(or any element of** E_{10} **root lattice) we define**

$$\Gamma(\alpha) := (\Gamma_1)^{p_1} \cdots (\Gamma_{10})^{p_{10}}$$

Then $\Gamma(\alpha)\Gamma(\beta) = \varepsilon_{\alpha,\beta}\,\Gamma(\alpha \pm \beta)$ **with cocycle** $\varepsilon_{\alpha,\beta} \equiv (-1)^{\sum_{\mathsf{a}<\mathsf{b}} q_{\mathsf{a}} p_{\mathsf{b}}} \ \Rightarrow$

$$\alpha \cdot \beta \in 2\mathbb{Z} \Longrightarrow \begin{cases} \big[\Gamma(\alpha), \Gamma(\beta)\big] \ = \ 0 \\ \{\Gamma(\alpha), \Gamma(\beta)\} \ = \ 2\epsilon_{\alpha,\beta}\Gamma(\alpha \pm \beta) \end{cases}$$

$$\alpha \cdot \beta \in 2\mathbb{Z} + 1 \Longrightarrow \begin{cases} \big[\Gamma(\alpha), \Gamma(\beta)\big] \ = \ 2\epsilon_{\alpha,\beta}\Gamma(\alpha \pm \beta) \\ \{\Gamma(\alpha), \Gamma(\beta)\} \ = \ 0 \end{cases}$$

Then $x_i \to \frac{1}{2}\Gamma(\alpha_i)$ **provides a realization of Serre-like relations!**

Multiple commutation shows that $\frac{1}{2}\Gamma(\alpha)$ **provides realisation** ***for all*** **real roots of** E_{10} **(of which there are infinitely many)!**

Higher spin realizations of $K(E_{10})$

$\rightarrow$ For $s > \frac{3}{2}$ these go beyond supergravity!

But first need to re-write spin-$\frac{3}{2}$ by means of crucial redefinition [Damour,Hillmann:0906.3116]

$$\phi_A^{\mathsf{a}} \equiv \sum_{B=1}^{32} \Gamma^a_{AB} \psi^a_B \quad \textbf{(no sum on } a\textbf{!)}$$

Re-definition breaks manifest Lorentz symmetry, but:

$$\{\psi^a_A\,,\,\psi^b_B\}_{\text{Dirac}} = \delta^{ab}\delta_{AB} - \frac{1}{9}(\Gamma^a\Gamma^b)_{AB} \quad \Rightarrow \quad \{\phi^{\mathsf{a}}_A\,,\,\phi^{\mathsf{b}}_B\} = G^{\mathsf{ab}}\delta_{AB}$$

$\Rightarrow$ manifest $SO(1,9)$ = invariance group of mini-superspace WDW Hamiltonian with DeWitt metric G_{ab} instead!

From analysis of known $K(E_{10})$ transformation acting in RS representation we extract a *second quantised realisation* of $\hat{J}(\alpha)$ *for all real roots* $\alpha \in \Delta(E_{10})$:

$$\hat{J}(\alpha) = \left(-\frac{1}{2}\alpha_{\mathsf{a}}\alpha_{\mathsf{b}} + \frac{1}{4}G_{\mathsf{ab}}\right)\phi^{\mathsf{a}}\Gamma(\alpha)\phi^{\mathsf{b}} \qquad \forall \textbf{ roots obeying } \alpha^2 = 2$$

[**NB: formula also valid for** $K(AE_3)$ [Damour,Spindel,1406.1309]]

There exists a *new* realization with 'spin-$\frac{5}{2}$' fermionic operators [Kleinschmidt,HN.:1307.0413]

$$\{\phi_A^{\mathsf{ab}}\,,\,\phi_B^{\mathsf{cd}}\} = G^{\mathsf{a(c}}G^{\mathsf{d)b}}\delta_{AB} \qquad (\phi_A^{\mathsf{ab}} = \phi_A^{\mathsf{ba}})$$

→ a fermionic Fock space $\mathcal{F}$ of dimension 2^{880}!

Then, Serre-like relations are satisfied on $\mathcal{F}$ with

$$\hat{J}(\alpha) = X(\alpha)_{\mathsf{ab\,cd}}\,\phi^{\mathsf{ab}}\Gamma(\alpha)\phi^{\mathsf{cd}}$$

and

$$X(\alpha)_{\mathsf{ab\,cd}} = \frac{1}{2}\alpha_{\mathsf{a}}\alpha_{\mathsf{b}}\alpha_{\mathsf{c}}\alpha_{\mathsf{d}} - \alpha_{(\mathsf{a}}G_{\mathsf{b})(\mathsf{c}}\alpha_{\mathsf{d})} + \frac{1}{4}G_{\mathsf{a(c}}G_{\mathsf{d)b}}$$

again *for all* real roots α!

⇒ novel realisation of $K(E_{10})$ *beyond* supergravity!

‘ Spin-$\frac{7}{2}$ ’

Construction also works for spin-$\frac{7}{2}$ fermions:

$$\left\{\phi_A^{\mathtt{abc}}, \phi_{\mathtt{def}\,B}\right\} = \delta_{(\mathtt{d}}^{(\mathtt{a}}\delta_{\mathtt{e}}^{\mathtt{b}}\delta_{\mathtt{f})}^{\mathtt{c})}\delta_{AB}$$

Then ‘Serre-like’ relations are again obeyed with

$$\hat{J}(\alpha) = X(\alpha)_{\mathtt{abc}\,\mathtt{def}}\,\phi^{\mathtt{abc}}\Gamma(\alpha)\phi^{\mathtt{def}}$$

and

$$\begin{aligned}
X_{\mathtt{abc}}{}^{\mathtt{def}}(\alpha) \;=\; & -\frac{1}{3}\alpha_{\mathtt{a}}\alpha_{\mathtt{b}}\alpha_{\mathtt{c}}\alpha^{\mathtt{d}}\alpha^{\mathtt{e}}\alpha^{\mathtt{f}} + \frac{3}{2}\alpha_{(\alpha}\alpha_{\mathtt{b}}\delta_{\mathtt{c})}^{(\mathtt{d}}\alpha^{\mathtt{d}}\alpha^{\mathtt{e}}\alpha^{\mathtt{f})} - \frac{3}{2}\alpha_{(\mathtt{a}}\delta_{\mathtt{b}}^{(\mathtt{d}}\delta_{\mathtt{c})}^{\mathtt{e}}\alpha^{\mathtt{f})} \\
& + \frac{1}{4}\delta_{(\mathtt{a}}^{(\mathtt{d}}\delta_{\mathtt{b}}^{\mathtt{e}}\delta_{\mathtt{c})}^{\mathtt{f})} + \frac{1}{12}\big(2-\sqrt{3}\big)\alpha_{(\mathtt{a}}G_{\mathtt{bc})}G^{(\mathtt{de}}\alpha^{\mathtt{f})} \\
& \frac{1}{12}\big(-1+\sqrt{3}\big)\Big(\alpha_{\mathtt{a}}\alpha_{\mathtt{b}}\alpha_{\mathtt{c}}G^{(\mathtt{de}}\alpha^{\mathtt{f})} + \alpha_{(\mathtt{a}}G_{\mathtt{bc})}\alpha^{\mathtt{d}}\alpha^{\mathtt{e}}\alpha^{\mathtt{f}}\Big)
\end{aligned}$$

Fermionic Fock space has dimension $\dim(\mathcal{F}) = 2^{3520}$.

As before, $\hat{J}(\alpha)$ provides a realisation *for all* real roots.

- **‘Higher spin’ *not* in ordinary space-time, but in (some variant of) Wheeler-DeWitt superspace!**
- **Restriction to** $\mathrm{E}_8 \subset E_{10}$ **must yield representations of** $\mathrm{K}(E_8) \equiv Spin(16)/Z_2$ $\rightarrow$ **for new realisations we find** $\mathbf{560}_v$ **for** $s = \frac{5}{2}$ **and** $\mathbf{1920}_s$ **for** $s = \frac{7}{2}$ $\rightarrow$ **implies strong restrictions beyond: e.g. no solution for** $s = \frac{9}{2}, \frac{11}{2}, \frac{13}{2}$ **!**
- **Another strange feature: decomposition under SO(10)** $\subset K(E_{10})$**:** $\mathbf{1760} \rightarrow \mathbf{1120} \oplus 2 \times \mathbf{288} \oplus 2 \times \mathbf{32}$**.** $\phi_A^{\mathsf{ab}} \rightarrow \psi_A^a$ **and** $\psi_A^{[ab]}$ **(= RS field strength?)**
- **Suggests nested structure of higher spin realizations that penetrate farther and farther into** $K(E_{10})$**...** **... but systematics (if any) is not known.**
- **Affine case** $\rightarrow$ **novel representations for** $K(\mathrm{E}_9)$**.**

SUSY Constraint and $K(E_{10})$

SUSY Constraint from canonical analysis:

$$\begin{aligned}\tilde{\mathcal{S}} \;&=\; \Gamma^{ab}\Big[\partial_a\psi_b + \frac{1}{4}\omega_{acd}\Gamma^{cd}\psi_b + \omega_{abc}\psi_c + \frac{1}{2}\omega_{ac0}\Gamma^c\Gamma^0\psi_b\Big] \\ &\quad + \frac{1}{4}F_{0abc}\Gamma^0\Gamma^{ab}\psi^c + \frac{1}{48}F_{abcd}\Gamma^{abcde}\psi_e\end{aligned}$$

Rewrite in terms of E_{10} coset variables (up to $\ell = 3$)

$$\begin{aligned}\mathcal{S} = \Big(P^{(0)}_{ab}\Gamma^a - P^{(0)}_{cc}\Gamma_b\Big)\Psi^b + \frac{1}{2}P^{(1)}_{abc}\Gamma^{ab}\Psi^c + \frac{1}{5!}P^{(2)}_{abcdef}\Gamma^{abcde}\Psi^f \\ + \frac{1}{6!}\Big(P^{(3)}_{a|ac_1\cdots c_7}\Gamma^{c_1\cdots c_6}\Psi^{c_7} - \frac{1}{28}P^{(3)}_{a|c_1\cdots c_8}\Gamma^{c_1\cdots c_8}\Psi^a\Big)\end{aligned}$$

Rewrite as a partial sum over (real and null) E_{10} roots:

$$\mathcal{S}_A = \pi_{\mathsf{a}}\phi^{\mathsf{a}}_A + \sum_{\substack{\alpha^2=2\\ \ell\leq 3,\alpha>0}} P_\alpha\Big(\Gamma(\alpha)\phi(\alpha)\Big)_A + \sum_{\substack{\delta^2=0\\ \ell=3}} P^r_\delta\Big(\Gamma(\delta)\phi(\epsilon^r)\Big)_A \quad (+\cdots ???)$$

with $\phi(v)_A \equiv v_{\mathsf{a}}\phi^{\mathsf{a}}_A$ $\rightarrow$ can we extend sum to imaginary roots?

$\rightarrow$ need higher-spin realisations to soak up polarisations?

SUSY constraint algebra

Canonical constraint superalgebra [Damour,Kleinschmidt,HN, CQG24(2007)046]

$$\{\mathcal{S}_A\,,\,\mathcal{S}_B\} = \delta_{AB}\mathcal{H} + \sum_\delta \mathfrak{L}_\delta\,\Gamma(\delta)_{AB} \;\; + \;\cdots$$

Supergravity Hamiltonian $\mathcal{H}$ and E_{10} Casimir H agree up to $\ell = 2$, but start to differ for $\ell \geq 3$ $\;\rightarrow$ more $K(E_{10})$ invariants???

The other (bosonic) canonical supergravity constraints $\mathfrak{L}_\delta$ are all associated with null roots of E_{10}: [Damour,Kleinschmidt,HN, CMP302(2011)755]

- **Diffeomorphisms:** $\delta = [0\,1\,2\,3\,4\,5\,6\,4\,2\,3]$ **= affine null root ($\ell = 3$).**
- **Gauss Constraint:** $\delta' = [1\,2\,3\,4\,5\,6\,7\,4\,2\,4]$ $\;(\ell = 4)$
- **'Dual Gauss Constraint' (Bianchi):** $\delta'' = [1\,2\,3\,4\,5\,7\,9\,6\,3\,5]$ $\;(\ell = 5)$
- **'Dual diffeomorphisms' (Bianchi):** $\delta''' = [1\,2\,3\,5\,7\,9\,11\,7\,3\,6]$ $\;(\ell = 6)$

Recall affine Sugawara $\mathfrak{L}_{m\delta} \propto \sum : J^a_{m-n} J^a_n :$ and δ = affine null root

$\rightarrow$ **is there a *hyperbolic* analog of the Sugawara construction?**

$N = 8$ Supergravity: a strange coincidence?

$SO(8) \to SU(3)\times U(1)$ **breaking and 'family-color locking'**

$$
\begin{array}{lcll}
(u\,,\,c\,,\,t)_L & : & \mathbf{3}_c \times \bar{\mathbf{3}}_f \to \mathbf{8} \oplus \mathbf{1}\,, & Q = \frac{2}{3} - q \\
(\bar{u}\,,\,\bar{c}\,,\,\bar{t})_L & : & \bar{\mathbf{3}}_c \times \mathbf{3}_f \to \mathbf{8} \oplus \mathbf{1}\,, & Q = -\frac{2}{3} + q \\
(d\,,\,s\,,\,b)_L & : & \mathbf{3}_c \times \mathbf{3}_f \to \mathbf{6} \oplus \bar{\mathbf{3}}\,, & Q = -\frac{1}{3} + q \\
(\bar{d}\,,\,\bar{s}\,,\,\bar{b})_L & : & \bar{\mathbf{3}}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{6}} \oplus \mathbf{3}\,, & Q = \frac{1}{3} - q \\
(e^-, \mu^-, \tau^-)_L & : & \mathbf{1}_c \times \mathbf{3}_f \to \mathbf{3}\,, & Q = -1 + q \\
(e^+, \mu^+, \tau^+)_L & : & \mathbf{1}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{3}}\,, & Q = 1 - q \\
(\nu_e\,,\,\nu_\mu\,,\,\nu_\tau)_L & : & \mathbf{1}_c \times \bar{\mathbf{3}}_f \to \bar{\mathbf{3}}\,, & Q = -q \\
(\bar{\nu}_e\,,\,\bar{\nu}_\mu\,,\,\bar{\nu}_\tau)_L & : & \mathbf{1}_c \times \mathbf{3}_f \to \mathbf{3}\,, & Q = q
\end{array}
$$

Supergravity and Standard Model assignments agree if spurion charge is chosen as $q = \frac{1}{6}$ [Gell-Mann (1983)]

Realized at $SU(3)\times U(1)$ **stationary point!** [Warner,HN, NPB259(1985)412]

Fixing the spurion charge

[Meissner,HN: Phys.Rev.D91(2015)065029; Kleinschmidt,HN: 1504.01586]

But need to go *beyond* $N=8$ supergravity!

Spurion charge shift can be realised via $\mathbf{U(1)}_q$

$$\mathcal{I} = \frac{1}{2}\big(T \wedge \mathbf{1} \wedge \mathbf{1} + \mathbf{1} \wedge T \wedge \mathbf{1} + \mathbf{1} \wedge \mathbf{1} \wedge T + T \wedge T \wedge T\big)$$

acting on 56 fermions χ^{ijk} in $\mathbf{8} \wedge \mathbf{8} \wedge \mathbf{8}$ of SU(8), with $T = \varepsilon \otimes \mathbf{1}_4$ (imaginary unit in $\mathbf{SU(3)} \times \mathbf{U(1)}$ breaking).

$\mathcal{I}$ is *not* in SU(8) $\equiv K(\mathrm{E}_7)$ ... but it is in $K(E_{10})$!

The proof requires over-extended root of $E_{10} \Rightarrow$ no way to realise q-shift with finite-dimensional R symmetries!

It would be rather striking if $K(E_{10})$ were needed to relate $N=8$ supergravity to Standard Model fermions...

Also: $K(E_{10}) \supset W(E_{10}) \supset W(E_7) \supset PSL_2(7)$

$\rightarrow$ **a new family symmetry?** [cf.: Chen,Perez,Ramond,1412.6107]

Summary and Outlook

- **All results obtained so far indicate that E_{10} requires a setting beyond known concepts of space and time.**
- **In this case space-time, and with it, concepts such as general covariance and local supersymmetry would have to be *emergent*.**
- **Fermionic sector: covariance in space-time replaced by covariance in generalized WDW moduli space.**
- **Need to resolve dichotomy between *finitely many fermionic* and *infinitely many bosonic* degrees of freedom $\rightarrow$ may require some kind of bosonization?**
- **SUGRA Hamiltonian *vs.* quadratic Casimir of E_{10}: a definite mismatch between E_{10} and maximal supersymmetry?**

Summary and Outlook

- **Apparent incompatibility of $K(E_{10})$ and supersymmetry for imaginary (null and timelike) roots $\rightarrow$ a new way to break, or rather *avoid*, supersymmetry with *even more* symmetry?**
- **$\Rightarrow$ Can E_{10} supersede SUSY as a unifying principle?**
- **Despite the existence of (at least) 10^{272000} string vacua**

  [most recent figures from: Taylor,Wang:1511.03209; Schellekens:1601.02462]

  $N = 8$ Supergravity remains the only theory that (after complete breaking of supersymmetry) gives 48 spin-$\frac{1}{2}$ fermions, and nothing more.